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ON SPREADING SEQUENCES AND ASYMPTOTIC STRUCTURES

D. FREEMAN, E. ODELL, B. SARI, AND B. ZHENG

Abstract. In the first part of the paper we study the structure of Banach spaces with a conditional spreading basis. The geometry of such spaces exhibits a striking resemblance to the geometry of James space. Further, we show that the averaging projections onto subspaces spanned by constant coefficient blocks with no gaps between supports are bounded. As a consequence, every Banach space with a spreading basis contains a complemented subspace with an unconditional basis. This gives an affirmative answer to a question of H. Rosenthal.

The second part contains two results on Banach spaces X whose asymptotic structures are closely related to c_0 and do not contain a copy of ℓ_1:

i) Suppose X has a normalized weakly null basis (x_i) and every spreading model (e_i) of a normalized weakly null block basis satisfies $\|e_1 - e_2\| = 1$. Then some subsequence of (x_i) is equivalent to the unit vector basis of c_0. This generalizes a similar theorem of Odell and Schlumprecht and yields a new proof of the Elton–Odell theorem on the existence of infinite $(1+\varepsilon)$-separated sequences in the unit sphere of an arbitrary infinite dimensional Banach space.

ii) Suppose that all asymptotic models of X generated by weakly null arrays are equivalent to the unit vector basis of c_0. Then X^* is separable and X is asymptotic-c_0 with respect to a shrinking basis (y_i) of $Y \supseteq X$.

1. Introduction

A basic sequence (x_i) in a Banach space is called *spreading* if it is equivalent to all of its subsequences. If, in addition, the sequence is unconditional, then it is called *subsymmetric*. When (x_i) is spreading and weakly null it is automatically suppression unconditional. In Section 2 we will focus most of our attention on spreading sequences that are not unconditional. A famous example is the boundedly complete basis of the James space J, and we shall see that much of the structure for J holds more generally for Banach spaces with a conditional spreading basis. We observe that if (e_i) is a normalized conditional spreading basis for X, then the difference sequence $(d_i) = (e_1, e_2 - e_1, e_3 - e_2, \ldots)$ is a skipped unconditional basis for X. This means that if (x_j) is a normalized block basis of (d_i) with $\mathrm{supp}(x_j) < i_j < \mathrm{supp}(x_{j+1})$ for some subsequence (i_j) of $\mathbb{N}$, then (x_j) is unconditional. Here

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Edward Odell (1947–2013).

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$\mathrm{supp}(x_j)$ refers to the basis (d_i); that is, if $x_j = \sum_i b_i^j d_i$, then $\mathrm{supp}(x_j) = \{i : b_i^j \neq 0\}$. It follows that in the case (e_i) is spreading but not weakly null, $\ell_1 \not\hookrightarrow X$ (ℓ_1 does not embed isomorphically into X) if and only if the difference basis (d_i) is shrinking. Also we show that $c_0 \not\hookrightarrow X$ if and only if (e_i) is boundedly complete. Furthermore, c_0 and ℓ_1 do not embed into X if and only if X is quasi-reflexive of order 1. It is interesting to note that these (except the skipped unconditionality result) were already observed in the 1970's by Brunel and Sucheston [BS] for ESA (equal sign additive) bases, which is a stronger property than spreading. However, our results are more general and the proofs are different. The crucial part of our approach is an unconditionality result, Theorem 2.3(a), which is of independent interest. We also show that the well-known averaging projection onto disjoint subsets of a subsymmetric basis remains bounded for the conditional spreading case as long as the subsets form a partition. One consequence is that X is isomorphic to $D \oplus X$ where D is the subspace spanned by $(d_{2n})_{n=1}^{\infty}$. Moreover, every Banach space with a spreading basis contains a complemented subspace with an unconditional basis. This answers an open problem of H. Rosenthal.

In Section 3 we make a few remarks on Banach spaces that admit conditional spreading models. Our study of the conditional spreading sequences was motivated by the problems discussed in this section.

In Section 4 we consider spaces whose asymptotic structure is closely related to c_0. In [OS] it was shown that if (x_i) is a basis for X and any spreading model (e_i) of a normalized block basis of (x_i) is 1-equivalent to the unit vector basis of c_0 (in fact, it is sufficient to assume that $\|e_1 + e_2\| = 1$), then c_0 embeds into X. Our first result of Section 4 generalizes this as follows. If (x_i) is weakly null and if every spreading model (e_i) generated by a *weakly null* block basis satisfies $\|e_1 - e_2\| = 1$ and $\ell_1 \not\hookrightarrow X$, then $c_0 \hookrightarrow X$. This yields a quick proof of the Elton–Odell theorem [EO]. Namely, for every Banach space X there exists an infinite sequence (z_i) in the unit sphere S_X and $\lambda > 1$ so that $\|z_i - z_j\| \geq \lambda$ for all $i \neq j$. Indeed, if X contains ℓ_1 or c_0 the result follows easily by the non-distortability of c_0 and ℓ_1. Otherwise, fix a weakly null normalized sequence (x_i). By our theorem, (x_i) must have a normalized block basis with a spreading model (e_i) with $\|e_1 - e_2\| > 1$ which yields an $\varepsilon > 0$ and an infinite $(1+\varepsilon)$-separated sequence. Note that one cannot similarly deduce this from [OS]. It is easy to construct examples of spreading (e_i) so that $\|e_i + e_j\| = 2$, while $\|e_i - e_j\| = 1$ for all $i < j$ (see Example 3.1 of [OS]).

One of the long-standing open problems on asymptotic structures of Banach spaces is the following. Suppose that every spreading model of X is equivalent to the unit vector basis of c_0 (or ℓ_p). Does X contain an asymptotic-c_0 (or asymptotic-ℓ_p) subspace? We solve the c_0 case with a somewhat stronger assumption. If all normalized *asymptotic models* (e_i) of normalized weakly null arrays in X are equivalent to the unit vector basis of c_0 and $\ell_1 \not\hookrightarrow X$, then X^* is separable and X is asymptotic-c_0 with respect to a shrinking basis (y_i) of $Y \supseteq X$. Recall that (e_i) is an asymptotic model of X, denoted by $(e_i) \in AM_w(X)$, if there exists a normalized array $(x_j^i)_{i,j\in\mathbb{N}}$ so that $(x_j^i)_{j=1}^{\infty}$ is weakly null for all $i \in \mathbb{N}$, and for some $\varepsilon_n \downarrow 0$, all n, and all $(a_i)_1^n \subseteq [-1,1]$ and $n \leq k_1 < k_2 < \cdots < k_n$,

$$\left| \left\| \sum_{i=1}^{n} a_i x_{k_i}^i \right\| - \left\| \sum_{i=1}^{n} a_i e_i \right\| \right| \leq \varepsilon_n. \tag{1.1}$$

The notion of asymptotic models is a direct generalization of spreading models and was introduced in [HO]. X is asymptotic-c_0 if for some $K < \infty$ for all n and all asymptotic spaces $(e_i)_{i=1}^n$ are K-equivalent to the unit vector basis of ℓ_∞^n [MMT]. These notions are recalled in Section 4.

2. Spreading bases

We begin with a result solving a problem asked of us by S. A. Argyros.

Theorem 2.1. *Let (e_n) be a normalized basis for X. If every subspace spanned by a skipped block basis of (e_n) is reflexive, then X is either reflexive or quasi-reflexive of order* 1.

Proof. The hypothesis yields that (e_n) is shrinking. If not, then for some normalized block basis (x_n) of (e_n) there exists $f \in B_{X^*}$ and $\varepsilon > 0$ with $f(x_n) > \varepsilon$ for all n. But then (x_{2n}) is a skipped block basis of (e_n) which cannot be shrinking, hence cannot span a reflexive space.

Let $F \in X^{**}$. Since the basis (e_i) is shrinking F is the w^*-limit of

$$(\sum_{i=1}^n F(e_i^*)e_i)_{n=1}^\infty,$$

where (e_i^*) is the biorthogonal sequence to (e_i) (a basis for X^*). We claim that if

$$\liminf_n |F(e_i^*)| = 0,$$

then $F \in i(X)$, where $i(X)$ is the natural embedding of X into X^{**}.

Indeed, pick a subsequence (i_j) such that $\sum_{j=1}^\infty |F(e_{i_j}^*)| < \infty$. Let

$$y = \sum_{j=1}^\infty F(e_{i_j}^*)e_{i_j}.$$

Then $y \in i(X)$. Let $G = F - y$. Then $G = w^* - \lim_n \sum_{j=1}^n \sum_{i_j<i<i_{j+1}} F(e_i^*)e_i$ and $(\sum_{i_j<i<i_{j+1}} F(e_i^*)e_i)_{j=1}^\infty$ is a skipped block sequence which spans a reflexive subspace. Thus $G \in i(X)$ and so is F.

Now suppose X is not reflexive and let $G \in X^{**}$ and $F \in X^{**} \setminus i(X)$. Choose $\lambda \in \mathbb{R}$ and a subsequence (i_n) of $\mathbb{N}$ so that $G(e_{i_n}^*) - \lambda F(e_{i_n}^*) \to 0$. Then by the claim above we conclude that $G - \lambda F \in i(X)$. Therefore $X^{**} = \mathbb{R}F \oplus i(X)$. ☐

Remark 2.2. A generalization of the above from a basis to finite dimensional decompositions (FDD) is false. Indeed, the Argyros-Haydon space $\mathfrak{X}_K$ has an FDD (M_n) with the property that every skipped blocking of (M_n) spans a reflexive subspace and yet its dual is isomorphic to ℓ_1 [AH, Theorem 9.1]. We thank Pavlos Motakis for pointing out the example.

We now turn to conditional spreading bases. Suppose that (e_i) is a normalized spreading basis for X which is not weakly null. Then the *summing functional*,

$$S(\sum_i a_i e_i) := \sum_i a_i,$$

is bounded on X. Indeed for some $\lambda \neq 0$, $f \in X^*$, and subsequence (i_n) of $\mathbb{N}$ we have that $f(e_{i_n}) - \lambda \to 0$ rapidly. So a perturbation of $\lambda^{-1}f$ is constantly 1 on the e_{i_n}'s. Then it follows from the spreading property that S is bounded on X.

By renorming we can assume that (e_i) is normalized, 1-spreading, and a bimonotone basis for X, and $\|S\| = 1$. This is easily achieved by replacing (e_i) by a spreading model of a subsequence and then by the renorming $|||x||| := \max(\|x\|, |S(x)|)$. With this we also get that the functional $S_I(\sum_i a_i e_i) := \sum_{i\in I} a_i$ is of norm one for any interval I. Note that the boundedness of S implies that the summing basis of c_0 is dominated by every conditional spreading sequence.

Theorem 2.3. *Let (e_i) be a normalized 1-spreading, non-weakly null, bimonotone basis for X.*

- (a) *If (x_i) is a normalized block basis of (e_i) with $S(x_i) = 0$ for all i, then (x_i) is suppression 1-unconditional.*
- (b) *Let $(d_i) = (e_1, e_2 - e_1, e_3 - e_2, \ldots)$. Then (d_i) is a skipped unconditional basis for X.*
- (c) *(e_i) is boundedly complete if and only if $c_0 \not\hookrightarrow X$.*
- (d) *(d_i) is shrinking if and only if $\ell_1 \not\hookrightarrow X$.*
- (e) *$\ell_1 \not\hookrightarrow X$ if and only if $X^* = \mathbb{R}S \oplus [(e_i^*)]$.*
- (f) *c_0 and ℓ_1 do not embed into X if and only if X is quasi-reflexive of order 1.*

Proof. For $x, y \in X$ which are finitely supported with respect to the basis (e_i), we write $x \sim y$ if

$$x = \sum_{i=1}^{k} a_i e_{n_i} \quad \text{and} \quad y = \sum_{i=1}^{k} a_i e_{m_i} \quad \text{where} \quad n_1 < \cdots < n_k,\ m_1 < \cdots < m_k.$$

(a) Let (x_i) be as in (a). We now need the following lemma.

Lemma 2.4. *For all $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $f \in S_{X^*}$ there exists $\tilde{x} \in X$, $\tilde{x} \sim x_{i_0}$ and $\operatorname{supp}(\tilde{x}) \subseteq [j, m]$, $j = \min\operatorname{supp}(x_{i_0})$, so that $|f(\tilde{x})| < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and $i_0 \in \mathbb{N}$. Since $|f(e_i)| \le 1$ for any $f \in S_{X^*}$, by the pigeonhole principle there exists m with the following property:

Let $j = \min\operatorname{supp}(x_{i_0})$. For all $f \in S_{X^*}$ there exists $\lambda \in [-1, 1]$ and $F \subseteq [j, m]$ with $|F| = k = |\operatorname{supp}(x_{i_0})|$ so that for $i \in F$, $|f(e_i) - \lambda| < \varepsilon/k$.

Place $\tilde{x} \equiv \sum_{i\in F} a_i e_i$ on F so that $\tilde{x} \sim x_{i_0}$. Then $S(\tilde{x}) = S(x_{i_0}) = 0$ and

$$|f(\tilde{x})| \le |f(\tilde{x} - \lambda S(\tilde{x}))| + |\lambda S(\tilde{x})| = \Big|\sum_{i\in F} a_i (f(e_i) - \lambda)\Big| < \varepsilon. \qquad \square$$

Now let $x = \sum_{i=1}^{k} a_i x_i$, $\|x\| = 1$, $\varepsilon > 0$. Let $F \subseteq \{1, 2, \ldots, k\}$. We will show that $\|\sum_{i\in F} a_i x_i\| \le 1 + \varepsilon$. Let $j_i = \min\operatorname{supp}(x_i)$ for $i \le k$ and choose m_i by Lemma 2.4 for ε/k and j_i. Since (e_i) is 1-spreading we may assume that $j_1 < m_1 < j_2 < m_2 < \cdots$. Let $f \in S_{X^*}$ with $f(\sum_{i\in F} a_i x_i) = \|\sum_{i\in F} a_i x_i\|$. For $i \notin F$, $i \le k$, we choose $\tilde{x}_i \sim x_i$ with $\operatorname{supp}(\tilde{x}_i) \subseteq [j_i, m_i]$ so that $|f(\tilde{x}_i)| < \varepsilon/k$. Then

$$\Big\|\sum_{i\in F} a_i x_i\Big\| \le \Big|f\Big(\sum_{i\in F} a_i x_i + \sum_{i\notin F} a_i \tilde{x}_i\Big)\Big| + \Big|f\Big(\sum_{i\notin F} a_i \tilde{x}\Big)\Big| \le \|x\| + \varepsilon = 1 + \varepsilon.$$

This proves (a).

(b) To see that (d_i) is a basis for X we need only note that it is basic. This is an easy calculation that holds for any difference sequence (d_i) obtained from a normalized basic (e_i) that dominates the summing basis (i.e., S is bounded). Indeed, for any $n < m$,

$$\begin{aligned}\Big\|\sum_{i=1}^{n} a_i d_i\Big\| &= \Big\|\sum_{i=1}^{n-1}(a_i - a_{i+1})e_i + a_n e_n\Big\| \le \Big\|\sum_{i=1}^{m} a_i d_i\Big\| + \|a_{n+1} e_n\| \\ &= \Big\|\sum_{i=1}^{m} a_i d_i\Big\| + \Big|S\Big(\sum_{i=n+1}^{m-1}(a_i - a_{i+1})e_i + a_m e_m\Big)\Big| \le 2\Big\|\sum_{i=1}^{m} a_i d_i\Big\|.\end{aligned}$$

That (d_i) is skipped unconditional follows from (a).

(c) We need only show that if (e_i) is not boundedly complete, then $c_0 \hookrightarrow X$. Suppose that there exists $(a_i) \subseteq \mathbb{R}$ so that $\sup_n \|\sum_{i=1}^{n} a_i e_i\| = 1$ and $\sum_{i=1}^{\infty} a_i e_i$ diverges. Choose $\delta > 0$ and a subsequence (k_i) of $\mathbb{N}$ so that $\|x_i\| > \delta$ where $x_i = \sum_{j=k_i}^{k_{i+1}-1} a_i e_i$ for $i \in \mathbb{N}$.

Choose a block sequence (y_i) of (e_i) so that $y_{2i-1} \sim x_i$ and $y_{2i} \sim x_i$ for all i. Then (y_{2i-1}) and (y_{2i}) are each equivalent to (x_i), and $(y_{2i-1} - y_{2i})$ is unconditional by (a). Furthermore

$$\sup_n \|\sum_{i=1}^{n}(y_{2i-1} - y_{2i})\| \le 2,$$

and $2 \ge \|y_{2i-1} - y_{2i}\| \ge \delta$ for all i. Thus $(y_{2i-1} - y_{2i})$ is equivalent to the unit vector basis of c_0.

(d) This follows easily since (d_i) is skipped unconditional.

(e) Suppose ℓ_1 does not embed into X. By Rosenthal's ℓ_1 theorem [R] and the fact that (e_i) is spreading, (e_i) is weak Cauchy.

Let $f \in X^*$. Then $f = w^* - \lim_n \sum_{i=1}^{n} f(e_i)e_i^*$, and $\lim_{i\to\infty} f(e_i) \equiv \lambda$ exists. Then $f - \lambda S \in [(e_i^*)]$. Indeed $f - \lambda S = w^* - \lim_{n\to\infty} \sum_{i=1}^{n} b_i e_i^*$, where $\lim_i b_i = 0$. If the series is not norm convergent there exists $\delta > 0$, $(n_i) \in [\mathbb{N}]^\omega$, and a normalized block basis (x_i) of (e_i) so that $x_1 < e_{n_1} < x_2 < e_{n_2} < \cdots$, so that $(f - \lambda S)x_i > \delta$ for all i and $b_{n_i} \to 0$ rapidly. In particular, $(x_i - S(x_i)e_{n_i})$ is unconditional and $(f - \lambda S)(x_i - S(x_i)e_{n_i}) > \delta/2$ for all i. Thus $(x_i - S(x_i)e_{n_i})$ is equivalent to the unit vector basis of ℓ_1, a contradiction.

(f) Let (u_n) be a skipped block basis of (d_i), and assume c_0 and ℓ_1 do not embed into X. Then (u_n) is unconditional and shrinking by (b) and (d) and is also boundedly complete since X does not contain c_0. Thus $[(u_n)]$ is reflexive and Theorem 2.1 yields the result. □

If X has an unconditional basis and $Y \subseteq X$ has non-separable dual, then $\ell_1 \hookrightarrow Y$ [BP]. This also holds if X has a spreading basis. In fact, the result holds more generally.

Proposition 2.5. *Suppose X has a skipped unconditional basis and let $Y \subseteq X$ with Y^* not separable. Then ℓ_1 embeds into Y.*

Proof. Assume that Y^* is not separable and ℓ_1 does not embed into Y. By Theorem 3.14 of [AJO] there exists an ℓ_1^+ weakly null tree $(y_\alpha)_{\alpha\in T_\omega}$ in Y. Here $T_\omega = \{(n_i)_1^k : n_1 < \cdots < n_k, n_i \in \mathbb{N}, k \in \mathbb{N}\}$. $(y_{(\alpha,n)})_n$ is weakly null and normalized for all $\alpha \in \{\emptyset\} \cup T_\omega$. Furthermore, for some $c > 0$, $\|\sum_i a_i y_{\alpha_i}\| \ge c\sum_i a_i$ for all branches (α_i) of T_ω and $a_i \ge 0$. Using that the tree is weakly null and X has a skipped unconditional basis it is easy to find a branch (y_{α_i}) which is unconditional, hence is equivalent to the unit vector basis of ℓ_1. This is a contradiction. □

Remark 2.6. The same proof also yields that if X is a subspace of a space with skipped unconditional finite dimensional decomposition and X^* is non-separable, then ℓ_1 embeds into X.

The next result answers a question asked of us by Rosenthal: If X has a spreading basis, does X contain a complemented subspace with an unconditional basis?

Proposition 2.7. *If (e_i) is a normalized spreading basis for X, then the subspace Y spanned by the unconditional block basis $[(e_{2n-1}-e_{2n})]$ is complemented in X.*

It suffices to prove that the complementary "projection" Q is bounded where

$$Q(\sum_i a_ie_i) = \sum_i \frac{a_{2i-1}+a_{2i}}{2}(e_{2i-1}+e_{2i}).$$

This is a consequence of the following more general result which is well known if the basis is subsymmetric.

Theorem 2.8. *Let (e_i) be a normalized bimonotone 1-spreading basis for X. Let $(\sigma_j)_{j=1}^\infty$ be a partition of $\mathbb{N}$ into successive intervals, $\sigma_1<\sigma_2<\cdots$, with $|\sigma_j|=n_j$ for $j\in\mathbb{N}$. Then the averaging operator*

$$Q\Big(\sum_i a_ie_i\Big) = \sum_{j=1}^\infty \Big(\big(\sum_{i\in\sigma_j} a_i\big)/n_j\Big)\Big(\sum_{i\in\sigma_j} e_i\Big)$$

is a bounded projection on X with $\|Q\|\le 3$.

It is important to note that, unlike the subsymmetric case, there are no gaps allowed between blocks in this averaging operator.

Proof. It suffices to prove that for all k, $\|Qx\|\le 3\|x\|$ if $\mathrm{supp}(x)\subseteq\bigcup_{i=1}^k\sigma_i$. Let $k\in\mathbb{N}$, $x=\sum_{j=1}^{\max(\sigma_k)}a_je_j$. Let M be the least common multiple of $(n_1,n_2,\dots,n_k)$ and set $m_j=M/n_j$ for $j\le k$.

We will construct vectors $(y_i)_{i=1}^{2M}$ so that $\frac{1}{2M}\sum_{j=1}^{2M}y_j = \bar{x}+\sum_{j=1}^M z_j$ where $y_i\sim x$, $2\bar{x}\sim Qx$, and $z_j\sim\frac{1}{2M}x$ for $j\le M$. It follows that

$$\|Qx\| = 2\|\bar{x}\| \le 2\Big(\|x\|+M\frac{1}{2M}\|x\|\Big)=3\|x\|.$$

To begin we spread x to obtain y_1 so that the coordinates of y_1 look like

$$y_1=(a_1,a_2,\dots,a_{n_1},0,\dots,0,a_{n_1+1},\dots,a_{n_1+n_2},0,\dots,0,a_{n_1+n_2+1},\dots).$$

For each $1\le j\le k-1$, we insert $2n_j-1$ zeros between the blocks of x corresponding to σ_j and σ_{j+1}, and let γ_j be the index set for the coordinates of the inserted block of zeros. The vectors $y_2,\dots,y_{2M}$ will be spreads of y_1. The position of the first block $(a_1,\dots,a_{n_1})$ is preserved for $y_2,\dots,y_{m_1}$. This block is then shifted one unit right for $y_{m_1+1},\dots,y_{2m_1}$. Then another unit to the right for $y_{2m_1+1},\dots,y_{3m_1}$ and so on n_1 times until reaching $y_{2M}=y_{2n_1m_1}$. The same scheme is followed for the second block $(a_{n+1},\dots,a_{n_1+n_2})$ and the subsequent blocks. Thus the second block is preserved for $y_2,\dots,y_{m_2}$ and then shifted once right for $y_{m_2+1},\dots,y_{2m_2}$.

When we average the y_j's, $\bar{x}$ will be the average of the vectors $y_1,\dots,y_{2M}$ restricted to the coordinates given by the union over $1\le j\le k$ of the first n_j coordinates of γ_j.

We give a simple example in the diagram below explaining this averaging procedure in the case $k = 2$, $n_1 = 2$, $n_2 = 3$, and so $M = 6$, $m_1 = 3$, and $m_2 = 2$.

$$\begin{array}{ccccccccccccc}
a_1 & a_2 & 0 & 0 & 0 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & 0 & 0 & 0 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & 0 & 0 & 0 & 0 & a_3 & a_4 & \mathbf{a_5} & 0 & 0 & 0 & 0 \\
0 & a_1 & \mathbf{a_2} & 0 & 0 & 0 & a_3 & a_4 & \mathbf{a_5} & 0 & 0 & 0 & 0 \\
0 & a_1 & \mathbf{a_2} & 0 & 0 & 0 & 0 & a_3 & \mathbf{a_4} & \mathbf{a_5} & 0 & 0 & 0 \\
0 & a_1 & \mathbf{a_2} & 0 & 0 & 0 & 0 & a_3 & \mathbf{a_4} & \mathbf{a_5} & 0 & 0 & 0 \\
0 & 0 & \mathbf{a_1} & \mathbf{a_2} & 0 & 0 & 0 & 0 & \mathbf{a_3} & \mathbf{a_4} & \mathbf{a_5} & 0 & 0 \\
0 & 0 & \mathbf{a_1} & \mathbf{a_2} & 0 & 0 & 0 & 0 & \mathbf{a_3} & \mathbf{a_4} & \mathbf{a_5} & 0 & 0 \\
0 & 0 & \mathbf{a_1} & \mathbf{a_2} & 0 & 0 & 0 & 0 & 0 & \mathbf{a_3} & \mathbf{a_4} & a_5 & 0 \\
0 & 0 & 0 & \mathbf{a_1} & a_2 & 0 & 0 & 0 & 0 & \mathbf{a_3} & \mathbf{a_4} & a_5 & 0 \\
0 & 0 & 0 & \mathbf{a_1} & a_2 & 0 & 0 & 0 & 0 & 0 & \mathbf{a_3} & a_4 & a_5 \\
0 & 0 & 0 & \mathbf{a_1} & a_2 & 0 & 0 & 0 & 0 & 0 & \mathbf{a_3} & a_4 & a_5
\end{array}$$

The vector $\bar{x}$ is the average of $y_1, \ldots, y_{2M}$ restricted to the coordinates given in bold type. The remaining coefficients are easily partitioned into M spreads of x. $\square$

Proposition 2.9. *Let (e_i) be a normalized conditional spreading basis for X. Let $D = [(d_{2n})]$, where (d_n) is the difference basis. Then $X \simeq D \oplus Y$ where $Y = [(e_1 + e_2, e_3 + e_4, \ldots)]$ is isomorphic to X.*

Proof. We may assume (e_i) is 1-spreading. By Proposition 2.7 and Theorem 2.8 it suffices to prove that $(e_{2n-1} + e_{2n})_{n=1}^{\infty}$ dominates (e_n). We will prove that if $x = \sum_{i=1}^n a_i e_i$, $\|x\| = 1$, then $\| \sum_{i=1}^n a_i (e_{2i-1} + e_{2i})\| \geq 2/3$. Write $x_1 = \sum_{i=1}^n a_i e_{3i-1}$, $x_2 = \sum_{i=1}^n a_i e_{3i-2}$, and $x_3 = \sum_{i=1}^n a_i e_{3i}$. Assume $\|x_1 + x_2\| = c$. Let $f \in S_{X^*}$, $1 = f(x_1)$. Then $f(x_1 + x_2) \leq c$ so $f(x_2) \leq c - 1$. Also using $\|x_1 + x_3\| = c$, $f(x_3) \leq c - 1$. Thus $c \geq -f(x_2 + x_3) \geq 2 - 2c$ and so $c \geq 2/3$. Thus, $\| \sum_{i=1}^n a_i (e_{2i-1} + e_{2i})\| = \| \sum_{i=1}^n a_i (e_{3i-2} + e_{3i-1})\| = c \geq 2/3$. Note that the argument can easily be generalized for all $\varepsilon > 0$ to get $c \geq 1 - \varepsilon$. $\square$

It has been shown that spaces X whose dual are isomorphic to ℓ_1 are quite plentiful and need not contain c_0 [BD]. Moreover, any Y with Y^* separable embeds into such a space [FOS]. But if X has a spreading basis, X^* is separable, and $\ell_1 \hookrightarrow X^*$, then $c_0 \hookrightarrow X$. This holds more generally if X^* is separable and X^{**} is not separable, assuming a spreading basis, by Theorem 2.3. More can be said if X^* is isomorphic to ℓ_1.

Theorem 2.10. *Let (e_i) be a normalized spreading basis for X and assume X^* is isomorphic to ℓ_1. Then (e_i) is equivalent to either the unit vector basis of c_0 or the summing basis.*

Proof. If (e_i) is weakly null, then it is unconditional. It follows that (e_i^*) is subsymmetric. Since $X^* \simeq \ell_1$ some subsequence of (e_i^*) is equivalent to the unit vector basis of ℓ_1, so (e_i^*) is such and so (e_i) is equivalent to the unit vector basis of c_0.

If (e_i) is not weakly null, then we consider the difference basis (d_i) of X. To show (e_i) is equivalent to the summing basis it suffices to show that (d_i) is equivalent to the unit vector basis of c_0. To do this, it suffices, by the triangle inequality, to show that (d_{2i}) is equivalent to the unit vector basis of c_0 since (d_{2i}) is equivalent to (d_{2i-1}). Now $D = [(d_{2n})]$ is complemented in X and (d_{2n}) is unconditional and shrinking. So $(d^*_{2n}|_D)$ is an unconditional basis for D^* which is isomorphic to ℓ_1, since it is complemented in $X^* \simeq \ell_1$. Thus $(d^*_{2n}|_D)$ is equivalent to the unit vector basis of ℓ_1. These are due to the fact that ℓ_1 is prime and has unique unconditional basis. Hence (d_{2n}) is equivalent to the unit vector basis of c_0. □

3. Remarks on conditional spreading models

Recall that a normalized basic sequence (e_i) is a spreading model of a sequence (x_i) if for some $\varepsilon_n \downarrow 0$, for all n, $(a_i)_1^n \subseteq [-1,1]$ positive integers $n \le k_1 < \cdots < k_n$,

$$\left| \Big\| \sum_{j=1}^n a_j x_{k_j} \Big\| - \Big\| \sum_{i=1}^n a_i e_i \Big\| \right| \le \varepsilon_n. \tag{3.1}$$

In this case (e_i) is 1-spreading, and if (x_i) is weakly null, then (e_i) is suppression 1-unconditional. We denote by $SP_w(X)$ the set of all spreading models of X generated by weakly null sequences. If (y_i) is normalized basic, then, via Ramsey theory, some subsequence (x_i) of (y_i) generates a spreading model (e_i) as in (3.1) above. If (y_i) is normalized but does not have a basic subsequence, then any basic spreading model admitted by (y_i) must be equivalent to the unit vector basis of ℓ_1. Indeed, by Rosenthal's ℓ_1 theorem we may assume (y_i) is weak Cauchy. Every non-trivial weak Cauchy sequence has a basic subsequence (see the proof of [Ro, Proposition 2.2]). Thus a subsequence (x_i) of (y_i) weakly converges to a non-zero element x_0, and $(x_i - x_0)$ generates an unconditional spreading model (u_i). So (e_i) is equivalent to $(x_0 + u_i)$ in $\langle x_0 \rangle \oplus [(u_i)]$. Since (e_i) is basic, (u_i) is not weakly null and therefore is equivalent to the unit vector basis of ℓ_1, and so is (e_i).

One of the questions of interest about spreading models is whether there exists a "small" space that is universal for all (or a large class of) spreading models. Recall that the space $C(\omega^\omega)$ is universal for all unconditional spreading models, that is, every subsymmetric basic sequence is a spreading model of $C(\omega^\omega)$ [O]. In [AM] a remarkable example of a *reflexive* space is constructed so that *every infinite dimensional subspace* of it is universal for all unconditional spreading models. For the case of conditional spreading models, S. A. Argyros raised the following problem, which partly motivated our study of conditional spreading sequences above.

Problem 3.1. Let (e_i) be a conditional normalized spreading sequence. Does there exist a quasi-reflexive of order 1 space X with a normalized basis (x_i) which generates (e_i) as a spreading model?

We show that the answer is affirmative for the summing basis of c_0. For a given basis (e_i), recall the space $J(e_i)$. For $x \in J(e_i)$, the norm is given by

$$\|x\| = \sup\Big\{ \Big\| \sum_{i=1}^k s_i(x) e_{p_i} \Big\| : s_1 < s_2 < \cdots < s_k \text{ are intervals in } \mathbb{N},\ \min s_i = p_i \Big\},$$

where $s_i(x) = \sum_{j \in s_i} a_j$, $s_i = [p_i, q_i)$, and $x = (a_j)$.

Proposition 3.2. *Let (e_i) be the unit vector basis of the dual Tsirelson space T^*. Then the space $J(e_i)$ is quasi-reflexive of order* 1 *and the spreading model generated by its natural basis is equivalent to the summing basis of c_0.*

Proof. In [BHO] it is shown that if (e_i) is a basis of a reflexive space, then $J(e_i)$ is quasi-reflexive of order 1. Thus the first assertion follows since T^* is reflexive.

Also it is easy to see that any subsequence of the basis (u_i) of $J(e_i)$ generates a spreading model equivalent to the summing basis (s_i). Indeed, to estimate the norm of a vector $x = \sum_{j=1}^{k} a_j u_{i_j}$ where $k \le i_1 < \cdots < i_k$ note that for an arbitrary $s_1 < \cdots < s_k$ we have

$$\Big\| \sum_{j=1}^{k} s_j(x) e_{i_j} \Big\|_{T^*} \le 2 \max_j \Big| \sum_{i \in s_j} a_i \Big|,$$

and the latter expression is at most twice the summing norm of x. The reverse inequality is trivial (consider intervals $s = [l, i_k], k \le l \le i_k$). □

In a follow-up work [AMS] constructions similar to the above are studied in more detail and, in particular, Problem 3.1 is solved affirmatively.

4. SPREADING AND ASYMPTOTIC MODELS

Our first result of this section is a strengthening of the c_0-part of the following theorem of Odell and Schlumprecht [OS]: *If X has a basis (x_i) so that every spreading model of a normalized block basis of (x_i) is 1-equivalent to the unit vector basis of c_0 (respectively, ℓ_1), then X contains an isomorphic copy of c_0 (respectively, ℓ_1).* Here we show that it is sufficient to restrict the assumption to those spreading models generated by weakly null block bases.

Theorem 4.1. *Let (x_i) be a normalized weakly null basis for X. Assume that ℓ_1 does not embed into X and whenever (y_i) is a normalized weakly null block basis of (x_i) with spreading model (e_i), then $\|e_1 - e_2\| = 1$. Then some subsequence of (x_i) is equivalent to the unit vector basis of c_0.*

Remark. The hypothesis yields that every spreading model (e_i) generated by a weakly null normalized sequence (y_i) is 1-equivalent to the unit vector basis of c_0. Indeed, we may assume (y_i) is a weakly null normalized block basis of (x_i). Then $\left(\frac{y_{2n-1}-y_{2n}}{\|y_{2n-1}-y_{2n}\|}\right)$ is a weakly null block basis generating the normalized spreading model $(e_{2n-1} - e_{2n})$ and so $\|e_1 - e_2 - e_3 + e_4\| = 1$. By iteration of this argument, 1-spreading, and the suppression 1-unconditionality of (e_i),

$$\Big\| \sum_{i=1}^{n} \pm e_i \Big\| = 1 \text{ for all } \pm 1 \text{ and all } n.$$

This implies (e_i) is 1-equivalent to the unit vector basis of c_0.

As was pointed out in the introduction this immediately implies the following well-known theorem of Elton and Odell [EO].

Theorem 4.2 (Elton–Odell). *Let X be an infinite dimensional Banach space. Then there exist $\lambda > 1$ and an infinite sequence $(x_i) \subset S_X$ such that $\|x_i - x_j\| \ge \lambda$ for all $i \ne j$.*

For the proof of Theorem 4.1 we need to recall some terminology. A collection $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is called *thin* if there do not exist $F, G \in \mathcal{F}$ with F being a proper initial segment of G. $\mathcal{F}$ is *large* in $M \in [\mathbb{N}]^\omega$ if for all $N \in [M]^\omega$ there exists an initial segment F of N with $F \in \mathcal{F}$. For a sequence $(x_i) \subseteq X$ and $E \in [\mathbb{N}]^{<\omega}$ we set $x_E = \sum_{i\in E} x_i$. For a thin $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ we let

$$\mathcal{F}^I = \{G \in [\mathbb{N}]^{<\omega} : G \text{ is an initial segment of some } F \in \mathcal{F}\}.$$

Lemma 4.3. *Let X and (x_i) be as in the hypothesis of Theorem* 4.1*. Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{N}$ satisfying*

$$\sup\{\|x_E\| : E \in \mathcal{F}\} < \infty. \tag{4.1}$$

Then there exists $M \in [\mathbb{N}]^\omega$ so that for all $E_1 < E_2 < \cdots$ with $E_i \in \mathcal{F} \cap [M]^{<\omega}$ for all $i \in \mathbb{N}$, the sequence (x_{E_i}) is weakly null.

Proof. By Elton's near unconditionality theorem [E], there exists $M \subseteq \mathbb{N}$ such that for some $C < \infty$ the subsequence $(x_i)_{i\in M}$ satisfies, for all $E \subseteq F \in [M]^{<\omega}$,

$$\Big\|\sum_{i\in E} \delta_i x_i\Big\| \le C\Big\|\sum_{i\in F} \delta_i x_i\Big\| \quad \text{for all choices of signs, } \delta_i = \pm 1. \tag{4.2}$$

Suppose that for some $E_1 < E_2 < \cdots$, $E_i \in \mathcal{F}$ with $E_i \subseteq M$ for all i, the sequence (x_{E_i}) is not weakly null. Then after passing to a subsequence, there exist $\varepsilon > 0$ and $f \in B_{X^*}$ so that $f(x_{E_j}) > \varepsilon$ for all $j \in \mathbb{N}$. Since X does not contain ℓ_1, by Rosenthal's ℓ_1 theorem and passing to a further subsequence, we may assume that (x_{E_j}) is weak Cauchy.

Let $z_j = x_{E_{2j-1}} - x_{E_{2j}}$ for $j \in \mathbb{N}$. Then (z_j) is weakly null, and moreover by (4.2)

$$n\varepsilon \le \Big\|\sum_{j\in G} x_{E_{2j-1}}\Big\| \le C\Big\|\sum_{j\in G} z_j\Big\|$$

for all $|G| = n$, $n \in \mathbb{N}$. Thus $(z_j/\|z_j\|)_j$ cannot have a c_0 spreading model since $\sup_j \|z_j\| < \infty$ by the assumption (4.1). ☐

Lemma 4.4. *Let X and (x_i) be as in the hypothesis of Theorem* 4.1*. Let $\mathcal{F}$ be a thin collection of finite subsets of $\mathbb{N}$ which is large in $\mathbb{N}$. Assume that (x_{E_i}) is weakly null for all $E_1 < E_2 < \cdots$ in $\mathcal{F}$ and*

$$\limsup_n \{\|x_E\| : E \in \mathcal{F}, n \le E\} = 1. \tag{4.3}$$

Then there exists $N = (n_i) \in [\mathbb{N}]^\omega$ so that $\mathcal{G}$, defined by

$$\mathcal{G} = \Big\{\bigcup_{i=1}^k E_i : k \in \mathbb{N}, n_k = \min(E_1),\ E_1 < \cdots < E_k, E_i \in \mathcal{F} \cap [N]^{<\omega} \text{ for } i \le k\Big\},$$

is thin and large in N and furthermore $\mathcal{G}$ satisfies (4.3) (*when $\mathcal{G}$ replaces $\mathcal{F}$*).

Proof. First we note that by passing to a subsequence, using that (x_i) is normalized and weakly null, we may assume that

$$\liminf_{n\to\infty} \{\|x_E\| : n \le E \in [\mathbb{N}]^{<\omega}\} \ge 1. \tag{4.4}$$

Indeed, for each $j \in \mathbb{N}$ we may choose $f_j \in X^*$ with $\|f_j\| = 1$ such that $f_j(x_j) = \|x_j\| = 1$. Fix $\delta_n \downarrow 0$, and after passing to a subsequence we may assume that $f_n(x_j) < \delta_n 2^{-j}$ for each $n < j$. Thus, $1 - \delta_n < f_{\min E}(x_E) \le \|x_E\|$, for all $n \le E$, and (4.4) follows.

Let $\varepsilon_k \downarrow 0$ and set

$$\mathcal{A}_k = \Big\{ M \in [\mathbb{N}]^\omega : \text{if } E_1 < \cdots < E_k,\ E_i \in \mathcal{F} \text{ for } i \le k,$$
$$E = \bigcup_{i=1}^k E_i \text{ is an initial segment of } M, \text{ then } \|x_E\| \le 1 + \varepsilon_k \Big\}.$$

Note that as $\mathcal{F}$ is thin and large in $\mathbb{N}$, for each $M \in [\mathbb{N}]^\omega$ there exists unique $E_1 < \cdots < E_k$ with $E_i \in \mathcal{F}$ for $1 \le i \le k$ such that $\bigcup_{i=1}^k E_i$ is an initial segment of M. Thus, whether or not a sequence $M \in [\mathbb{N}]^\omega$ is contained in $\mathcal{A}_k$ depends entirely on a unique initial segment of M. This makes $\mathcal{A}_k \subset [\mathbb{N}]^\omega$ open in the product topology. Open sets are Ramsey, so we can find subsequences of $\mathbb{N}$, $M_1 \supset M_2 \supset \cdots$, so that either $[M_k]^\omega \subseteq \mathcal{A}_k$ or $[M_k]^\omega \cap \mathcal{A}_k = \emptyset$ for each k.

By the 1-equivalent to c_0 spreading model hypothesis we must always have $[M_k]^\omega \subseteq \mathcal{A}_k$. Let $N = (n_i)$ be a diagonal sequence, $(n_i)_{i=k}^\infty \in M_k$ for all k. Define $\mathcal{G}$ as in the statement of the lemma with respect to N. □

Proof of Theorem 4.1. We may assume, using [E] as in the proof of Lemma 4.3, that for some $C < \infty$,

$$\|x_E\| \le C\|x_F\| \text{ for all } E \subseteq F \in [\mathbb{N}]^{<\omega}. \tag{4.5}$$

We will show that for $\alpha < \omega_1$ there exists $N_\alpha = (n_i^\alpha)_i \in [\mathbb{N}]^\omega$ and $\mathcal{G}_\alpha \subseteq [N_\alpha]^{<\omega}$ so that $\mathcal{G}_\alpha$ is thin and large in N_α. Moreover, $\mathcal{G}_\alpha^I$ has Cantor-Bendixson index $CB(\mathcal{G}_\alpha^I) \ge \omega^\alpha$ and

$$\sup\{\|x_E\| : E \in \mathcal{G}_\alpha,\ n_k^\alpha \le E\} \le 1 + \varepsilon_k, \tag{4.6}$$

where $\varepsilon_k \downarrow 0$ is fixed. By (4.5) we have that

$$\sup\{\|x_E\| : E \in \mathcal{G}_\alpha^I,\ n_k^\alpha \le E\} \le C(1 + \varepsilon_k). \tag{4.7}$$

Recall that if K is a countable set, then its Cantor-Bendixson index will be a countable ordinal. Thus, the Cantor-Bendixson index of $\bigcup_{\alpha<\omega_1} \mathcal{G}_\alpha^I$ is uncountable, and it follows that for some $N = (n_i) \in [\mathbb{N}]^\omega$, 1_N is in the pointwise closure of

$$\{1_E : \|x_E\| \le 2C,\ E \in [\mathbb{N}]^{<\omega}\} \text{ in } \{0,1\}^{\mathbb{N}}.$$

Thus $\sup_k \|\sum_{i=1}^k x_{n_i}\| < \infty$, and by (4.5) we obtain that (x_{n_i}) is equivalent to the unit vector basis of c_0.

To begin we use Lemma 4.4 applied to $\{\{j\} : j \in \mathbb{N}\}$ to obtain $N_1 = (n_i^1)$ and $\mathcal{G}_1 = \{E : n_k^1 = \min E,\ |E| = k,\ E \subseteq N_1\}$ satisfying (4.6) and note that $CB(\mathcal{G}_1^I) = \omega$. Assume N_α and $\mathcal{G}_\alpha$ are chosen to satisfy the given conditions. Choose $\tilde{N}_{\alpha+1} \subseteq N_\alpha$ by Lemma 4.3. Then apply Lemma 4.4 to $\tilde{N}_{\alpha+1}$ and $\mathcal{G}_\alpha$ to obtain $N_{\alpha+1}$ and $\mathcal{G}_{\alpha+1}$. By the definition of $\mathcal{G}_{\alpha+1}$, $CB(\mathcal{G}_{\alpha+1}^I) \ge \omega^{\alpha+1}$.

If α is a limit ordinal, choose $\beta_n \uparrow \alpha$, and let $\tilde{N}_\alpha$ be a diagonal sequence of (N_{β_n}) so that $(\tilde{n}_i^\alpha)_{i=k}^\infty \subseteq N_{\beta_k}$ and (4.6) holds. Let $\tilde{\mathcal{G}}_\alpha = \{E \subseteq \tilde{N}_\alpha : E \subseteq \mathcal{G}_{\beta_n} \text{ for some } n\}$. Apply Lemmas 4.3 and 4.4 as above. □

Recall that the *n-dimensional asymptotic structure* of X (with respect to a fixed filter $\mathrm{cof}(X)$ of finite co-dimensional subspaces of X) is the collection $\{X\}_n$ of normalized basic sequences $(e_i)_1^n$ satisfying the following. For all $\varepsilon > 0$ and all $X_1 \in \mathrm{cof}(X)$ there exists $x_1 \in S_{X_1}$ such that for all $X_2 \in \mathrm{cof}(X)$ there exists $x_2 \in S_{X_2}$ so that for all $X_n \in \mathrm{cof}(X)$ there exists $x_n \in S_{X_n}$ so that $(x_i)_1^n$ is $(1+\varepsilon)$-equivalent to $(e_i)_1^n$ [MMT]. X is asymptotic-c_0 if for some $K < \infty$ and all

n, $(e_i)_1^n \in \{X\}_n$ implies that $(e_i)_1^n$ is K-equivalent to the unit vector basis of ℓ_∞^n. In this case X^* must be separable, and the condition can be described in terms of weakly null trees. Namely, X is asymptotic-c_0 (assuming X^* is separable) if and only if for some $K < \infty$ for all $n \in \mathbb{N}$ and all normalized weakly null trees $(x_\alpha)_{\alpha\in T_n}$ in X, some branch is K-equivalent to the unit vector basis of ℓ_∞^n where $T_n = \{(k_1, k_2, \ldots, k_i) : 1 \le k_1 < \cdots < k_i, i \le n\}$. Recall that $(x_\alpha)_{\alpha\in T_n}$ is weakly null if for all $\alpha = (k_1, \ldots, k_i) \in T_{n-1}$, the sequence of successors $(x_{(\alpha,k)})_{k>k_i}$ to x_α is weakly null.

The following question is open.

Problem 4.5. Suppose that ℓ_1 does not embed into X and every spreading model generated by weakly null normalized sequences in X is equivalent to the unit vector basis of c_0. Does X contain an asymptotic-c_0 subspace? Does X contain a subspace Y with Y^* separable?

Note that the space JH constructed by Hagler [H] has non-separable dual, does not contain ℓ_1, and every weakly null normalized sequence has a subsequence equivalent to the unit vector basis of c_0. So if the problem has an affirmative answer it is necessary to pass to a subspace. We will prove a weaker theorem.

Theorem 4.6. *Suppose that a Banach space X does not contain an isomorphic copy of ℓ_1 and every asymptotic model (e_i) generated by weakly null arrays in X is equivalent to the unit vector basis of c_0. Then:*

(i) *X^* is separable, and thus X embeds into a space Y with a shrinking basis (y_i).*

(ii) *X is asymptotic-c_0 (with respect to the basis (y_i)).*

Recall that (e_i) is an *asymptotic model of* X, denoted by $(e_i) \in AM_w(X)$, generated by a normalized weakly null array $(x_j^i)_{i,j\in\mathbb{N}}$ if $(x_j^i)_{j=1}^\infty$ is weakly null for all $i \in \mathbb{N}$, and for some $\varepsilon_n \downarrow 0$, all n, and all $(a_i)_1^n \subseteq [-1, 1]$ and $n \le k_1 < k_2 < \cdots < k_n$,

$$\left| \left\| \sum_{i=1}^n a_i x_{k_i}^i \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| \le \varepsilon_n. \tag{4.8}$$

Asymptotic models were introduced in [HO]. If every $(e_i) \in AM_w(X)$ is equivalent to the unit vector basis of c_0, then there exists $K < \infty$ so that every $(e_i) \in AM_w(X)$ is K-equivalent to the unit vector basis of c_0 [HO].

The hypothesis of the theorem can be contrasted with being asymptotic-c_0 as follows. The asymptotic model condition implies that for some K, every $n \in \mathbb{N}$, and normalized weakly null tree $(x_\alpha)_{\alpha\in T_n}$ of a certain type, some branch is K-equivalent to the unit vector basis of ℓ_∞^n. The "certain type" condition is: there exist n normalized weakly null sequences $(x_j^i)_{j=1}^\infty$, $1 \le i \le n$ so that if $\alpha = (\ell_1, \ldots, \ell_k)$, then $x_{\ell_k}^k = x_\alpha$. In short, the successor sequences to each $|\beta| = k - 1$ are tails of the same sequence, depending only on k, for all $1 \le k \le n$. Theorem 4.6 states that if these specific normalized weakly null trees in X each have a branch K-equivalent to the unit vector basis of ℓ_∞^n, then all normalized weakly null trees $(x_\alpha)_{\alpha\in T_n}$ in X do as well.

Proof. (i) We first show that X^* is separable. Assume not. By a result of Stegall [S] for all $\varepsilon > 0$ there exists $\Delta \subseteq S_{X^*}$, Δ is w^*-homeomorphic to the Cantor set, and a Haar-like system $(x_{n,i}) \subseteq X$. More precisely, there exists a sequence $(A_{n,i})$

of subsets of Δ for $n = 0, 1, 2, \dots$ and $i = 0, 1, \dots, 2^n - 1$ such that $A_{0,0} = \Delta$ and each $A_{n,i}$ is the union of disjoint, non-empty, clopen subsets $A_{n+1,2i}$ and $A_{n+1,2i+1}$ with $\lim_{n\to\infty} \sup_{0\le i<2^n} \operatorname{diam}(A_{n,i}) = 0$, and Haar functions $h_{n,i} \subseteq C(\Delta)$ (relative to $(A_{n,i})$) so that

$$h_{2^n+i} := 1_{A_{n+1,2i}} - 1_{A_{n+1,2i+1}}, \quad n = 0, 1, \dots, \quad i = 0, 1, \dots, 2^n - 1.$$

Finally, $(x_{n,i}) \subseteq X$ is a Haar-like system (relative to $(A_{n,i})$) if, indexing above Haar functions as $h_{2^n+i} = h_{n,i}$, we have $\|x_{n,i}\| \le 1 + \varepsilon$ for all (n, i) so that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|x_{n,i}|_\Delta - h_{n,i}\|_{C(\Delta)} < \varepsilon.$$

For simplicity in what follows we will assume $x_{n,i}|_\Delta = h_{n,i}$ and ignore the tiny perturbations, and we will refer to the sets $A_{n,i}$ as intervals. We will construct a Rademacher-type system (r_n) from the $x_{n,i}$'s and conclude that $\ell_1 \hookrightarrow X$ to get a contradiction.

Begin with $r_1 \equiv x_{0,0}$ and suppose $r_1, \dots, r_n \in \operatorname{span}(x_{k,i})$ have been constructed so that for each choice of signs $(\varepsilon_i)_1^n$ there is an interval I in Δ on which for $i \le n$, $r_i|_I = \varepsilon_i$. Fix such an I and consider the subsequence $(x_{k,l})$ that is 'supported' on I, that is, $\operatorname{supp} x_{k,l}|_\Delta \subseteq I$. A further subsequence has pairwise disjoint support, and a further subsequence of that is weak Cauchy. Thus the corresponding difference sequence is weakly null. The difference sequence has norm in $[1, 2]$ and takes values $-1, 0, 1$ on I.

Now consider that this has been done for all 2^n such I's. Label the sequences as $(d_j^i)_{j=1}^\infty$ for $i \le 2^n$. By the asymptotic model hypothesis (applied to the weakly null array $(d_j^i)_{j=1}^\infty$, $i \le 2^n$) we can form $r_{n+1} = \sum_{i=1}^{2^n} d_{j_i}^i$ with $1 \le \|r_{n+1}\| \le 2K$.

If $(a_n)_{n=1}^N \subseteq \mathbb{R}$ we choose an interval $I \subseteq \Delta$ such that $r_n|_I = sign(a_n)$ for all $1 \le n \le N$. Thus, $\|\sum_n a_n r_n\| \ge \|\sum_n a_n r_n|_I\|_{C(I)|} = \sum_n |a_n|$. Thus (r_n) is a seminormalized sequence which dominates the unit vector basis of ℓ_1. This contradicts that ℓ_1 does not embed into X and hence X^* must be separable. By Zippin's theorem X embeds into a space Y with a shrinking basis (y_i).

(ii) We proceed to show that X is an asymptotic-c_0 space with respect to the basis (y_i). We need to prove that there exists a constant C such that for all n every asymptotic space $(e_i)_1^n \in \{X\}_n$ is C-equivalent to the unit vector basis of ℓ_∞^n. If $(e_i)_1^n \in \{X\}_n$, then also $(\varepsilon_i e_i)_1^n \in \{X\}_n$ for all sequences of signs $(\varepsilon_i)_1^n$. Therefore, it is sufficient to show that there exists C such that for all $n \in \mathbb{N}$ and for every asymptotic space $(e_i)_1^n \in \{X\}_n$ we have $\|\sum_{i=1}^n e_i\| \le C$.

Suppose this is not the case. Then for all $C \ge 1$ there exists n and a normalized asymptotic tree (i.e., countably branching block tree) $(x_\alpha)_{\alpha\in T_n}$ in X so that for every branch $\beta = (x_i)_{i=1}^n$ of $(x_\alpha)_{\alpha\in T_n}$ there exists $f_\beta \in S_{X^*}$ with $f_\beta(\sum_{i=1}^n x_i) > C$.

We will construct weakly null seminormalized sequences $(y_i^1)_{i\ge1}, (y_i^2)_{i\ge2}, \dots, (y_i^n)_{i\ge n}$ from the linear combinations of carefully chosen nodes of $(x_\alpha)_{\alpha\in T_n}$ so that, after passing to subsequences in each and relabeling, the array $\{y_i^k : 1 \le k \le n, i \ge 1\}$ satisfies $\|y_i^k\| \le K$ for all $1 \le k \le n$, $i \ge 1$, and $\|\sum_{k=1}^n y_{i_k}^k\| > C$ for all $i_1 < \dots < i_n$. This will contradict the assumption that all asymptotic models generated by weakly null arrays are K-equivalent to the unit vector basis of c_0.

We first describe a general procedure of extracting an array of weakly null sequences from a tree. The actual array will be obtained by applying this procedure to a carefully pruned tree (using our assumptions) that we describe later.

Extracting arrays from trees. The main idea of the construction is that each y_i^k is chosen to be a linear combination of nodes of $(x_\alpha)_{\alpha\in T_n}$ from the kth level so that for every $i_1<\cdots<i_n$ the union of the supports (with respect to the tree T_n) of $y_{i_1}^1,\dots,y_{i_n}^n$ contains a (unique) full branch of the tree T_n.

Let $(x_\alpha)_{\alpha\in T_n}$ be the tree above. For $(i_1,\dots,i_k)\in T_n$ we label the node $x^k(i_1,\dots,i_k):=x_{(i_1,\dots,i_k)}$. The superscript (which denotes the kth level in the tree) is redundant, but we keep it for the sake of clarity.

We will construct the desired n-array so that all rows $(y_i^k)_{i\geq k}$ and all diagonal sequences $(y_{i_k}^k)_{k=1}^n$, $i_1<\cdots<i_n$, are block sequences. We will often prune the tree $(x_\alpha)_{\alpha\in T_n}$ by deleting nodes and then relabeling the remaining nodes. The pruned tree will always be a full (sub)tree. Moreover, to ease the notation for later constructions we will relabel the full subtree to match the indices so that the resulting array will have the property that for every diagonal sequence $(y_{i_k}^k)_{k=1}^n$, $i_1<\cdots<i_n$, the corresponding unique full branch is $(x^1(i_1),x^2(i_1,i_2),\dots,x^n(i_1,i_2,\dots,i_n))$.

The array is to be labeled as follows and constructed in diagonal order:

$$\begin{array}{cccccc}
y_1^1 & y_2^1 & y_3^1 & y_4^1 & y_5^1 & \cdots\\
 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & \cdots\\
 & & \ddots & \ddots & \ddots & \\
 & & & y_n^n & y_{n+1}^n & \cdots .
\end{array}$$

Let $y_i^1=x^1(i)$ for all $i\geq 1$. So $(y_i^1)_i$ is the sequence of initial nodes of $(x_\alpha)_{\alpha\in T_n}$. For the first diagonal sequence $(y_1^1,\dots,y_n^n)$ take the leftmost branch of $(x_\alpha)_{\alpha\in T_n}$, that is,

$$y_1^1=x^1(1),\qquad y_2^2=x^2(1,2),\ \dots,\qquad y_n^n=x^n(1,\dots,n). \tag{4.9}$$

The node y_3^2 will be a sum of two successors to the nodes $x^1(1)$ and $x^1(2)$ that comprise y_1^1 and y_2^1, respectively. To do this we pick $i_1>2$ and $i_2>2$ large enough so that $x^2(1,i_1)$ and $x^2(2,i_2)$ are supported after $x^1(2)$ (and hence after $x^1(1)$). Delete the nodes $x^2(1,j)$ for $2<j<i_1$ and the nodes $x^2(2,j)$ for $3\leq j<i_2$ and relabel the remaining sequences so that the chosen nodes become $x^2(1,i_1)=x^2(1,3)$ and $x^2(2,i_2)=x^2(2,3)$. Put

$$y_3^2=x^2(1,3)+x^2(2,3). \tag{4.10}$$

We proceed in similar fashion so that each vector y_j^k of the kth row ($j>k>1$) is defined as a sum of nodes from the kth level of the tree $(x_\alpha)_{\alpha\in T_n}$ and are successors to the nodes that comprise the previously chosen vectors $y_{k-1}^{k-1},y_k^{k-1},\dots,y_{j-1}^{k-1}$. We pick the nodes so that the block conditions are satisfied and relabel the tree after deleting finitely many nodes. Thus y_4^3 is a sum of nodes successor to the nodes of y_2^2 and y_3^2, and after relabeling the nodes it becomes

$$y_4^3=x^3(1,2,4)+x^3(1,3,4)+x^3(2,3,4). \tag{4.11}$$

In general, suppose that y_j^{k-1} for $k-1\leq j<i$ and $k\leq n$ are defined. Let

$$y_j^{k-1}=x^{k-1}(\bar t_1)+x^{k-1}(\bar t_2)+\cdots=\sum_{m\in A_j^{k-1}}x^{k-1}(\bar t_m)\quad\text{ for some } A_j^{k-1}\subset\mathbb{N}$$

be the enumeration of the (finitely many) nodes comprising y_j^{k-1}'s in the order they appear and where each $\bar t_s$ is a $k-1$-tuple with maximal entry j.

We denote concatenation by $(a_1,\dots,a_n)\frown a_{n+1}=(a_1,\dots,a_n,a_{n+1})$. By passing to subsequences and relabeling the sequences of successor nodes

$$(x^k(\bar t_1\frown l))_{l\geq j},(x^k(\bar t_2\frown l))_{l\geq j},(x^k(\bar t_3\frown l))_{l\geq j},\dots,$$

we may assume that each of these vectors is supported after the previously chosen ones. We define y^k_i as a sum of successors to the nodes comprising $y^{k-1}_{k-1},\dots,y^{k-1}_{i-1}$. That is, we put

$$y^k_i=\sum_{j=k-1}^{i-1}\sum_{m\in A^{k-1}_j}x^k(\bar t_m\frown i). \tag{4.12}$$

Note that j is the maximal entry of $\bar t_m\in A^{k-1}_j$ and hence $x^k(\bar t_m\frown i)$ is a successor of $x^k(\bar t_m)$ as $j<i$.

This completes the construction of the array. It follows that the support of any diagonal sequence $(y^k_{i_k})^n_{k=1}$, $i_1<\dots<i_n$, contains the unique full branch $(x^1(i_1),x^2(i_1,i_2),\dots,x^n(i_1,i_2,\dots,i_n))$ as desired.

Pruning the tree. For notational convenience we will denote branches

$$\beta=(x^1(i_1),x^2(i_1,i_2),\dots,x^n(i_1,i_2,\dots,i_n))$$

of the tree by $\beta=(i_1,i_2,\dots,i_n)$. From the construction the support of (the sum of) each sequence $y^1_{i_1},\dots,y^n_{i_n}$ consists of the unique full branch $\beta=(i_1,i_2,\dots,i_n)$ and other off-branch nodes whose numbers add up quickly as i_n gets large. By our assumption there is a branch functional f_β so that

$$f_\beta\Big(\sum_{k=1}^n x^k(i_1,\dots,i_k)\Big)>C. \tag{4.13}$$

Our goal here is to show that for all $\varepsilon>0$ we can prune the tree so that the array (with respect to the pruned tree) satisfies

$$\|y^k_i\|\leq K+\varepsilon,\ \text{for all } 1\leq k\leq n,\ i\geq k, \tag{4.14}$$

and

$$f_\beta\Big(\sum_{k=1}^n y^k_{i_k}\Big)\geq C-\varepsilon. \tag{4.15}$$

Let $\varepsilon>0$. Fix $(\varepsilon_k)^n_{k=1}$ so that $\sum_{k=1}^n\varepsilon_k<\varepsilon$. Let $(x_\alpha)_{\alpha\in T_n}$ be a full subtree satisfying block conditions described in the above construction. That is, every sequence of successor nodes of $(x_\alpha)_{\alpha\in T_n}$ is a block basis and whenever y^k_i is defined as in (4.25) the sequences $(y^1_{i_1},\dots,y^n_{i_n})$ are blocks as well.

As before we will proceed in diagonal order (of the array). Let $y^1_i=x^1(i)$ for all $i\geq 1$. For the first diagonal sequence $(y^1_1,\dots,y^n_n)$ again we take the leftmost branch of $(x_\alpha)_{\alpha\in T_n}$, that is,

$$y^1_1=x^1(1),\ y^2_2=x^2(1,2),\ \dots,\ y^n_n=x^n(1,\dots,n). \tag{4.16}$$

The condition (4.14) is clearly satisfied since the tree is normalized and the condition (4.15) follows from the assumption (4.13).

We wish to define y^2_3 as in (4.10). This will require two steps. First consider the sequences of level 2 successor nodes

$$(x^2(1,l))_{l\geq 3},(x^2(2,l))_{l\geq 3}.$$

By our main assumption, the array formed by these sequences can be refined to generate an asymptotic model K-equivalent to the unit vector basis of ℓ_∞^2. Thus by passing to subsequences, relabeling, and ignoring tiny perturbations we can assume that for all $3 \le l_1 < l_2$,

$$\|x^2(1,l_1) + x^2(2,l_2)\| \le K. \tag{4.17}$$

This will ensure that whenever y_3^2 is defined as in (4.10) the condition (4.14) is satisfied. The second refinement towards ensuring (4.15) is somewhat more complicated.

Consider again the sequences of successor nodes $(x^2(1,l))_{l\ge 3}$ and $(x^2(2,l))_{l\ge 3}$. By the main assumption each of these sequences generates spreading models which are K-equivalent to the unit vector basis of c_0. Fix $N \ge 1 + K^2/\varepsilon_1^2 + 2K/\varepsilon_1$. By passing to subsequences and relabeling we can assume that both $(x^2(1,l))_{l=3}^{N+3}$ and $(x^2(2,l))_{l=3}^{N+3}$ are K-equivalent to the unit vector basis of ℓ_∞^N. For every branch $\beta = (i_1,\ldots,i_n)$ of T_n we let $f_{(i_1,i_2,\ldots,i_n)}$ denote the corresponding branch functional satisfying (4.13). For each $3 \le l \le N+3$, $f_{(1,l)\frown\bar{j}}$ and $f_{(2,l)\frown\bar{j}}$ are the branch functionals for branches extending $(1,l)$ and $(2,l)$ respectively, where $\bar{j}$ is an $(n-2)$-tuple. We stabilize the values of these functionals on the chosen nodes. That is, by passing to subsequences and ignoring tiny perturbations we can assume that for all $\bar{j}, \bar{j}'$ we have

$$f_{(1,l)\frown\bar{j}}(x^2(2,t)) = f_{(1,l)\frown\bar{j}'}(x^2(2,t)) \quad \text{and} \quad f_{(2,l)\frown\bar{j}}(x^2(1,t)) = f_{(2,l)\frown\bar{j}'}(x^2(1,t)),$$

for all $3 \le l, t \le N+3$.

Claim. There exist $3 \le l_1, l_2 \le N+3$ so that for all $\bar{j}$,

$$|f_{(1,l_1)\frown\bar{j}}(x^2(2,l_2))| < \varepsilon_1 \quad \text{and} \quad |f_{(2,l_2)\frown\bar{j}}(x^2(1,l_1))| < \varepsilon_1. \tag{4.18}$$

For any functional f of norm at most 1 and sequence $(x_t)_{t=1}^n$ which is K-equivalent to the unit vector basis of ℓ_∞^n there is a sequence of signs $\delta_t = \pm 1$ so that

$$\sum_{t=1}^n |f(x_t)| = \left| f\Big(\sum_{t=1}^n \delta_t x_t\Big)\right| \le K. \tag{4.19}$$

It follows that the cardinality $|\{t : |f(x_t)| \ge \varepsilon_1\}| \le K/\varepsilon_1$. Thus for each l and $\bar{j}$,

$$|A_l| := \Big|\{t : |f_{(1,l)\frown\bar{j}}(x^2(2,t))| < \varepsilon_1\}\Big| \ge N - K/\varepsilon_1.$$

Then for any $B \subset \{3,\ldots,N+3\}$ with $K/\varepsilon_1 + 1 \le |B| < K/\varepsilon_1 + 2$ we have

$$\Big|\bigcap_{l\in B} A_l\Big| \ge 1.$$

Indeed, $N - |B|K/\varepsilon_1 \ge N - K^2/\varepsilon_1^2 - 2K/\varepsilon_1 \ge 1$. Fix such a subset B and let $l_2 \in \bigcap_{l\in B} A_l$. Then $|f_{(1,l)\frown\bar{j}}(x^2(2,l_2))| < \varepsilon_1$ for all $l \in B$. Now consider the functionals $f_{(2,l_2)\frown\bar{j}}$. Since $(x^2(1,l))_{l\in B}$ is K-equivalent to the unit vector basis of $\ell_\infty^{|B|}$ and $|B| \ge K/\varepsilon_1 + 1$, by a similar argument as above, there is $l_1 \in B$ such that $|f_{(2,l_2)\frown\bar{j}}(x^2(1,l_1))| < \varepsilon_1$, proving the claim.

Now we relabel the nodes as $x^2(1,l_1) = x^2(1,3)$ and $x^2(2,l_2) = x^2(2,3)$ (by deleting finitely many nodes) and put

$$y_3^2 = x^2(1,3) + x^2(2,3). \tag{4.20}$$

At this stage the pruned tree has the following *gap property* of the branch functionals $f_{(1,3)\frown\bar{j}}$ and $f_{(2,3)\frown\bar{j}}$:

$$f_{(1,3)\frown\bar{j}}\Big((y_1^1 + y_3^2) - (x^1(1) + x^2(1,3))\Big) = f_{(1,3)\frown\bar{j}}\Big(x^2(2,3)\Big) < \varepsilon_1,$$
$$f_{(2,3)\frown\bar{j}}\Big((y_2^1 + y_3^2) - (x^1(2) + x^2(2,3))\Big) = f_{(2,3)\frown\bar{j}}\Big(x^2(1,3)\Big) < \varepsilon_1.$$

We have that $x^1(1)$ and $x^2(1,3)$ are the nodes on the branch of $(1,3)\frown\bar{j}$. Thus the first inequality above states that the branch functional $f_{(1,3)\frown\bar{j}}$ is small on the off-branch part of $y_1^1 + y_3^2$, and the second inequality above states that the branch functional $f_{(2,3)\frown\bar{j}}$ is small on the off-branch part of $y_2^1 + y_3^2$. This will be important for us as the branch functionals f_β are defined to be large on their branch. We will eventually be able to obtain (4.15) by showing that f_β is greater than C on the branch part of $\sum_{k=1}^n y_{i_k}^k$ and f_β is smaller than ε on the off-branch part of $\sum_{k=1}^n y_{i_k}^k$ where $\beta = (i_1, \dots, i_n)$.

For the sake of clarity we also show how to define y_4^3 as in (4.11) before proceeding with the inductive step. The array formed by the sequences of level 3 successor nodes

$$(x^3(1,2,l))_{l\geq 4}, \qquad (x^3(1,3,l))_{l\geq 4}, \qquad (x^3(2,3,l))_{l\geq 4}$$

can be refined to generate an asymptotic model K-equivalent to the unit vector basis of ℓ_∞^3. Thus by passing to subsequences, relabeling, and ignoring tiny perturbations we get that for all $4 \leq l_1 < l_2 < l_3$,

$$\|x^3(1,2,l_1) + x^3(1,3,l_2) + x^3(2,3,l_3)\| \leq K. \tag{4.21}$$

This will ensure condition (4.14).

The second refinement is done as before. Fix a large $N = N(K, \varepsilon_2/2)$ and using the c_0 spreading models assumption pick sequences $(x^3(1,2,l))_{l=4}^{N+4}$, $(x^3(1,3,l))_{l=4}^{N+4}$, and $(x^3(2,3,l))_{l=4}^{N+4}$ that are K-equivalent to the unit vector basis of ℓ_∞^N. Refine the tree by passing to subsequences of the successors of these so that the branch functionals $f_{(1,2,l)\frown\bar{j}}$, $f_{(1,3,l)\frown\bar{j}}$, and $f_{(2,3,l)\frown\bar{j}}$ are stabilized. That is, their values on the chosen nodes are independent of $\bar{j}$. Then a similar combinatorial argument as before yields (see the gap lemma below) a node from each sequence which we relabel as $x^3(1,2,4)$, $x^3(1,3,4)$, and $x^3(2,3,4)$ so that

$$|f_{(1,2,4)\frown\bar{j}}(x^3(1,3,4))| < \varepsilon_2/2, \ |f_{(1,2,4)\frown\bar{j}}(x^3(2,3,4))| < \varepsilon_2/2,$$
$$|f_{(1,3,4)\frown\bar{j}}(x^3(1,2,4))| < \varepsilon_2/2, \ |f_{(1,3,4)\frown\bar{j}}(x^3(2,3,4))| < \varepsilon_2/2, \text{ and}$$
$$|f_{(2,3,4)\frown\bar{j}}(x^3(1,2,4))| < \varepsilon_2/2, \ |f_{(2,3,4)\frown\bar{j}}(x^3(1,3,4))| < \varepsilon_2/2.$$

Let

$$y_4^3 = x^3(1,2,4) + x^3(1,3,4) + x^3(2,3,4).$$

Then the branch functionals through these nodes satisfy the desired gap properties: For $1 \leq t_1 < t_2 < 4$, denoting $\mathbf{x}_{(t_1,t_2,4)} = x^1(t_1) + x^2(t_1,t_2) + x^3(t_1,t_2,4)$, and

$\mathbf{y}_{(t_1,t_2,4)} = y^1_{t_1} + y^2_{t_2} + y^3_4$ we have

$$\begin{aligned}\Big|f_{(1,2,4)\frown\bar{j}}\Big(\mathbf{y}_{(1,2,4)} - \mathbf{x}_{(1,2,4)}\Big)\Big| &\le \Big|f_{(1,2,4)\frown\bar{j}}\big(x^3(1,3,4)\big)\Big| + \Big|f_{(1,2,4)\frown\bar{j}}\big(x^3(2,3,4)\big)\Big| \\ &< \varepsilon_2/2 + \varepsilon_2/2,\end{aligned}$$

$$\begin{aligned}&\Big|f_{(1,3,4)\frown\bar{j}}\Big(\mathbf{y}_{(1,3,4)} - \mathbf{x}_{(1,3,4)}\Big)\Big| \\ \le\ &\Big|f_{(1,3,4)\frown\bar{j}}\big(x^2(2,3)\big)\Big| + \Big|f_{(1,3,4)\frown\bar{j}}\big(x^3(1,2,4)\big)\Big| + \Big|f_{(1,3,4)\frown\bar{j}}\big(x^3(2,3,4)\big)\Big| \\ <\ &\varepsilon_1 + \varepsilon_2/2 + \varepsilon_2/2,\end{aligned}$$

and

$$\begin{aligned}&\Big|f_{(2,3,4)\frown\bar{j}}\Big(\mathbf{y}_{(2,3,4)} - \mathbf{x}_{(2,3,4)}\Big)\Big| \\ \le\ &\Big|f_{(2,3,4)\frown\bar{j}}\big(x^2(1,3)\big)\Big| + \Big|f_{(2,3,4)\frown\bar{j}}\big(x^3(1,2,4)\big)\Big| + \Big|f_{(2,3,4)\frown\bar{j}}\big(x^3(1,3,4)\big)\Big| \\ <\ &\varepsilon_1 + \varepsilon_2/2 + \varepsilon_2/2.\end{aligned}$$

As before, the idea is that f_β is large on the branch part of $y^1_{t_1} + y^2_{t_2} + y^3_4$ and is small on the off-branch part where $\beta = (t_1, t_2, 4)$.

We now proceed inductively. Suppose that for $k-1 \le j < i$ and $k \le n$,

$$y^{k-1}_j = \sum_{m \in A^{k-1}_j} x^{k-1}(\bar{t}_m)$$

are defined where $x^{k-1}(\bar{t}_m)$ are $(k-1)$-level nodes and $A^{k-1}_j \subset \mathbb{N}$ is finite. For each $\bar{t}_m = (t_1, \ldots, t_{k-1})$ denote the sum of the initial segment of a diagonal sequence of the array constructed thus far by

$$\mathbf{y}_{\bar{t}_m} = \sum_{i=1}^{k-1} y^i_{t_i}$$

and the sum of the initial segment of the tree by

$$\mathbf{x}_{\bar{t}_m} = \sum_{i=1}^{k-1} x^i(t_1, \ldots, t_{k-1}).$$

For the induction hypothesis we also assume that the branch functionals $f_{\bar{t}_m\frown\bar{j}}$ for the branches whose initial segments are $\bar{t}_m$ satisfy the gap property:

$$\Big|f_{\bar{t}_m\frown\bar{j}}(\mathbf{y}_{\bar{t}_m} - \mathbf{x}_{\bar{t}_m})\Big| < \sum_{i=1}^{k-1} \varepsilon_i. \tag{4.22}$$

Consider the array formed by the sequences of successor nodes

$$(x^k(\bar{t}_1\frown l))_{l>\max\bar{t}_1}, (x^k(\bar{t}_2\frown l))_{l>\max\bar{t}_2}, \ldots, (x^k(\bar{t}_M\frown l))_{l>\max\bar{t}_M}$$

for $m \in \bigcup_{j=k-1}^{i-1} A^{k-1}_j$ and where $M = |\bigcup_{j=k-1}^{i-1} A^{k-1}_j|$. The array is formed in the order the nodes appear in the support of $y^{k-1}_{k-1}, \ldots, y^{k-1}_{i-1}$. By the main assumption the array generates an asymptotic model K-equivalent to the unit vector basis of ℓ^M_∞. Thus by passing to subsequences and relabeling we can assume that for all $\max_{1\le m\le M}\max\bar{t}_m < l_1 < l_2 < \cdots < l_M$,

$$\Big\|\sum_{m=1}^{M} x^k(\bar{t}_m\frown l_m)\Big\| \le K. \tag{4.23}$$

Fix a large $N = N(K, \varepsilon_k/M)$ (determined by the lemma below). For each $1 \leq m \leq M$, using the fact that every sequence of successor nodes generates a c_0 spreading model, pick $\big(x^k(\bar{t}_m \frown l)\big)_{l \in B_m}$, $|B_m| = N$, which is K-equivalent to the unit vector basis of ℓ_∞^N. For all m and $l \in B_m$, by passing to a subsequence of the successors $(x^{k+1}(\bar{t}_m \frown l \frown j))_j$ of $x^k(\bar{t}_m \frown l)$ we can assume that all the branch functionals $f_{\bar{t}_m \frown l \frown \bar{j}}$ are stabilized on the chosen nodes. That is, for all $\bar{j}$ and $\bar{j}'$, ignoring tiny perturbations, we have

$$f_{\bar{t}_m \frown l \frown \bar{j}}(x^k(\bar{t}_{m'} \frown l')) = f_{\bar{t}_m \frown l \frown \bar{j}'}(x^k(\bar{t}_{m'} \frown l'))$$

for all $m \neq m'$ and $l \in B_m, l' \in B_{m'}$. (Note: If $k = n$, the last level of the tree, then all the branch functionals are already determined.)

Claim. For all $1 \leq m \leq M$ there exist $l_m \in B_m$ such that for all $m \neq m'$,

$$\Big| f_{\bar{t}_m \frown l_m \frown \bar{j}}(x^k(\bar{t}_{m'} \frown l_{m'})) \Big| < \varepsilon_k/M. \tag{4.24}$$

This is a consequence of the following combinatorial lemma (for $\varepsilon = \varepsilon_k/M$).

Gap lemma. *Let $\varepsilon > 0$, $M \in \mathbb{N}$. Then there exists $N = N(\varepsilon, M, K)$ such that given sequences $(x_l^1)_{l=1}^N, \dots, (x_l^M)_{l=1}^N$ each K-equivalent to the unit vector basis of ℓ_∞^N and functionals $(f_l^1)_{l=1}^N, \dots, (f_l^M)_{l=1}^N$ of norm at most 1 there exists $l_1, \dots, l_M$ such that*

$$\Big| f_{l_j}^j(x_{l_i}^i) \Big| < \varepsilon, \text{ for all } i \neq j.$$

Proof. The proof is by induction on M. For the base case $M = 2$ we prove the following, which is a slight generalization of (4.18): For all $N_0 \in \mathbb{N}$ there exists $N = N(N_0, \varepsilon, K)$ so that whenever $(x_l^1)_{l=1}^N, (x_l^2)_{l=1}^N$ and $(f_l^1)_{l=1}^N, (f_l^2)_{l=1}^N$ are as in the statement there exist $A_1, A_2 \subset \{1, \dots, N\}$ with $|A_1|, |A_2| \geq N_0$ such that for all $j \in A_1$ and $i \in A_2$ we have $|f_j^1(x_i^2)| < \varepsilon$ and $|f_i^2(x_j^1)| < \varepsilon$.

Fix $N \geq N_0(1 + K/\varepsilon + K^2/\varepsilon^2)$. For any functional f of norm at most 1 and sequence $(x_i)_1^n$ K-equivalent to the unit vector basis ℓ_∞^n, we have, by (4.19), $|\{i : |f(x_i)| \geq \varepsilon\}| \leq K/\varepsilon$. Thus for $N_0(1 + K/\varepsilon) \leq N_1 \leq N_0(1 + K/\varepsilon) + 1$,

$$\Big| \bigcap_{l=1}^{N_1} \Big\{ 1 \leq i \leq N : |f_l^1(x_i^2)| < \varepsilon \Big\} \Big| \geq N - N_1 K/\varepsilon \geq N_0.$$

Let A_2 be a subset of $\bigcap_{l=1}^{N_1} \big\{ 1 \leq i \leq N : |f_l^1(x_i^2)| < \varepsilon \big\}$ with cardinality N_0, and we have

$$|A_1| = \Big| \bigcap_{l \in A_2} \Big\{ 1 \leq i \leq N_1 : |f_l^2(x_i^1)| < \varepsilon \Big\} \Big| \geq N_1 - N_0 K/\varepsilon \geq N_0,$$

as desired.

For the induction suppose that for all $N_0 \in \mathbb{N}$ there exists N and $A_1, \dots, A_m$ with $|A_i| \geq N_0$ so that for all $l_i \in A_i$,

$$\Big| f_{l_j}^j(x_{l_i}^i) \Big| < \varepsilon, \text{ for all } 1 \leq i \neq j \leq m.$$

Fix $N_0 \geq 1 + K/\varepsilon + K^2/\varepsilon^2$ and apply the argument in the base case for the pairs $(x_l^i)_{l \in A_i}, (x_l^{m+1})_{l=1}^{N_0}$ and $(f_l^i)_{l \in A_i}, (f_l^{m+1})_{l=1}^{N_0}$ for $1 \leq i \leq m$ to get the desired $(m+1)$-tuple $l_1, \dots, l_{m+1}$ so that

$$\Big| f_{l_j}^j(x_{l_i}^i) \Big| < \varepsilon, \text{ for all } 1 \leq i \neq j \leq m + 1.$$

□

Consider $l_1, \dots, l_M$ from the Claim. We discard the nodes $x^l(\bar{t}_m \frown l)$, $l \in B_m$, and $l \neq l_m$ and relabel the rest so that for all $1 \leq m \leq M$, $x^k(\bar{t}_m \frown l_m) = x^k(\bar{t}_m \frown i)$, where $i = \max_m \max \bar{t}_m + 1$, and put

$$y_i^k = \sum_{j=k-1}^{i-1} \sum_{m \in A_j^{k-1}} x^k(\bar{t}_m \frown i). \tag{4.25}$$

By (4.23) $\|y_i^k\| \leq K$. By the induction hypothesis (4.22) and the Claim (4.24) we have

$$\left| f_{(\bar{t}_m, i, \bar{j})}\Big((\mathbf{y}_{\bar{t}_m} + y_i^k) - (\mathbf{x}_{\bar{t}_m} + x^k(\bar{t}_m \frown i)) \Big) \right| < \sum_{i=1}^{k-1} \varepsilon_i + \sum_{m=1}^{M} \varepsilon_k / M = \sum_{i=1}^{k} \varepsilon_i \tag{4.26}$$

for all $1 \leq m \leq M$ and $\bar{j}$, as desired. This concludes the construction of the array.

Now let $\beta = (i_1, \dots, i_n)$ be arbitrary. Then by the construction and our main assumption we have

$$\begin{aligned} f_\beta\Big(\sum_{k=1}^n y_{i_k}^k\Big) &\geq f_\beta\Big(\sum_{k=1}^n x^k(i_1, \dots, i_k)\Big) - \Big| f_\beta\Big(\sum_{k=1}^n x^k(i_1, \dots, i_k) - \sum_{k=1}^n y_{i_k}^k\Big)\Big| \\ &\geq C - \sum_{k=1}^n \varepsilon_k > C - \varepsilon. \end{aligned}$$

□

The proof is completed.

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Department of Mathematics and Computer Science, Saint Louis University , St. Louis, Missouri 63103
Email address: `dfreema7@slu.edu`

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, Texas 78712-0257

Department of Mathematics, University of North Texas, Denton, Texas 76203-5017
Email address: `bunyamin@unt.edu`

Department of Mathematics, University of Memphis, Memphis, Tennessee 38152-3240
Email address: `bzheng@memphis.edu`