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A GAP THEOREM FOR THE COMPLEX GEOMETRY OF CONVEX DOMAINS

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Abstract. In this paper we establish a gap theorem for the complex geometry of smoothly bounded convex domains which informally says that if the complex geometry near the boundary is close to the complex geometry of the unit ball, then the domain must be strongly pseudoconvex.

One consequence of our general result is the following: for any dimension there exists some $\epsilon > 0$ so that if the squeezing function on a smoothly bounded convex domain is greater than $1-\epsilon$ outside a compact set, then the domain is strongly pseudoconvex (and hence the squeezing function limits to one on the boundary). Another consequence is the following: for any dimension d there exists some $\epsilon > 0$ so that if the holomorphic sectional curvature of the Bergman metric on a smoothly bounded convex domain is within ϵ of $-4/(d+1)$ outside a compact set, then the domain is strongly pseudoconvex (and hence the holomorphic sectional curvature limits to $-4/(d+1)$ on the boundary).

1. Introduction

There are many results showing that the asymptotic complex geometry of a strongly pseudoconvex domain coincides with the complex geometry of the unit ball. In this paper we consider the following related question.

Question. Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^∞ boundary. If the asymptotic complex geometry of Ω coincides with the complex geometry of the unit ball, is Ω strongly pseudoconvex?

We will restrict our attention to convex domains and for such domains answer the above question in the affirmative (see Theorems 1.14 and 1.15 below). We begin by stating some consequences of our general results.

1.1. The squeezing function.

Let $\mathbb{B}_d \subset \mathbb{C}^d$ denote the unit ball. Given a domain $\Omega \subset \mathbb{C}^d$ let $s_\Omega : \Omega \to (0,1]$ be the *squeezing function on* Ω, that is,

$$s_\Omega(p) = \sup\{r : \text{there exists a one-to-one holomorphic map } f : \Omega \to \mathbb{B}_d \text{ with } f(p) = 0 \text{ and } r\,\mathbb{B}_d \subset f(\Omega)\}.$$

The squeezing function has a number of applications (see for instance [12, 14]).

Theorems of Diederich, Fornæss, and Wold [5, Theorem 1.1] and Deng, Guan, and Zhang [4, Theorem 1.1] imply the following asymptotic result for the squeezing function on strongly pseudoconvex domains.

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Theorem 1.1 ([4,5]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. Then*

$$\lim_{z \to \partial\Omega} s_\Omega(z) = 1.$$

Based on the above theorem, it is natural to ask if the converse holds.

Question (Fornæss). Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^∞ boundary. If

$$\lim_{z \to \partial\Omega} s_\Omega(z) = 1,$$

is Ω strongly pseudoconvex?

In this paper we answer the question for convex domains and in this case prove the following stronger assertion.

Theorem 1.2 (See Section 7). *For any $d > 0$, there exists some $\epsilon = \epsilon(d) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and*

$$s_\Omega(z) \geq 1 - \epsilon$$

outside a compact subset of Ω, then Ω is strongly pseudoconvex.

1.2. **Holomorphic sectional curvature of the Bergman metric.** Let (X, J) be a complex manifold with Kähler metric g. If R is the Riemannian curvature tensor of (X, g), then the *holomorphic sectional curvature* $H_g(v)$ of a non-zero vector v is defined to be the sectional curvature of the 2-plane spanned by v and Jv, that is,

$$H_g(v) := \frac{R(v, Jv, Jv, v)}{\|v\|_g^4}.$$

It is a classical result of Hawley [8] and Igusa [9] that if (X, g) is a complete simply connected Kähler manifold with constant negative holomorphic sectional curvature, then X is biholomorphic to the unit ball (also see Chapter IX, Section 7 in [11]). Moreover, if $b_{\mathbb{B}_d}$ is the Bergman metric on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$, then $(\mathbb{B}_d, b_{\mathbb{B}_d})$ has constant holomorphic sectional curvature $-4/(d+1)$.

Klembeck proved the following asymptotic result for the Bergman metric on a strongly pseudoconvex domain.

Theorem 1.3 (Klembeck [10]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. Then*

$$\lim_{z \to \partial\Omega} \max_{v \in T_z\Omega \setminus \{0\}} |H_{b_\Omega}(v) - \frac{-4}{d+1}| = 0,$$

where b_Ω is the Bergman metric on Ω.

In this paper we will prove the following converse to Klembeck's theorem.

Theorem 1.4 (See Section 9). *For any $d > 0$, there exists some $\epsilon = \epsilon(d) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and*

$$\max_{v \in T_z\Omega \setminus \{0\}} |H_{b_\Omega}(v) - \frac{-4}{d+1}| \leq \epsilon$$

outside a compact subset of Ω, then Ω is strongly pseudoconvex.

1.3. **Kähler metrics with bounded geometry.** Theorem 1.4 actually holds for a much larger class of Kähler metrics with bounded geometry. Before stating our result we will need to rigorously define what we mean by "Kähler metrics with bounded geometry".

Definition 1.5. Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain and $M > 1$. Let $\mathcal{G}_M(\Omega)$ be the set of Kähler metrics g on Ω (with respect to the standard complex structure) with the following properties:

(1) g is a C^2 metric.
(2) For all $z \in \Omega$ and $v \in \mathbb{C}^d$,
$$\frac{1}{M}\sqrt{g_z(v,v)} \leq k_\Omega(z;v) \leq M\sqrt{g_z(v,v)},$$
where k_Ω is the infinitesimal Kobayashi metric on Ω.
(3) If $X, v, w \in \mathbb{C}^d$, then
$$|X(g_z(v,w))| \leq M k_\Omega(z;X) k_\Omega(z;v) k_\Omega(z;w).$$
(4) If $X, Y, v, w \in \mathbb{C}^d$, then
$$|Y(X(g_z(v,w)))| \leq M k_\Omega(z;Y) k_\Omega(z;X) k_\Omega(z;v) k_\Omega(z;w).$$
(5) If $X, Y, v, w \in \mathbb{C}^d$ and $z_1, z_2 \in \Omega$, then
$$\begin{aligned}&|Y(X(g_{z_1}(v,w))) - Y(X(g_{z_2}(v,w)))|\\ &\quad \leq M k_\Omega(z;Y) k_\Omega(z;X) k_\Omega(z;v) k_\Omega(z;w) K_\Omega(z_1,z_2),\end{aligned}$$
where K_Ω is the Kobayashi distance on Ω.

Remark 1.6.

(1) The definition above essentially says that $\mathcal{G}_M(\Omega)$ is the set of all Kähler metrics whose second derivative is bounded and Lipschitz with respect to the Kobayashi metric.
(2) Here we use the standard notation: if $f : \Omega \to \mathbb{C}$ is a C^1 function and $X \in \mathbb{C}^d$ is some vector, then we define the function $X(f) : \Omega \to \mathbb{C}$ by
$$X(f)(z) = \left.\frac{d}{dt}\right|_{t=0} f(z + tX).$$
(3) We will prove that for any $d > 0$ there exists an $M = M(d) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain in $\mathbb{C}^d$, then $b_\Omega \in \mathcal{G}_M(\Omega)$.

Theorem 1.4 will be a consequence of the following more general result.

Theorem 1.7 (See Section 8)**.** *For any $d > 0$ and $M > 1$, there exists some $\epsilon = \epsilon(M,d) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and there exists a metric $g \in \mathcal{G}_M(\Omega)$ with*
$$\max_{v,w \in T_z\Omega\setminus\{0\}} |H_g(v) - H_g(w)| \leq \epsilon$$
outside a compact subset of Ω, then Ω is strongly pseudoconvex.

1.4. **The general theorem.** Theorems 1.2, 1.4, and 1.7 are particular cases of a more general gap theorem which we now describe. In order to state our main result we need to define the space of convex domains and intrinsic functions on them.

Definition 1.8. Let $\mathbb{X}_d$ be the set of convex domains in $\mathbb{C}^d$ which do not contain a complex affine line and let $\mathbb{X}_{d,0}$ be the set of pairs (Ω, x), where $\Omega \in \mathbb{X}_d$ and $x \in \Omega$.

Remark 1.9. When Ω is a convex domain, Barth [1] proved that the following are equivalent:

(1) Ω contains no complex affine lines,
(2) the Kobayashi metric is non-degenerate,
(3) the Kobayashi metric is Cauchy complete.

Thus, from a complex geometric point of view, it is natural to study the convex domains which do not contain any affine lines.

Definition 1.10. A function $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is called *intrinsic* if $f(\Omega_1, p_1) = f(\Omega_2, p_2)$ whenever there exists a biholomorphism $\varphi : \Omega_1 \to \Omega_2$ with $\varphi(p_1) = p_2$.

Example 1.11. There are many examples of intrinsic functions, for instance the functions

$$(\Omega, x) \to s_\Omega(x)$$

and

$$(\Omega, x) \to \max_{v \in T_x\Omega \setminus \{0\}} |H_{b_\Omega}(v) - \frac{-4}{d+1}|$$

are intrinsic.

Using the fact that the unit ball is a homogeneous domain we have the following.

Observation 1.12. If $\mathbb{B}_d \subset \mathbb{C}^d$ is the unit ball and $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is an intrinsic function, then $f(\mathbb{B}_d, x) = f(\mathbb{B}_d, 0)$ for all $x \in \mathbb{B}_d$.

The set $\mathbb{X}_{d,0}$ has a topology coming from the local Hausdorff topology (see Section 2 below) and when an intrinsic function is continuous in this topology we obtain the following generalized version of Klembeck's Theorem for convex domains.

Proposition 1.13 (See Section 5). *Suppose $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is a continuous intrinsic function and Ω is a bounded convex domain with C^2 boundary. If $\xi \in \partial\Omega$ is a strongly pseudoconvex point of $\partial\Omega$, then*

$$\lim_{z \to \xi} f(\Omega, z) = f(\mathbb{B}_d, 0).$$

The main result of this paper is the following two converses to the above proposition.

Theorem 1.14 (See Section 6). *Suppose that $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is a continuous intrinsic function with the following property: if $\Omega \in \mathbb{X}_d$ and $f(\Omega, x) = f(\mathbb{B}_d, 0)$ for all $x \in \Omega$, then Ω is biholomorphic to $\mathbb{B}_d$.*

Then there exists some $\epsilon = \epsilon(d, f) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and

$$|f(\Omega, z) - f(\mathbb{B}_d, 0)| \leq \epsilon$$

outside some compact subset of Ω, then Ω is strongly pseudoconvex and thus

$$\lim_{z \to \partial\Omega} f(\Omega, z) = f(\mathbb{B}_d, 0).$$

Some interesting intrinsic functions, for instance the squeezing function, do not appear to be continuous on $\mathbb{X}_{d,0}$ but are upper-semicontinuous. So we will also establish the following.

Theorem 1.15 (See Section 6). *Suppose that $f : \mathbb{X}_{d,0} \to \mathbb{R}$ is an upper semicontinuous intrinsic function with the following property: if $\Omega \in \mathbb{X}_d$ and $f(\Omega, x) \geq f(\mathbb{B}_d, 0)$ for all $x \in \Omega$, then Ω is biholomorphic to $\mathbb{B}_d$.*

Then there exists some $\epsilon = \epsilon(d, f) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and

$$f(\Omega, z) \geq f(\mathbb{B}_d, 0) - \epsilon$$

outside some compact subset of Ω, then Ω is strongly pseudoconvex.

1.5. **Outline of proof.** The key step in the proof of Theorems 1.14 and 1.15 is establishing the following.

Proposition 1.16 (See Theorems 3.5, 3.7, and 4.1). *For any $d > 0$, there exists a compact set $\mathbb{L} \subset \mathbb{X}_d$ with the following properties:*

(1) *if $\Omega \in \mathbb{L}$, then Ω is not biholomorphic to the unit ball,*
(2) *if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary which is not strongly pseudoconvex, then there exists a sequence $x_n \in \Omega$ approaching the boundary and affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A_n(\Omega, x_n)$ converges to $(\Omega_\infty, 0)$, where $\Omega_\infty \in \mathbb{L}$.*

One then deduces Theorem 1.14 from the proposition by contradiction: assume for a contradiction that for any $n > 0$ there exists a bounded convex domain $\Omega_n \subset \mathbb{C}^d$ with C^∞ boundary so that

(1) outside some compact set of Ω_n

$$|f(\Omega_n, z) - f(\mathbb{B}_d, 0)| \leq 1/n,$$

(2) Ω_n is not strongly pseudoconvex.

Then for each $n > 0$, there exist points $x_m^{(n)} \in \Omega_n$ and affine maps $A_m^{(n)} \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A_{n,m}(\Omega_n, x_{n,m})$ converges to $(\widehat{\Omega}_n, \widehat{x}_n)$ and $\widehat{\Omega}_n \in \mathbb{L}$. A simple argument will show that

$$|f(\widehat{\Omega}_n, z) - f(\mathbb{B}_d, 0)| \leq 1/n$$

for all $z \in \widehat{\Omega}_n$. Now using the fact that $\mathbb{L}$ is compact we can pass to a subsequence and assume that $\widehat{\Omega}_n$ converges to some $\widehat{\Omega} \in \mathbb{L}$. Now

$$f(\widehat{\Omega}, z) = f(\mathbb{B}_d, 0)$$

for all $z \in \widehat{\Omega}$. So $\widehat{\Omega}$ is biholomorphic to $\mathbb{B}_d$, which is a contradiction since $\widehat{\Omega} \in \mathbb{L}$.

The proof of Theorem 1.15 is nearly identical.

1.6. **Notation.** We end the introduction by fixing some very basic notation.

(1) For $v, w \in \mathbb{C}^d$, $\langle v, w \rangle$ will denote the standard Hermitian inner product and $\|v\|$ will denote the standard Euclidean distance.
(2) For a domain $\Omega \subset \mathbb{C}^d$ we will let b_Ω denote the Bergman metric, k_Ω denote the Kobayashi infinitesimal metric, and K_Ω denote the Kobayashi distance.
(3) $\mathrm{Aff}(\mathbb{C}^d)$ will denote the group of affine automorphisms of $\mathbb{C}^d$.

2. The space of convex sets

2.1. **The local Hausdorff topology.** Given a set $A \subset \mathbb{C}^d$, let $\mathcal{N}_\epsilon(A)$ denote the *ϵ-neighborhood of A* with respect to the Euclidean distance. The *Hausdorff distance* between two compact sets $A, B \subset \mathbb{C}^d$ is given by

$$d_H(A,B) = \inf\left\{\epsilon > 0 : A \subset \mathcal{N}_\epsilon(B) \text{ and } B \subset \mathcal{N}_\epsilon(A)\right\}.$$

Equivalently,

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

The Hausdorff distance is a complete metric on the space of compact sets in $\mathbb{C}^d$.

The space of all closed convex sets in $\mathbb{C}^d$ can be given a topology from the local Hausdorff seminorms. For $R > 0$ and a set $A \subset \mathbb{C}^d$ let $A^{(R)} := A \cap B_R(0)$. Then define the *local Hausdorff seminorms* by

$$d_H^{(R)}(A,B) := d_H(A^{(R)}, B^{(R)}).$$

Since an open convex set is completely determined by its closure, we say a sequence of open convex sets A_n converges in the local Hausdorff topology to an open convex set A if there exists some $R_0 \geq 0$ so that

$$\lim_{n\to\infty} d_H^{(R)}(\overline{A}_n, \overline{A}) = 0$$

for all $R \geq R_0$.

Finally we introduce a topology on $\mathbb{X}_d$ and $\mathbb{X}_{d,0}$ using the local Hausdorff topology:

(1) A sequence Ω_n converges to Ω_∞ in $\mathbb{X}_d$ if $\Omega_n \to \Omega_\infty$ in the local Hausdorff topology.
(2) A sequence (Ω_n, x_n) converges to $(\Omega_\infty, x_\infty)$ in $\mathbb{X}_{d,0}$ if $\Omega_n \to \Omega_\infty$ in the local Hausdorff topology and $x_n \to x_\infty$.

2.2. **Continuity of the Kobayashi distance and metric.** Unsurprisingly, the Kobayashi distance is continuous with respect to the local Hausdorff topology.

Theorem 2.1. *Suppose Ω_n converges to Ω in $\mathbb{X}_d$. Then*

$$K_\Omega(x,y) = \lim_{n\to\infty} K_{\Omega_n}(x,y)$$

for all $x, y \in \Omega$ uniformly on compact sets of $\Omega \times \Omega$.

See Theorem 4.1 in [16] for a detailed argument. The proof of Theorem 4.1 in [16] also implies that the Kobayashi metric is continuous.

Theorem 2.2. *Suppose Ω_n converges to Ω in $\mathbb{X}_d$. Then*

$$k_\Omega(x;v) = \lim_{n\to\infty} k_{\Omega_n}(x;v)$$

for all $x \in \Omega$ and $v \in \mathbb{C}^d$ uniformly on compact sets of $\Omega \times \mathbb{C}^d$.

2.3. **The affine group acts co-compactly.** Let $\mathrm{Aff}(\mathbb{C}^d)$ be the group of complex affine isomorphisms of $\mathbb{C}^d$. Then $\mathrm{Aff}(\mathbb{C}^d)$ acts on $\mathbb{X}_d$ and $\mathbb{X}_{d,0}$. Remarkably, the action of $\mathrm{Aff}(\mathbb{C}^d)$ on $\mathbb{X}_{d,0}$ is co-compact.

Theorem 2.3 (Frankel [6]). *The group $\mathrm{Aff}(\mathbb{C}^d)$ acts co-compactly on $\mathbb{X}_{d,0}$, that is, there exists a compact set $K \subset \mathbb{X}_{d,0}$ so that $\mathrm{Aff}(\mathbb{C}^d) \cdot K = \mathbb{X}_{d,0}$.*

This subsection is devoted to producing a particular compact set whose $\mathrm{Aff}(\mathbb{C}^d)$-translates cover $\mathbb{X}_{d,0}$.

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{C}^d$ and for $1 \le i \le d$ define the complex $(d-i)$-plane Z_i by

$$Z_i = \left\{ e_i + \sum_{j=1}^{d-i} z_j e_{i+j} : z_1, \dots, z_{d-i} \in \mathbb{C} \right\}.$$

Definition 2.4. Let $\mathbb{K} \subset \mathbb{X}_{d,0}$ be the set of pairs $(\Omega, 0)$, where Ω is a convex domain so that:

1. $\mathbb{D}\, e_i \subset \Omega$ for each $1 \le i \le d$,
2. $Z_i \cap \Omega = \emptyset$ for each $1 \le i \le d$.

Theorem 2.5. *With the notation above:*

1. *$\mathbb{K}$ is a compact subset of $\mathbb{X}_{d,0}$.*
2. *For any $(\Omega, x) \in \mathbb{X}_{d,0}$ there exists some $A \in \mathrm{Aff}(\mathbb{C}^d)$ so that $A(\Omega, x) \in \mathbb{K}$.*

We begin the proof of Theorem 2.5 by proving the following.

Lemma 2.6. *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain so that:*

1. *$\mathbb{D}\, e_i \subset \Omega$ for each $1 \le i \le d$,*
2. *$Z_i \cap \Omega = \emptyset$ for each $1 \le i \le d$.*

Then Ω contains no complex affine lines (and hence is in $\mathbb{K}$).

Proof. Since Ω is convex, for each $1 \le i \le d$ there exists a real hyperplane H_i so that $Z_i \subset H_i$ and $H_i \cap \Omega = \emptyset$. Then for each $1 \le i \le d$ there exists some $v_i \in \mathbb{C}^d$ and $a_i \in \mathbb{R}$ so that

$$H_i = \{ z \in \mathbb{C}^d : \mathrm{Re}\, \langle v_i, z \rangle = a_i \}.$$

Since $0 \notin H_i$, we see that $a_i \neq 0$ and so we can assume that $a_i = 1$. Now suppose that $v_i = (v_{i,1}, \dots, v_{i,d})$. Then since $Z_i \subset H_i$, we see that $v_{i,j} = 0$ for $j > i$ and $v_{i,i} = 1$. Thus $v_1, \dots, v_d$ forms a $\mathbb{C}$-basis for $\mathbb{C}^d$.

Now suppose that L is a complex line; we claim that L is not contained in Ω. Fix $a, b \in \mathbb{C}^d$ so that

$$L = \{ b + az : z \in \mathbb{C} \}.$$

Then, since $v_1, \dots, v_d$ is a basis of $\mathbb{C}^d$, there exists $1 \le i \le d$ so that $\langle v_i, a \rangle \neq 0$. But then $L \cap H_i \neq \emptyset$ and so L is not contained in Ω. Since L was an arbitrary complex line, we see that Ω contains no complex affine lines. $\square$

Proof of Theorem 2.5. Suppose $(\Omega_n, 0)$ is a subsequence in $\mathbb{K}$. By passing to a subsequence we can assume that $\overline{\Omega}_{n_k}$ converges in the local Hausdorff topology to some closed convex set $\mathcal{C} \subset \mathbb{C}^d$. Then $\mathbb{D}\, e_i \subset \mathcal{C}$ for $1 \le i \le d$ and so $\mathcal{C}$ has nonempty interior. Let Ω be the interior of $\mathcal{C}$. Then Ω_n converges to Ω in the local Hausdorff topology. Moreover,

(1) $\mathbb{D}\, e_i \subset \Omega$ for each $1 \leq i \leq d$,
(2) $Z_i \cap \Omega = \emptyset$ for each $1 \leq i \leq d$.

So by Lemma 2.6, $\Omega \in \mathbb{K}$ and thus $(\Omega_n, 0)$ converges to $(\Omega, 0)$ in $\mathbb{K}$.

Now suppose that $(\Omega, x) \in \mathbb{X}_{d,0}$. We will find some $A \in \operatorname{Aff}(\mathbb{C}^d)$ so that $A(\Omega, x) \in \mathbb{K}$. Pick $y_1, \dots, y_d \in \partial\Omega$ as follows: first let $y_1 \in \partial\Omega$ be the closest point to x in $\partial\Omega$. Then supposing $y_1, \dots, y_{k-1}$ have already been selected, let W be the maximal complex subspace through x which is orthogonal to the complex lines

$$x + \mathbb{C}(y_1 - x), \dots, x + \mathbb{C}(y_{k-1} - x)$$

and let y_k be the closest point to x in $\partial\Omega \cap W$.

Next let $T : \mathbb{C}^d \to \mathbb{C}^d$ be the translation $T(z) = z - x$ and let $U : \mathbb{C}^d \to \mathbb{C}^d$ be the unitary map defined by

$$(UT)(y_i) = \|x - y_i\|\, e_i.$$

Next consider the diagonal matrix

$$\Lambda = \begin{pmatrix} 1/\left\|x - y_1\right\| & & \\ & \ddots & \\ & & 1/\left\|x - y_d\right\| \end{pmatrix}$$

and the affine map $A = \Lambda U T$. Then, by construction,

(1) $\mathbb{D}\, e_i \subset A\Omega$ for each $1 \leq i \leq d$,
(2) $Z_i \cap (A\Omega) = \emptyset$ for each $1 \leq i \leq d$.

Hence $A(\Omega, x) \in \mathbb{K}$. $\square$

3. Rescaling convex domains

Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain and $x_n \in \Omega$ is a sequence converging to the boundary. Then by Theorem 2.5 it is always possible to find affine maps A_n so that (after possibly passing to a subsequence) $A_n(\Omega, x_n)$ converges in $\mathbb{X}_{d,0}$ to some pair $(\Omega_\infty, x_\infty)$. In this section we describe some of the possible limit domains Ω_∞ when Ω has C^∞ boundary.

In the following it will be notationally convenient to work in $\mathbb{C}^{d+1}$ instead of $\mathbb{C}^d$.

3.1. Line type.

In this subsection we recall the line type of a point in the boundary of a convex domain. This discussion is based on a paper of Yu [15].

Suppose $U \subset \mathbb{C}^d$ is a neighborhood of 0 and $f : U \to \mathbb{R}$ is a C^∞ function with $f(0) = 0$. If $v \in \mathbb{C}^d$, let $U_v = \{z \in \mathbb{C} : zv \in U\}$ and let $f_v : U_v \to \mathbb{C}$ be the function defined by $f_v(z) = f(zv)$. Finally let $\operatorname{ord}(f; v) \in \mathbb{N} \cup \{\infty\}$ denote the order of vanishing of the function f_v at $z = 0$.

Observation 3.1. With the notation above and $r \in \mathbb{N}$, the following are equivalent:

(1) $\operatorname{ord}(f; v) \geq r$,
(2) $\frac{\partial^{k+\ell}}{\partial z^k \partial \bar{z}^\ell} f_v(0) = 0$ for all $k + \ell \leq r - 1$,
(3) there exists $C > 0$ so that $|f_v(z)| \leq C|z|^r$ for z sufficiently close to 0.

Now suppose that $f : U \subset \mathbb{C}^d \to \mathbb{R}$ is C^∞, convex, non-negative, and $f(0) = 0$. For any $r \in \mathbb{N} \cup \{\infty\}$ let

$$V_r = \{v \in \mathbb{C}^d : \operatorname{ord}(f; v) \geq r\}.$$

Clearly $V_\infty \subset V_r$ and $V_{r+1} \subset V_r$ for all $r \in \mathbb{N}$.

Observation 3.2. With the notation above, V_r is a complex linear subspace of $\mathbb{C}^d$.

Proof. First suppose that $r \in \mathbb{N}$. If $v \in V_r$, then clearly $\lambda v \in V_r$ for $\lambda \in \mathbb{C}$. Moreover, if $v, w \in V_r$, then there exists $C > 0$ so that

$$|f(zv)|, |f(zw)| \leq C|z|^r$$

for z sufficiently small. Then for z sufficiently small we have

$$|f(z(v+w))| = f(zv + zw) \leq \frac{1}{2} f(2zv) + \frac{1}{2} f(2zw) \leq 2^r C|z|^r$$

since f is convex and non-negative.

Next consider the case in which $r = \infty$. Then $V_\infty = \bigcap_{r \in \mathbb{N}} V_r$ and so V_∞ is also a subspace. $\square$

Then there exists $r_1 < r_2 < \cdots < r_k$ in $\mathbb{N} \cup \{\infty\}$ so that

(1) $r_1 = \max\{r \in \mathbb{N} \cup \{\infty\} : V_r = \mathbb{C}^d\}$,
(2) $r_k = \max\{r \in \mathbb{N} \cup \{\infty\} : V_r \neq (0)\}$,
(3) $V_{r_k} \subsetneq V_{r_{k-1}} \subsetneq \cdots \subsetneq V_{r_1}$, and
(4) $\operatorname{ord}(f; v) = r_i$ for all $v \in V_{r_i} \setminus V_{r_{i+1}}$.

Now for $0 \leq i \leq k$ let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ \dim V_{r_i} & \text{if } 1 \leq i \leq k. \end{cases}$$

Finally define the *type of f at* 0 to be the d-tuple $m = (m_1, \dots, m_d)$, where

$$m_j = r_i \quad \text{if} \quad d_{i-1} < j \leq d_i.$$

Now suppose that $\Omega \subset \mathbb{C}^{d+1}$ is a convex domain with C^∞ boundary. Then we can associate to every point $\xi \in \partial\Omega$ a tuple $m(\xi) = (m_1, \dots, m_d)$, where $m_i \in \mathbb{N} \cup \{\infty\}$ as follows. Using an affine transformation we can assume that $\xi = 0$, $T_\xi \partial\Omega = \mathbb{R} \times \mathbb{C}^d$, and $\Omega \subset \{(z_1, \dots, z_{d+1}) \in \mathbb{C}^{d+1} : \operatorname{Im}(z_1) > 0\}$. Now there exist neighborhoods $U \subset \mathbb{R}$, $V \subset \mathbb{R}$, $W \subset \mathbb{C}^d$, and a C^∞ function $F : U \times W \to V$ so that

$$\Omega \cap \Big((U + iV) \times W \Big) = \{(x + iy, z) : y > F(x, z)\}.$$

Then the function $f = F|_{\{0\} \times \mathbb{C}^d} : W \to \mathbb{R}$ is C^∞, convex, non-negative, and $f(0) = 0$. Finally, let $m(\xi)$ be the type of f at $z = 0$.

Definition 3.3. Suppose that $\Omega \subset \mathbb{C}^{d+1}$ is a convex domain with C^∞ boundary. The *line type* of $\xi \in \partial\Omega$ is the d-tuple $m(\xi) \in (\mathbb{N} \cup \{\infty\})^d$.

Remark 3.4.

(1) In the definition of $m(\xi)$, we made one choice: the initial affine transformation. However the value of $m(\xi)$ does not depend on this choice; see for instance [15, Proposition 2].
(2) Ω is strongly pseudoconvex at ξ if and only if $m(\xi) = (2, \dots, 2)$.

3.2. **The finite type case.** Let $e_0, e_1, \dots, e_d$ be the standard basis of $\mathbb{C}^{d+1}$ and for $0 \leq i \leq d$ define the complex $(d-i)$-plane Z_i by

$$Z_i = \left\{ e_i + \sum_{j=1}^{d-i} z_j e_{i+j} : z_1, \dots, z_{d-i} \in \mathbb{C} \right\}.$$

Theorem 3.5. *Suppose that $\Omega \subset \mathbb{C}^{d+1}$ is a bounded convex domain with C^∞ boundary, $\xi \in \partial\Omega$, and $m(\xi) = (m_1, \dots, m_d) \in \mathbb{N}^d$. Then there exist $x_n \in \Omega$ converging to ξ and affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^{d+1})$ so that*

$$A_n(\Omega, x_n) \to (\widehat{\Omega}, 0) \text{ in } \mathbb{X}_{d+1,0},$$

where

$$\widehat{\Omega} = \{(x+iy, z) : x < 1 - P(z)\}$$

and

1. *$P : \mathbb{C}^d \to \mathbb{R}$ is a convex, non-negative polynomial with $P(0) = 0$,*
2. *for all $t > 0$ and $(z_1, \dots, z_d) \in \mathbb{C}^d$ we have*
$$P(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) = tP(z_1, \dots, z_d),$$
3. *$\mathbb{D}\, e_i \subset \widehat{\Omega}$ for all $0 \leq i \leq d$,*
4. *$Z_i \cap \widehat{\Omega} = \emptyset$ for all $0 \leq i \leq d$.*

Remark 3.6. The last two conditions imply that $\widehat{\Omega}$ is contained in the compact set $\mathbb{K}$ defined in Subsection 2.3.

Proof. By Proposition 2 in [7], there exist $x_n \in \Omega$ converging to ξ and affine maps $B_n \in \mathrm{Aff}(\mathbb{C}^{d+1})$ so that

$$B_n(\Omega, x_n) \to (\widehat{\Omega}_0, 0) \text{ in } \mathbb{X}_{d+1,0},$$

where

$$\widehat{\Omega}_0 = \{(x+iy, z) : x < 1 - P_0(z)\}$$

and

1. $P_0 : \mathbb{C}^d \to \mathbb{R}$ is a convex, non-negative polynomial with $P_0(0) = 0$,
2. for all $t > 0$ and $(z_1, \dots, z_d) \in \mathbb{C}^d$ we have
$$P_0(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) = tP_0(z_1, \dots, z_d).$$

The rest of the argument is devoted to producing an affine map A so that $A_n = AB_n$ satisfies the rest of the conditions in the theorem.

Pick $0 = d_0 < d_1 < \dots < d_{r+1} = d$ so that

$$m_1 = \dots = m_{d_1} < m_{d_1+1} = \dots = m_{d_2} < \dots < m_{d_{r-1}+1} = \dots = m_d.$$

Then define vector spaces

$$W_\ell = \mathrm{Span}_{\mathbb{C}}\{e_i : d_\ell < i \leq d_{\ell+1}\}.$$

Now for each $0 \leq \ell \leq r$ we select an orthogonal basis $x_{d_\ell+1}, x_{d_\ell+2}, \dots, x_{d_{\ell+1}} \in W_\ell$ as follows: first pick $x_{d_\ell+1} \in W_\ell$ so that $x_{d_\ell+1}$ is the closest point in $\partial\widehat{\Omega}_0 \cap W_\ell$ to 0. Then assuming $x_{d_\ell+1}, \dots, x_{d_\ell+k-1}$ have been selected pick $x_{d_\ell+k} \in W_\ell$ so that $x_{d_\ell+k}$ is a closest point in

$$\partial\widehat{\Omega}_0 \cap \mathrm{Span}_{\mathbb{C}}\{x_{d_\ell+1}, \dots, x_{d_\ell+k-1}\}^{\perp} \cap W_\ell$$

to 0.

Let $U : \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}$ be the unitary map with $U(e_0) = e_0$ and $U(x_k) = \|x_k\| e_k$ for $1 \leq k \leq d$. Next let

$$\Lambda = \begin{pmatrix} 1 & & & \\ & 1/\|x_1\| & & \\ & & \ddots & \\ & & & 1/\|x_d\| \end{pmatrix}.$$

Finally let $A = \Lambda U$ and consider $\widehat{\Omega} = A\widehat{\Omega}_0$. By construction

$$\widehat{\Omega} = \{(x+iy, z) : x < 1 + P(z)\},$$

where $P : \mathbb{C}^d \to \mathbb{R}$ is a convex, non-negative polynomial with $P(0) = 0$. Moreover, since U and Λ preserve the subspaces W_ℓ we see that

$$P(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) = tP(z_1, \dots, z_d)$$

for all $t > 0$ and $(z_1, \dots, z_d) \in \mathbb{C}^d$. By the way we picked the x_i, we see that $e_i \in \partial\widehat{\Omega}$ and $\mathbb{D}\, e_i \subset \widehat{\Omega}$ for all $1 \leq i \leq d$.

We now show that

$$Z_i \cap \widehat{\Omega} = \emptyset$$

for all $0 \leq i \leq d$. By construction $Z_0 \cap \widehat{\Omega} = \emptyset$. Now since $\partial\widehat{\Omega}$ is C^1 (actually C^∞) each $\xi \in \partial\widehat{\Omega}$ is contained in a unique tangent real hyperplane H and since $\widehat{\Omega}$ is convex $H \cap \widehat{\Omega} = \emptyset$. Now this hyperplane is given by

$$H = \xi + \left\{ z \in \mathbb{C}^{d+1} : \mathrm{Re}\left(z_0 + 2\sum_{j=1}^{d} \frac{\partial P}{\partial z_j}(\xi) z_j \right) = 0 \right\}.$$

Thus it is enough to show that

$$\frac{\partial P}{\partial z_j}(e_i) = 0$$

for all $1 \leq i < j \leq d$.

Fix $1 \leq i < d$ and assume that $d_\ell < i \leq d_{\ell+1}$. If $d_\ell < i < j \leq d_{\ell+1}$, then the way we selected $x_1, \dots, x_d$ implies that

$$\frac{\partial P}{\partial z_j}(e_i) = 0.$$

In the case in which $i \leq d_{\ell+1} < j$ we have $m_i < m_j$. Since

$$P(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) = tP(z_1, \dots, z_d)$$

we see that

$$t^{1/m_j} \frac{\partial P}{\partial z_j}(t^{1/m_i} e_i) = t \frac{\partial P}{\partial z_j}(e_i). \tag{3.1}$$

Now suppose for a contradiction that $\frac{\partial P}{\partial z_j}(e_i) \neq 0$. Then

$$s \in \mathbb{R} \to \frac{\partial P}{\partial z_j}(se_i)$$

is a non-zero polynomial in s. Suppose this polynomial has degree D. Then comparing sides in (3.1) we see that

$$\frac{1}{m_j} + \frac{D}{m_i} = 1.$$

So $D = m_i - m_i/m_j$. But D is an integer and $m_j > m_i$, so we have a contradiction. Thus

$$\frac{\partial P}{\partial z_j}(e_i) = 0$$

for all $1 \leq i < j \leq d$. And so

$$Z_i \cap \widehat{\Omega} = \emptyset$$

for all $0 \leq i \leq d$. □

3.3. **The infinite type case.**

Theorem 3.7. *Suppose that $\Omega \subset \mathbb{C}^{d+1}$ is a bounded convex domain with C^∞ boundary, $\xi \in \partial\Omega$, and $m(\xi) \notin \mathbb{N}^d$. Then there exist $x_n \in \Omega$ converging to ξ and affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^{d+1})$ so that*

$$A_n(\Omega, x_n) \to (\widehat{\Omega}, 0) \text{ in } \mathbb{X}_{d+1,0},$$

where

(1) $\mathbb{D}\, e_i \subset \widehat{\Omega}$ *for* $0 \leq i \leq d$,
(2) $Z_i \cap \widehat{\Omega} = \emptyset$ *for* $0 \leq i \leq d$,
(3) $e_0 + \mathbb{D}\, e_1 \subset \partial\widehat{\Omega}$.

Proof. By (the proof of) Proposition 6.1 in [16], there exist $x_n \in \Omega$ converging to ξ and affine maps $B_n \in \mathrm{Aff}(\mathbb{C}^{d+1})$ so that

$$B_n(\Omega, x_n) \to (\widehat{\Omega}_0, 0) \text{ in } \mathbb{X}_{d+1,0},$$

where

(1) $\mathbb{D}\, e_0 \cup \mathbb{D}\, e_1 \in \widehat{\Omega}_0$,
(2) $(e_0 + \mathbb{D}\, e_1) \cup \{e_1\} \subset \partial\widehat{\Omega}_0$.

The rest of the argument is devoted to producing an affine map A so that $A_n = AB_n$ satisfies the rest of the conditions in the theorem.

We begin by selecting $y_0, \dots, y_d \in \partial\widehat{\Omega}_0$ and subspaces $W_0, \dots, W_d \subset \mathbb{C}^{d+1}$ as follows: first let $y_0 = e_0$ and let W_0 be a complex hyperplane so that:

(1) $\mathbb{D}\, e_1 \subset W_0$,
(2) $(y_0 + W_0) \cap \widehat{\Omega}_0 = \emptyset$.

Since $\widehat{\Omega}_0$ is convex, such a hyperplane exists. Next pick $y_1 \in \mathbb{C}\, e_1 \cap \partial\widehat{\Omega}_0$ so that

$$\|y_1\| = \inf\{\|z\| : z \in \mathbb{C}\, e_1 \cap \partial\widehat{\Omega}_0\}.$$

Since $e_1 \in \partial\widehat{\Omega}_0$, we have that $\|y_1\| \leq 1$. Then let W_1 be a complex subspace so that

(1) $\dim_{\mathbb{C}} W_1 = \dim_{\mathbb{C}} W_0 - 1 = d - 1$,

(2) $W_1 \subset W_0$,
(3) $(y_1 + W_1) \cap \widehat{\Omega}_0 = \emptyset$.

Now supposing that $y_1, \dots, y_k \in \partial\widehat{\Omega}_0$ and $W_0, \dots, W_k \subset \mathbb{C}^{d+1}$ have already been selected, select $y_{k+1} \in W_k \cap \partial\widehat{\Omega}_0$ so that

$$\|y_{k+1}\| = \inf\{\|z\| : z \in W_k \cap \partial\widehat{\Omega}_0\}$$

and let W_{k+1} be a complex subspace so that

(1) $\dim_{\mathbb{C}} W_{k+1} = \dim_{\mathbb{C}} W_k - 1 = d - k - 1$,
(2) $W_{k+1} \subset W_k$,
(3) $(y_{k+1} + W_{k+1}) \cap \widehat{\Omega}_0 = \emptyset$.

Next let $A \in \mathrm{GL}_{d+1}(\mathbb{C})$ be the linear map so that

$$A(y_i) = e_i \text{ for all } 0 \leq i \leq d.$$

Then

$$A(W_i) = \mathrm{Span}_{\mathbb{C}}\{e_{i+1}, \dots, e_d\}.$$

So if $\widehat{\Omega} := A\widehat{\Omega}_0$ and $A_n := AB_n$, then

$$A_n(\Omega, x_n) \to (\widehat{\Omega}, 0) \text{ in } \mathbb{X}_{d+1,0},$$

where

(1) $\mathbb{D}\, e_i \subset \widehat{\Omega}$ for $0 \leq i \leq d$,
(2) $Z_i \cap \widehat{\Omega} = \emptyset$ for $0 \leq i \leq d$,
(3) $e_0 + \mathbb{D}\, e_1 \subset \partial\widehat{\Omega}$. □

4. The rescaled domains

In this section we show that the domains obtained by rescaling in Section 3 are contained in a compact subset of $\mathbb{X}_{d+1}$.

Let $\mathbb{F} \subset \mathbb{X}_{d+1}$ be the set of domains $\Omega \in \mathbb{X}_{d+1}$ with the following properties:

(1) $\mathbb{D}\, e_i \subset \Omega$ for $0 \leq i \leq d$,
(2) $Z_i \cap \Omega = \emptyset$ for $0 \leq i \leq d$,
(3) $e_0 + \mathbb{D}\, e_1 \subset \partial\Omega$.

Next suppose that $m = (m_1, \dots, m_d)$ and $2 \leq m_1 \leq m_2 \leq \dots \leq m_d < \infty$. Let $\mathbb{P}(m) \subset \mathbb{X}_{d+1}$ be the set of domains Ω, where

$$\Omega = \{(x + iy, z) : x < 1 - P(z)\}$$

and

(1) $P : \mathbb{C}^d \to \mathbb{R}$ is a convex, non-negative polynomial with $P(0) = 0$,
(2) for all $t > 0$ and $(z_1, \dots, z_d) \in \mathbb{C}^d$ we have

$$P(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) = tP(z_1, \dots, z_d),$$

(3) $\mathbb{D}\, e_i \subset \Omega$ for all $0 \leq i \leq d$, and
(4) $Z_i \cap \Omega = \emptyset$ for all $0 \leq i \leq d$.

Finally, define

$$\mathbb{L} := \mathbb{F} \cup \bigcup_{m \neq (2,\dots,2)} \mathbb{P}(m).$$

Theorem 4.1. *The subset $\mathbb{L} \subset \mathbb{X}_{d+1}$ is compact and if $\Omega \in \mathbb{L}$, then Ω is not biholomorphic to $\mathbb{B}_{d+1}$.*

Proof. Let $\mathbb{F}_0 = \{(\Omega, 0) : \Omega \in \mathbb{F}\}$. Then $\mathbb{F}_0$ is a closed subset of $\mathbb{K}$ and so Theorem 2.5 implies that $\mathbb{F}_0$ is compact. Hence $\mathbb{F}$ is compact. Thus, to show that $\mathbb{L}$ is compact it is enough to consider a sequence $\Omega_n \in \mathbb{P}(m^{(n)})$ and show that there exists a subsequence Ω_{n_k} which converges to some domain in $\mathbb{L}$.

Since $(\Omega_n, 0) \in \mathbb{K}$, we can pass to a subsequence so that Ω_n converges to some Ω in $\mathbb{X}_{d+1}$.

Suppose that P_n is the polynomial so that

$$\Omega_n = \{(x+iy, z) : x < 1 - P_n(z)\}.$$

By passing to a subsequence we can suppose that

$$m_i := \lim_{n\to\infty} m_i^{(n)}$$

exists in $\mathbb{N}\cup\{\infty\}$ for all $1 \leq i \leq d$.

Case 1. $m_d < \infty$. Then by passing to a subsequence we can assume that

$$\left(m_1^{(n)}, \dots, m_d^{(n)}\right) = m = (m_1, \dots, m_d)$$

for all $n \in \mathbb{N}$. This implies that there exists an $N > 0$ so that

$$P_n(z) = \sum_{2\leq|\alpha|+|\beta|\leq N} c_{\alpha,\beta}^{(n)} z^\alpha \overline{z}^\beta.$$

Now since $\mathbb{D}\, e_i \subset \Omega_n$ we see that $P_n(ze_i) \leq 1$ for $1 \leq i \leq d$ and $|z| \leq 1$. Then, by convexity, $P_n(z) \leq 1$ for

$$z \in \mathcal{C} := \operatorname{ConvexHull}\left(\mathbb{D}\, e_1 \cup \dots \cup \mathbb{D}\, e_d\right).$$

Since P_n is non-negative, we then see that

$$\sup_{z\in\mathcal{C}} |P_n(z)| \leq 1.$$

Now by the equivalence of finite-dimensional norms, there exists some $C > 0$ so that

$$\sup_{2\leq|\alpha|+|\beta|\leq N} |c_{\alpha,\beta}^{(n)}| \leq C \sup_{z\in\mathcal{C}} \Big| \sum_{2\leq|\alpha|+|\beta|\leq N} c_{\alpha,\beta}^{(n)} z^\alpha \overline{z}^\beta \Big| \leq C.$$

Thus, after passing to a subsequence, we can suppose that

$$\lim_{n\to\infty} c_{\alpha,\beta}^{(n)} = c_{\alpha,\beta}$$

for all $2 \leq |\alpha| + |\beta| \leq N$. Then P_n converges locally uniformly to the polynomial

$$P(z) = \sum_{2\leq|\alpha|+|\beta|\leq N} c_{\alpha,\beta} z^\alpha z^\beta.$$

Then, since Ω_n converges to Ω in the local Hausdorff topology and

$$\Omega_n = \{(x+iy, z) : x < 1 - P_n(z)\},$$

we have

$$\Omega = \{(x+iy, z) : x < 1 - P(z)\}.$$

We next claim that $\Omega \in \mathbb{P}(m)$. Since the sequence P_n converges locally uniformly to P and each P_n is convex and non-negative, we see that $P : \mathbb{C}^d \to \mathbb{R}$ is also convex and non-negative. Further, for $t > 0$ and $(z_1, \dots, z_d) \in \mathbb{C}^d$ we have

$$\begin{aligned} P(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) &= \lim_{n \to \infty} P_n(t^{1/m_1} z_1, \dots, t^{1/m_d} z_d) \\ &= \lim_{n \to \infty} t P_n(z_1, \dots, z_d) \\ &= t P(z_1, \dots, z_d). \end{aligned}$$

Since the sequence Ω_n converges to Ω in the local Hausdorff topology we see that

(1) $\mathbb{D}\, e_i \subset \Omega$ for all $0 \leq i \leq d$ and
(2) $Z_i \cap \Omega = \emptyset$ for all $0 \leq i \leq d$.

Thus $\Omega \in \mathbb{P}(m)$.

Case 2. $m_d = \infty$. Then using the fact that

$$P_n(0, \dots, 0, z_d) = |z_d|^{m_d^{(n)}} P\left(0, \dots, 0, \frac{z_d}{|z_d|}\right) \leq |z_d|^{m_d^{(n)}}$$

we see that $e_0 + \mathbb{D}\, e_d \subset \partial\Omega$. Thus $\Omega \in \mathbb{F}$.

We now prove the second assertion of the theorem.

Claim. If $\Omega \in \mathbb{L}$, then Ω is not biholomorphic to $\mathbb{B}_{d+1}$.

If $\Omega \in \mathbb{P}(m)$ for some $m \neq (2, \dots, 2)$, then Ω is not biholomorphic to $\mathbb{B}_{d+1}$ by work of Coupet and Pinchuk [2]. So suppose that $\Omega \in \mathbb{F}$ and let K_Ω be the Kobayashi distance on Ω. Since $\partial\Omega$ contains a complex affine disk, Theorem 3.1 in [16] implies that the metric space (Ω, K_Ω) is not Gromov hyperbolic. However, $\mathbb{B}_{d+1}$ endowed with its Kobayashi metric is a model of complex hyperbolic space which is Gromov hyperbolic. Since Gromov hyperbolicity is an isometric invariant, we see that (Ω, K_Ω) and $(\mathbb{B}_{d+1}, K_{\mathbb{B}_{d+1}})$ cannot be isometric and thus Ω cannot be biholomorphic to $\mathbb{B}_{d+1}$. $\square$

5. The proof of Proposition 1.13

Proposition 1.13 is a consequence of Pinchuk's rescaling method (see [13]).

Theorem 5.1. *Suppose $\Omega \subset \mathbb{C}^d$ is a domain whose boundary is C^2 and strongly pseudoconvex in a neighborhood of some point $\xi \in \partial\Omega$. If $x_n \in \Omega$ is a sequence converging to ξ, then there exist affine maps $A_n \in \operatorname{Aff}(\mathbb{C}^d)$ so that $A_n\Omega$ converges in the local Hausdorff topology to*

$$\mathcal{U} = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) > \sum_{i=2}^{d} |z_i|^2 \right\}$$

and $A_n x_n = (i, 0, \dots, 0)$.

See the proof of Theorem 2 in [7] for a detailed argument.

6. The proofs of Theorem 1.14 and Theorem 1.15

Proof of Theorem 1.14. Suppose for a contradiction that there does not exist such an $\epsilon > 0$. Then for each $n > 0$ there exists some bounded convex domain $\Omega_n \subset \mathbb{C}^d$ with C^∞ boundary so that

$$\limsup_{z \to \partial\Omega_n} |f(\Omega_n, z) - f(\mathbb{B}_d, 0)| < 1/n$$

and Ω_n is not strongly pseudoconvex. Fix some $\xi_n \in \partial\Omega_n$ with $m(\xi_n) = (m_1^{(n)}, \dots, m_{d-1}^{(n)})$ and $m_{d-1}^{(n)} > 2$.

Now using Theorems 3.5 and 3.7, for each $n \in \mathbb{N}$ we can find a sequence $x_m \to \xi_n$ and affine maps A_m so that $A_m(\Omega_n, x_m)$ converges in $\mathbb{X}_{d,0}$ to $(\widehat{\Omega}_n, 0)$, where $\widehat{\Omega}_n \in \mathbb{L}$.

We claim that

$$|f(\widehat{\Omega}_n, x) - f(\mathbb{B}_d, 0)| < 1/n$$

for all $x \in \widehat{\Omega}_n$. Now there exists $y_m \in \Omega_n$ so that $A_m y_m \to x$. Moreover, by Theorem 2.1,

$$\lim_{m\to\infty} K_{\Omega_n}(x_m, y_m) = \lim_{m\to\infty} K_{A_m\Omega_n}(A_m x_m, A_m y_m) = K_{\widehat{\Omega}_n}(x, 0).$$

Since K_{Ω_n} is a proper metric and $x_m \to \xi_n \in \partial\Omega_n$, we then see that

$$\lim_{m\to\infty} d_{\mathrm{Euc}}(y_m, \partial\Omega_n) = 0.$$

Hence

$$\begin{aligned} |f(\widehat{\Omega}_n, x) - f(\mathbb{B}_d, 0)| &= \limsup_{m\to\infty} |f(A_m\Omega_n, A_m y_m) - f(\mathbb{B}_d, 0)| \\ &= \limsup_{m\to\infty} |f(\Omega_n, y_m) - f(\mathbb{B}_d, 0)| < 1/n. \end{aligned}$$

Now since $\mathbb{L}$ is compact in $\mathbb{X}_d$ we can pass to a subsequence so that $\widehat{\Omega}_n$ converges to some $\widehat{\Omega}$ in $\mathbb{L}$. Now for $x \in \widehat{\Omega}$ we have

$$f(\widehat{\Omega}, x) = \lim_{n\to\infty} f(\widehat{\Omega}_n, x) = f(\mathbb{B}_d, 0).$$

Thus by hypothesis $\widehat{\Omega}$ is biholomorphic to $\mathbb{B}_d$. But $\widehat{\Omega} \in \mathbb{L}$ and so we have a contradiction. □

Proof of Theorem 1.15. This is essentially identical to the proof of Theorem 1.14. □

7. The proof of Theorem 1.2

To deduce Theorem 1.2 from our general results we only need to show the following.

Proposition 7.1. *Suppose* (Ω_n, x_n) *is a sequence converging to* (Ω, x) *in* $\mathbb{X}_{d,0}$. *Then*

$$\limsup_{n\to\infty} s_{\Omega_n}(x_n) \leq s_\Omega(x).$$

Proof. By passing to a subsequence we may assume that

$$\lim_{n\to\infty} s_{\Omega_n}(x_n) = \limsup_{n\to\infty} s_{\Omega_n}(x_n).$$

Let $r_n = s_{\Omega_n}(x_n)$ and $r = \lim_{n\to\infty} s_{\Omega_n}(x_n)$. We may assume that $r > 0$ (otherwise there is nothing to prove).

By Theorem 2.1 in [3], there exists an injective holomorphic map $f_n : \Omega_n \to \mathbb{B}_d$ with $f(x_n) = 0$ and

$$r_n \mathbb{B}_d \subset f_n(\Omega_n).$$

Now by Theorem 2.1, K_{Ω_n} converges locally uniformly to K_Ω and so we can pass to a subsequence so that f_n converges locally uniformly to a holomorphic function $f : \Omega \to \mathbb{B}_d$.

Now fix some $w \in r\,\mathbb{B}_d$. Then for large n, we have $f_n^{-1}(w) = \{z_n\}$ for some $z_n \in \Omega_n$. Then

$$K_{\Omega_n}(z_n, x_n) = K_{f(\Omega_n)}(f(z_n), 0) \leq K_{r_n \mathbb{B}_d}(w, 0)$$

and so

$$\limsup_{n\to\infty} K_{\Omega_n}(z_n, x_n) \leq K_{r\,\mathbb{B}_d}(w, 0).$$

So, using the fact that K_{Ω_n} converges locally uniformly to K_Ω, we can pass to a subsequence so that z_n converges to some $z \in \Omega$. Then $f(z) = w$. Since $w \in r\,\mathbb{B}_d$ was arbitrary, we see that $r\,\mathbb{B}_d \subset f(\Omega)$.

Finally, by Theorem 2.2 in [3] the map $f : \Omega \to \mathbb{B}_d$ is injective. Thus

$$\lim_{n\to\infty} s_{\Omega_n}(x_n) \leq s_\Omega(x)$$

and the proof is complete. □

8. The proof of Theorem 1.7

We begin by showing the following normal family type result.

Lemma 8.1. *Suppose Ω_n converges to some Ω in $\mathbb{X}_d$ and $g_n \in \mathcal{G}_M(\Omega_n)$. Then there exist $n_k \to \infty$ and a metric $g \in \mathcal{G}_M(\Omega)$ so that g_{n_k} converges to g locally uniformly and for all vectors $X, Y, v, w \in \mathbb{C}^d$*

$$X(Y(g_{n_k}(v, w))) \to X(Y(g(v, w)))$$

locally uniformly.

Proof. Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{C}^d$ and let $u_1 = e_1, \dots, u_d = e_d$, $u_{d+1} = ie_1, \dots, u_{2d} = ie_d$. Then for $1 \leq i, j, k, \ell \leq 2d$ define the function $f_{n,i,j,k,\ell} : \Omega_n \to \mathbb{C}$ by

$$f_{n,i,j,k,\ell} = u_\ell(u_k(g_n(u_i, u_j))).$$

Now if $K \subset \Omega$ is a compact set, there exists $N > 0$ so that $K \subset \Omega_n$ for $n > N$. Using the definition of $\mathcal{G}_M(\Omega_n)$, there exists some $C > 0$ so that for all $n > N$ and all $1 \leq i, j, k, \ell \leq 2d$, the function $f_{n,i,j,k,\ell}$ is C-Lipschitz on K and $\sup_{z\in K} |f_{n,i,j,k,\ell}(z)| < C$.

Since $K \subset \Omega$ is an arbitrary compact set, we can pass to a subsequence of the g_n and assume that

$$\lim_{n\to\infty} f_{n,i,j,k,\ell}$$

exists for all i, j, k, ℓ and the convergence is locally uniform.

Now for $1 \leq i, j, k \leq 2d$ define the function $h_{n,i,j,k} : \Omega_n \to \mathbb{C}$ by

$$h_{n,i,j,k} = u_k(g(u_i, u_j)).$$

Fix some $z_0 \in \Omega$. Then there exists some $N > 0$ so that $z_0 \in \Omega_n$ for all $n > N$. Using part (3) of the definition of $\mathcal{G}_M$, we see that $\sup_{n>N} h_{n,i,j}(z_0)$ is finite. Then we can pass to a subsequence and assume that

$$\lim_{n\to\infty} h_{n,i,j}(z_0)$$

exists for all $1 \leq i, j \leq d$. Then using the fact that $u_\ell(h_{n,i,j}) = f_{n,i,j,k,\ell}$, we can pass to a subsequence so that

$$\lim_{n\to\infty} h_{n,i,j}$$

exists and the convergence is locally uniformly in the C^1 topology.

Repeating the argument above, we can then pass to a subsequence of the g_n which converges locally uniformly to some C^2 symmetric 2-form g and for all vectors $X, Y, v, w \in \mathbb{C}^d$

$$X(Y(g_{n_k}(v,w))) \to X(Y(g(v,w)))$$

locally uniformly. Part (2) of Definition 1.5 implies that g is an actual metric and hence is in $\mathcal{G}_M(\Omega)$. ☐

Now for $M, d > 0$, define a function $h_M : \mathbb{X}_{d,0} \to \mathbb{R}$ by letting $h_M(\Omega, x)$ be the infimum of all numbers $\epsilon > 0$ so that there exists a metric $g \in \mathcal{G}_M(\Omega)$ with

$$\max_{v,w \in T_z\Omega \setminus \{0\}} |H_g(v) - H_g(w)| \leq \epsilon \text{ for all } z \in B_\Omega(x; 1/\epsilon).$$

Here $B_\Omega(x;r)$ is the closed ball of radius r about the point $x \in \Omega$ with respect to the Kobayashi distance.

Proposition 8.2. *The function $(-h_M) : \mathbb{X}_{d,0} \to \mathbb{R}_{\leq 0}$ is an upper semicontinuous intrinsic function.*

Proof. We first argue that h_M is intrinsic. Let $\varphi : \Omega_1 \to \Omega_2$ be a biholomorphism with $\varphi(x_1) = x_2$. Suppose that $g \in \mathcal{G}_M(\Omega_1)$. Then consider the metric $\varphi^*(g)$ on Ω_2 given by

$$\varphi^*(g)_z(v,w) = g_{\varphi^{-1}}\left(d(\varphi^{-1})_z(v), d(\varphi^{-1})_z(w)\right).$$

Since φ is a biholomorphism the metric $\varphi^*(g)$ is Kähler and since φ is an isometry with respect to the Kobayashi metric and distance, it is easy to show that $\varphi^*(g) \in \mathcal{G}_M(\Omega_2)$. Finally, we observe that $H_g(v) = H_{\varphi^*(g)}(d(\varphi)(v))$ and $\varphi(B_{\Omega_1}(x;r)) = B_{\Omega_2}(\varphi(x);r)$. So we see that $h_M(\Omega_1, x_1) \geq h_M(\Omega_2, x_2)$. Repeating the above argument with $\varphi^{-1} : \Omega_2 \to \Omega_2$ shows that $h_M(\Omega_1, x_1) \leq h_M(\Omega_2, x_2)$. Hence h_M is intrinsic.

The fact that $(-h_M)$ is upper semicontinuous follows immediately from Lemma 8.1 and Theorems 2.1 and 2.2. ☐

Proposition 8.3. *Suppose $\Omega \in \mathbb{X}_d$, $M > 1$, and $h_M(\Omega, x) = 0$ for some $x \in \Omega$. Then Ω is biholomorphic to $\mathbb{B}_d$.*

Proof. Using Lemma 8.1, there exists a metric $g \in \mathcal{G}_M(\Omega)$ with holomorphic sectional curvature which only depends on the base point. Then Theorem 7.5 in [11, Chapter IX, Section 7] implies that g has constant holomorphic sectional curvature. Thus by Theorems 7.8 and 7.9 in [11, Chapter IX, Section 7], Ω is biholomorphic to the complex projective space, $\mathbb{B}_d$, or $\mathbb{C}^d$. Since Ω is non-compact, clearly Ω is not biholomorphic to the complex projective space. Since the Kobayashi metric is non-degenerate on Ω (see Remark 1.9), Ω cannot be biholomorphic to $\mathbb{C}^d$. Thus Ω is biholomorphic to $\mathbb{B}_d$. $\square$

Combining Proposition 8.2, Proposition 8.3, and Theorem 1.15 we obtain the following corollary.

Corollary 8.4. *For any $d > 0$ and $M > 1$, there exists some $\epsilon = \epsilon(M, d) > 0$ so that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and*

$$h(\Omega, z) \leq \epsilon$$

outside a compact subset of Ω, then Ω is strongly pseudoconvex.

Finally we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Fix $\epsilon > 0$ with the following property: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary and

$$h(\Omega, z) \leq 2\epsilon$$

outside a compact set of Ω, then Ω is strongly pseudoconvex.

Now suppose that $\Omega \in \mathbb{X}_d$, $K \subset \Omega$ is compact, and there exists a metric $g \in \mathcal{G}_M(\Omega)$ so that

$$\max_{v \in T_z\Omega \setminus \{0\}} |H_g(v) - \frac{-4}{d+1}| \leq \epsilon \text{ for all } z \in \Omega \setminus K.$$

We claim that Ω is strongly pseudoconvex.

Since K_Ω is a proper distance on Ω (see Remark 1.9), there exists some compact subset $K' \subset \Omega$ so that $B_\Omega(x; 1/\epsilon) \subset \Omega \setminus K$ for all $x \in \Omega \setminus K'$. Then, with this choice of K',

$$h_M(\Omega, x) \leq 2\epsilon$$

for all $x \in \Omega \setminus K'$. So by our choice of $\epsilon > 0$, Ω is strongly pseudoconvex. $\square$

9. Proof of Theorem 1.4

To deduce Theorem 1.4 from Theorem 1.7 it is enough to prove the following.

Proposition 9.1. *For any $d > 0$, there exists $M = M(d) > 1$ so that: $b_\Omega \in \mathcal{G}_M(\Omega)$ for every $\Omega \in \mathbb{X}_d$.*

Proof. By a result of Frankel [6], there exists some $C_0 = C_0(d) > 1$ so that

$$\frac{1}{C_0}\sqrt{b_\Omega} \leq k_\Omega \leq C_0\sqrt{b_\Omega} \tag{9.1}$$

for all $\Omega \in \mathbb{X}_d$. So we only need to prove conditions (3), (4), and (5) in the definition of $\mathcal{G}_M(\Omega)$. We will actually prove a stronger assertion: for any $n > 0$, there exists a $C_n = C_n(d) > 0$ so that: for all vectors $X_1, \dots, X_n, v, w \in \mathbb{C}^d$ we have

$$X_1 \cdots X_n(b_{\Omega,z}(v, w)) \leq C_n k_\Omega(z; v) k_\Omega(z; w) \prod_{i=1}^n k_\Omega(z; X_i).$$

Suppose not; then there exist some $n > 0$ and sequences $(\Omega_m, x_m) \in \mathbb{X}_{d,0}$, $X_{m,1}, \dots, X_{m,n}, v_m, w_m \in \mathbb{C}^d$ so that

$$\lim_{m\to\infty} \frac{X_{m,1} \cdots X_{m,n}(b_{\Omega_m,x_m}(v_m, w_m))}{k_{\Omega_m}(x_m; v_m) k_{\Omega_m}(x_m; w_m) \prod_{i=1}^n k_{\Omega_m}(x_m; X_{m,i})} = \infty.$$

Now, for any affine isomorphism $A \in \mathrm{Aff}(\mathbb{C}^d)$ and vectors $X_1, \dots, X_n, v, w \in \mathbb{C}^d$ we have

$$X_1 \cdots X_n(b_\Omega(v,w)) = (AX_1)\cdots(AX_n)(b_{A\Omega}(Av,Aw)).$$

So using the invariance of k_{Ω_n} we can replace each tuple

$$(\Omega_m, x_m), X_{m,1}, \dots, X_{m,n}, v_m, w_m$$

by

$$(A_m\Omega_m, A_m x_m), A_m X_{m,1}, \dots, A_m X_{m,n}, A_m v_m, A_m w_m,$$

where A_m is some affine isomorphism. So using Theorem 2.3, we can suppose that (Ω_m, x_m) converges to some (Ω, x) in $\mathbb{X}_{d,0}$. We also note that the ratio

$$\frac{X_{m,1}\cdots X_{m,n}(b_{\Omega_m,x_m}(v_m,w_m))}{k_{\Omega_m}(x_m;v_m)k_{\Omega_m}(x_m;w_m)\prod_{i=1}^n k_{\Omega_m}(x_m;X_{m,i})}$$

is invariant under scaling any of the $X_{m,1}, \dots, X_{m,n}, v_m, w_m$ so we may assume that these are all unit vectors with respect to the Euclidean metric. Then, after passing to a subsequence, we can suppose that

$$X_{m,1}, \dots, X_{m,n}, v_m, w_m \to X_1, \dots, X_n, v, w.$$

Now by Theorem 2.2

$$\lim_{m\to\infty} k_{\Omega_m}(x_m;v_m)k_{\Omega_m}(x_m;w_m)\prod_{i=1}^n k_{\Omega_m}(x_m;X_{m,i})$$
$$= k_\Omega(x;v)k_\Omega(x;w)\prod_{i=1}^n k_\Omega(x;X_i) > 0.$$

So to obtain a contradiction we must show that

$$\limsup_{m\to\infty} |X_{m,1}\cdots X_{m,n}(b_{\Omega_m,x_m}(v_m,w_m))| < +\infty.$$

Let $\kappa_{\mathcal{O}} : \mathcal{O}\times\mathcal{O} \to \mathbb{C}$ be the Bergman kernel on a domain $\mathcal{O}$. Then (by definition)

$$b_{\mathcal{O},z}(v,w) = \sum_{i=1}^d v_i \left(\frac{\partial^2}{\partial z_i \partial \overline{z}_i} \log \kappa_{\mathcal{O}}(z,z) \right) \overline{w}_i.$$

We also have the following basic properties:

(1) If $\mathcal{O}' \subset \mathcal{O}$, then $\kappa_{\mathcal{O}}(z,z) \leq \kappa_{\mathcal{O}'}(z,z)$.
(2) If $z, w \in \mathcal{O}$, then $\kappa_{\mathcal{O}}(z,w) \leq \kappa_{\mathcal{O}}(z,z)\kappa_{\mathcal{O}}(w,w)$.

We claim that after passing to a subsequence κ_{Ω_m} converges to some C^∞ function $\kappa : \Omega\times\Omega \to \mathbb{C}$ and the convergence is locally uniform for each partial derivative. Using basic properties of harmonic functions, it is enough to show that κ_{Ω_m} converges locally uniformly to some function $\kappa : \Omega\times\Omega\to\mathbb{C}$. Using Montel's theorem, it is enough to show that κ_{Ω_m} is locally bounded.

So fix a compact set $K \subset \Omega$ and an open neighborhood $\mathcal{O}$ of K which is relatively compact in Ω. Then for m large, $\mathcal{O} \subset \Omega_m$ and so

$$\kappa_{\Omega_m}(z,w) \leq \kappa_{\Omega_m}(z,z)\kappa_{\Omega_m}(w,w) \leq \kappa_{\mathcal{O}}(z,z)\kappa_{\mathcal{O}}(w,w)$$

for $z, w \in K$. So κ_{Ω_m} is locally bounded.

Then by Montel's theorem we can pass to a subsequence and assume that κ_{Ω_m} converges to some C^∞ function $\kappa : \Omega\times\Omega \to \mathbb{C}$ and the convergence is locally uniform for each partial derivative. Now using estimate (9.1), we see that $1/\kappa_{\Omega_m}(z,z)$ is locally bounded and thus

$$\limsup_{m\to\infty} |X_{m,1}\cdots X_{m,n}(b_{\Omega_m,x_m}(v_m,w_m))| < +\infty.$$

So we have a contradiction. □

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