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RENORMALIZATION OF THE TWO-DIMENSIONAL STOCHASTIC NONLINEAR WAVE EQUATIONS

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Abstract. We study the two-dimensional stochastic nonlinear wave equations (SNLW) with an additive space-time white noise forcing. In particular, we introduce a time-dependent renormalization and prove that SNLW is pathwise locally well-posed. As an application of the local well-posedness argument, we also establish a weak universality result for the renormalized SNLW.

1. Introduction

1.1. **Stochastic nonlinear wave equations.** We consider the following stochastic nonlinear wave equations (SNLW) on $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ with an additive space-time white noise forcing:

(1.1)

$$\begin{cases} \partial_t^2 u - \Delta u \pm u^k = \xi, \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2), \end{cases} \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R}_+,$$

where $k \geq 2$ is an integer and $\xi(x,t)$ denotes a (Gaussian) space-time white noise on $\mathbb{T}^2 \times \mathbb{R}_+$. In view of the time reversibility of the deterministic nonlinear wave equations, one can also consider (1.1) on $\mathbb{T}^2 \times \mathbb{R}$ by extending the white noise ξ onto $\mathbb{T}^2 \times \mathbb{R}$.[1] For simplicity, however, we only consider positive times in the following. Moreover, we restrict our discussion to the real-valued setting.

The stochastic wave equations with space-time white noise and with general nonlinearity have already been considered by Albeverio, Haba, Oberguggenberger, and Russo in a series of works [1, 22, 23, 28] for spatial dimensions going from one to three. In particular, they showed that, in two and three dimensions, solutions have to be distributions. Moreover, they highlighted a phenomenon of *triviality*; let u_ε be a smooth solution of SNLW obtained by replacing the space-time white noise ξ in (1.1) by a suitable regularized noise ξ_ε. Then, it was shown that as the regularization is removed, u_ε converges to a limiting process u satisfying a *linear* wave equation. The nonlinear behavior does not appear any more in the limiting equation due to the extreme oscillations of prelimit solutions u_ε. This phenomenon has already been noticed in parabolic equations, for example, in the stochastic

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[1]Namely, replace β_n in (1.3) by the sum of two independent Brownian motions, one forward in time on $\mathbb{T}^2 \times [0, \infty)$ and the other backward in time $\mathbb{T}^2 \times (-\infty, 0]$, both starting at $t = 0$.

quantization problem of Euclidean scalar fields in two and three dimensions, where a renormalization is needed in order to obtain a nontrivial limiting behavior. In this paper, we will present the first analysis of the renormalization problem for the stochastic nonlinear wave equation (1.1) in two dimensions. In order to implement this renormalization at the algebraic level, we restrict the form of the nonlinearity to a polynomial one. For the sake of simplicity, we consider a single monomial, although more general polynomial interactions could be considered. Other possible models for nonlinearity which should be amenable to renormalization are those given by trigonometric or exponential functions. In this case, however, we expect the renormalization problem to be more subtle and thus we leave it aside for the moment.

By letting $v = \partial_t u$, we can write (1.1) in the following Ito formulation:

$$\begin{cases} d\begin{pmatrix} u \\ v \end{pmatrix} + \left\{ \begin{pmatrix} 0 & -1 \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \pm u^k \end{pmatrix} \right\} dt = d\begin{pmatrix} 0 \\ W \end{pmatrix} \\ (u, v)|_{t=0} = (\phi_0, \phi_1). \end{cases} \tag{1.2}$$

Here, W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^2)$. More precisely, by letting

$$e_n(x) = e^{2\pi i n\cdot x}, \quad \mathcal{I} = (\mathbb{Z}_+ \times \{0\}) \cup (\mathbb{Z} \times \mathbb{Z}_+), \quad \text{and} \quad \mathcal{J} = \mathcal{I} \cup \{(0,0)\},$$

we have[2]

$$\begin{aligned} W(t) &= \beta_0(t)e_0 + \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \beta_n(t)e_n \\ &= \beta_0(t)e_0 + \sum_{n\in\mathcal{I}} \Big[\operatorname{Re}(\beta_n(t)) \cdot \sqrt{2}\cos(2\pi n\cdot x) - \operatorname{Im}(\beta_n(t)) \cdot \sqrt{2}\sin(2\pi n\cdot x) \Big], \end{aligned} \tag{1.3}$$

where $\{\beta_n\}_{n\in\mathcal{J}}$ is a family of mutually independent complex-valued Brownian motions[3] on a fixed probability space $(\Omega, \mathcal{F}, P)$ and $\beta_{-n} := \overline{\beta_n}$ for $n \in \mathcal{J}$. Note that $\operatorname{Var}(\beta_n(t)) = 2t$ for $n \in \mathbb{Z}^2 \setminus \{0\}$, while $\operatorname{Var}(\beta_0(t)) = t$. It is easy to see that W almost surely lies in $C^\alpha(\mathbb{R}_+; H^{-1-\varepsilon}(\mathbb{T}^2))$ for any $\alpha < \frac{1}{2}$ and $\varepsilon > 0$.

Let $S(t)$ be the propagator for the linear wave equation defined by

$$S(t)(\phi_0, \phi_1) := \cos(t|\nabla|)\phi_0 + \frac{\sin(t|\nabla|)}{|\nabla|}\phi_1$$

as a Fourier multiplier operator. Then, the mild formulation of the Cauchy problem (1.1) (and (1.2)) is given by

$$u(t) = S(t)(\phi_0, \phi_1) \mp \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} u^k(\tau)d\tau + \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} dW(\tau).$$

In fact, as it is written, this problem is ill-posed since solutions are expected to be merely distributions in the space variable, raising the problem of controlling the nonlinear term. The problem is already apparent at the level of the stochastic convolution:

$$\Psi(t) \stackrel{\text{def}}{=} (\partial_t^2 - \Delta)^{-1}\xi = \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} dW(\tau). \tag{1.4}$$

[2] Note that $\{1, \sqrt{2}\cos(2\pi n\cdot x), \sqrt{2}\sin(2\pi n\cdot x) : n \in \mathcal{I}\}$ forms an orthonormal basis of $L^2(\mathbb{T}^2)$ in the real-valued setting.

[3] Here, we take β_0 to be real-valued.

It can be shown that for each $t > 0$, $\Psi(t) \notin L^2(\mathbb{T}^2)$ almost surely. In particular, there is an issue in making sense of powers Ψ^k and a fortiori of the full nonlinearity u^k. As we discussed above, we need to modify the equation in order to take into account a proper renormalization and a different nonlinearity has to be considered.

1.2. **Renormalized SNLW.** In order to explain the renormalization process, we first regularize the equation (1.1) by a Fourier truncation of the noise term and of initial data:[4]

$$\begin{cases} \partial_t^2 u_N - \Delta u_N \pm (u_N)^k = \mathbf{P}_N \xi, \\ (u_N, \partial_t u_N)|_{t=0} = (\mathbf{P}_N \phi_0, \mathbf{P}_N \phi_1), \end{cases} \tag{1.5}$$

where $\mathbf{P}_N$ is the Dirichlet projection onto the spatial frequencies $\mathbb{Z}_N^2 \stackrel{\text{def}}{=} \{|n| \le N\}$. In the following, we discuss the renormalization for (1.5).

We define the truncated stochastic convolution $\Psi_N(t)$ by

$$\Psi_N(t) \stackrel{\text{def}}{=} \mathbf{P}_N \Psi(x,t) = \sum_{n \in \mathbb{Z}_N^2} e_n \int_0^t \frac{\sin((t-\tau)|n|)}{|n|} d\widetilde{\beta}_n(\tau) \tag{1.6}$$

with the understanding that

$$\frac{\sin((t-\tau)|0|)}{|0|} \stackrel{\text{def}}{=} t - \tau \qquad \text{and} \qquad \widetilde{\beta}_n = \begin{cases} \frac{1}{\sqrt{2}} \beta_n & \text{if } n \ne 0, \\ \beta_0 & \text{if } n = 0. \end{cases} \tag{1.7}$$

Then, for each fixed $x \in \mathbb{T}^2$ and $t \ge 0$, it follows from Ito isometry that the random variable $\Psi_N(x,t)$ is a mean-zero real-valued Gaussian random variable with variance

$$\begin{aligned} \sigma_N(t) \stackrel{\text{def}}{=} \mathbb{E}\big[\Psi_N^2(x,t)\big] &= \int_0^t (t-\tau)^2 \, d\tau + 2 \sum_{n \in \mathcal{I} \cap \mathbb{Z}_N^2} \int_0^t \left[\frac{\sin((t-\tau)|n|)}{|n|} \right]^2 d\tau \\ &= \frac{t^3}{3} + \sum_{0 < |n| \le N} \left\{ \frac{t}{2|n|^2} - \frac{\sin(2t|n|)}{4|n|^3} \right\} \sim t \log N. \end{aligned} \tag{1.8}$$

Note that $\sigma_N(t)$ is independent of $x \in \mathbb{T}^2$. The structure of the equation makes it clear that any solution can be decomposed as

$$u_N = \Psi_N + v_N, \tag{1.9}$$

where the residual term v_N solves a nonlinear wave equation (NLW) with the following polynomial nonlinearity with random coefficients depending on Ψ_N:

$$u_N^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \Psi_N^\ell v_N^{k-\ell}. \tag{1.10}$$

Note, however, that the monomial Ψ_N^ℓ does not have nice limiting behavior as $N \to \infty$. Despite this difficulty, the decomposition (1.9) is motivated by the heuristics that, in two dimensions, the only singularities which have to be dealt with in the renormalization process are related to the powers of the random field Ψ. We are going to prove that this is indeed the case and that the residual term v_N can be controlled in a nice space. The decomposition (1.9) usually takes the name of Da Prato-Debussche trick [9] in the field of stochastic parabolic PDEs. Note

[4] Strictly speaking, the regularization of initial data is not necessary here but it allows us to consider smooth solutions to the regularized equation (1.5).

that such an idea also appears in McKean [18] and Bourgain [4] in the context of (deterministic) dispersive PDEs with random initial data, predating [9]. See also Burq-Tzvetkov [6].

In order to renormalize the nonlinearity u_N^k in (1.10), we need to introduce suitable counter-terms. We will show that in order to renormalize each random monomial Ψ_N^ℓ, it is enough to replace it with its *Wick ordered* counterpart:

$$:\Psi_N^\ell(x,t): \stackrel{\text{def}}{=} H_\ell(\Psi_N(x,t);\sigma_N(t)). \tag{1.11}$$

Here, $H_\ell(x;\sigma)$ is given by

$$H_\ell(x;\sigma) = \sigma^{\frac{\ell}{2}} H_\ell(\sigma^{-\frac{1}{2}}x),$$

where $H_\ell(\cdot)$ is the ℓth Hermite polynomial for the standard Gaussian measure. Combining this with the following standard identity:

$$H_k(x+y) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y),$$

we have

$$\begin{aligned} H_k(x+y;\sigma) &= \sigma^{\frac{k}{2}} \sum_{\ell=0}^{k} \binom{k}{\ell} \sigma^{-\frac{k-\ell}{2}} x^{k-\ell} H_\ell(\sigma^{-\frac{1}{2}}y) \\ &= \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y;\sigma). \end{aligned} \tag{1.12}$$

In our situation, this gives

$$H_k(u_N(x,t);\sigma_N(t)) = \sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(\Psi_N(x,t);\sigma_N(t)) \big(v_N(x,t)\big)^{k-\ell}.$$

From this, we see that Wick ordering all the monomials Ψ_N^ℓ in (1.10) is equivalent to replacing the original nonlinearity u_N^k by the kth Hermite polynomial $H_k(u_N(x,t);\sigma_N(t))$. Note that there is no reason for u_N to be a Gaussian random variable. By common abuse of language, however, we refer to the function $H_k(u_N(x,t);\sigma_N(t))$ as a Wick ordered nonlinearity[5] of u_N^k. Compare this with the usual Wick ordered (deterministic) NLW on $\mathbb{T}^2$ considered in Oh-Thomann [27].

As in the case of the usual (time-independent) Wick ordered monomial, this time-dependent renormalization allows us to define

$$:\Psi^k: \stackrel{\text{def}}{=} \lim_{N\to\infty} :\Psi_N^k: \tag{1.13}$$

in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$ for any $p<\infty$ and $\varepsilon>0$ (and for any $k\in\mathbb{N}$).[6] See Proposition 2.1 below. This convergence result allows us to describe the limiting problem we are going to solve. Consider a function $u = \Psi + v$, where

[5]We expect the variance of the solution $u_N(t)$ grows in time. See Oh-Quastel-Sosoe [25] in the context of the stochastic KdV equation. Hence, the renormalization must depend on time. This is different from the situation where one expects an invariant measure for a given dynamic so that a renormalization is time-independent.

[6]Here, $W^{s,r}(\mathbb{T}^2)$ denotes the usual L^r-based Sobolev space (Bessel potential space) defined by the norm:

$$\|u\|_{W^{s,r}} = \|\langle\nabla\rangle^s u\|_{L^r} = \big\|\mathcal{F}^{-1}(\langle n\rangle^s \widehat{u}(n))\big\|_{L^r}.$$

When $r=2$, we have $H^s(\mathbb{T}^2) = W^{s,2}(\mathbb{T}^2)$.

$v \in L^q([0,T]; W^{s,r}(\mathbb{T}^2))$ for some appropriate $q, r \geq 1$ and $s > 0$. Then, as $N \to \infty$, we have

$$H_k(\mathbf{P}_N u(x,t); \sigma_N(t)) \\ \longrightarrow \,:\! u^k(x,t)\!: = F_\Psi(v)(x,t) \overset{\text{def}}{=} \sum_{\ell=0}^{k} \binom{k}{\ell} :\!\Psi^\ell(x,t)\!: \big(v(x,t)\big)^{k-\ell}. \tag{1.14}$$

We insist that the nonlinear (random) function $u \mapsto \,:\! u^k \!: = F_\Psi(v)$ is only defined for u of the form $\Psi + v$ with suitable v. With this in mind, we set our main goal to prove local well-posedness of the following Wick ordered SNLW:

$$\begin{cases} \partial_t^2 u - \Delta u \pm :\!u^k\!: = \xi, \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1). \end{cases} \tag{1.15}$$

In the following, we concentrate on the following mild formulation of the Wick ordered SNLW (1.15):

$$u(t) = S(t)(\phi_0, \phi_1) \mp \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} :\!u^k(\tau)\!: d\tau + \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} dW(\tau), \tag{1.16}$$

where the Wick ordered nonlinearity $:\!u^k\!:$ is defined by (1.14).

We point out that such a solution u to (1.15) and (1.16) can also be given as the limit of solutions to the following truncated Wick ordered SNLW:

$$\begin{cases} \partial_t^2 u_N - \Delta u_N \pm H_k(u_N; \sigma_N) = \mathbf{P}_N \xi, \\ (u_N, \partial_t u_N)|_{t=0} = (\mathbf{P}_N \phi_0, \mathbf{P}_N \phi_1), \end{cases} \tag{1.17}$$

as $N \in \mathbb{N}$. More precisely, one can study the following mild formulation of the truncated Wick ordered SNLW (1.17):

$$u_N(t) = S(t)(\mathbf{P}_N \phi_0, \mathbf{P}_N \phi_1) \mp \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} H_k(u_N; \sigma_N)(\tau) d\tau + \Psi_N(t) \tag{1.18}$$

and prove (i) (1.18) is locally well-posed "uniformly in $N \in \mathbb{N}$" and (ii) u_N converges to a stochastic process u such that the Wick ordered nonlinearity $:\!u^k(x,t)\!:$ in (1.15) is well defined and the following limit holds:

$$:\!u^k(x,t)\!: \overset{\text{def}}{=} \lim_{N\to\infty} H_k(u_N(x,t); \sigma_N(t)).$$

One can then define this limit u to be a solution to (1.15). This solution u constructed as a limit of u_N as above agrees with the solution to the mild formulation (1.16) in a suitable sense. See Remark 1.2 below.

1.3. **Main result.** Before we state our main result, we first need to discuss critical regularities associated to the deterministic NLW:

$$\partial_t^2 u - \Delta u \pm u^k = 0.$$

On the one hand, NLW on $\mathbb{R}^d$ enjoys the scaling symmetry, which induces the so-called scaling critical Sobolev index: $s_{\text{scaling}} = \frac{d}{2} - \frac{2}{k-1}$. On the other hand, NLW also enjoys the Lorentzian invariance (conformal symmetry), which yields its own critical regularity $s_{\text{conf}} = \frac{d+1}{4} - \frac{1}{k-1}$ (at least in the focusing case); see [17]. In particular, when $d = 2$, we define s_{crit} for a given integer $k \geq 2$ by

$$s_{\text{crit}} := \max(s_{\text{scaling}}, s_{\text{conf}}, 0) = \max\left(1 - \frac{2}{k-1}, \frac{3}{4} - \frac{1}{k-1}, 0\right). \tag{1.19}$$

Note that the third regularity restriction 0 appears in making sense of powers of u. See also (1.24) and Figure 1 below.

We now state our main result.

Theorem 1.1. *Given an integer $k \geq 2$, let s_{crit} be as in* (1.19). *Then, the Wick ordered SNLW* (1.15) *is pathwise locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$ for*

$$\text{(i) } k \geq 4: \ s \geq s_{\text{crit}} \qquad \textit{or} \qquad \text{(ii) } k = 2, 3: \ s > s_{\text{crit}}.$$

More precisely, given any $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists a stopping time $T = T_\omega(\phi_0, \phi_1)$ (which is positive almost surely) such that there exists a unique solution u to the mild formulation (1.16) *on $[0, T]$ with*

$$u \in \Psi + C([0, T]; H^\sigma(\mathbb{T}^2)) \subset C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, where $\sigma = \min(s, 1 - \varepsilon)$.

In Theorem 1.1, the uniqueness holds only in $\Psi + X^\sigma(T)$, where $X^\sigma(T)$ is given by

$$X^\sigma(T) = C([0, T]; H^\sigma(\mathbb{T}^2)) \cap C^1([0, T]; H^\sigma(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2))$$

for some suitable σ-admissible pair (q, r). See Section 3 for more on this point.

In [24], the third author proved pathwise local well-posedness of the following stochastic KdV with an additive space-time white noise forcing:

$$du + (\partial_x^3 u + u \partial_x u)dt = dW, \qquad (x, t) \in \mathbb{T} \times \mathbb{R}_+,$$

where W denotes a cylindrical Wiener process on $L^2(\mathbb{T})$. Theorem 1.1 is the second example on pathwise local well-posedness of rough stochastic dispersive PDEs with an additive space-time white noise forcing.

As we already mentioned in the previous subsection, the Wick ordered SNLW is *defined* only for functions

$$u = \Psi + v \tag{1.20}$$

with v of suitable positive regularity. The main strategy for proving Theorem 1.1 is then to consider the following fixed point problem for $v = u - \Psi$:

$$v(t) = S(t)(\phi_0, \phi_1) \mp \int_0^t \frac{\sin((t - \tau)|\nabla|)}{|\nabla|} F_\Psi(v(\tau)) d\tau, \tag{1.21}$$

where F_Ψ is as in (1.14).

The proof is based on a fixed point argument via the Strichartz estimates for the wave equations and the general structure of the proof is similar to that for stochastic parabolic equations. The key point is to use function spaces where the wave equation allows for a gain in regularity. This gain is sufficient to prove that v has better regularity than Ψ and gives a well-defined nonlinearity for which suitable local-in-time estimates can be established. In Section 2, we prove the necessary stochastic estimates for the random terms and then we give the deterministic nonlinear estimates and the proof of Theorem 1.1 in Section 3.

As an application of the local well-posedness argument, we show a weak universality result for the Wick ordered SNLW in Section 4. Given small $\varepsilon > 0$, we consider the following SNLW equation on a dilated torus $(\varepsilon^{-1}\mathbb{T})^2$ with a smooth noise η^ε:

$$\begin{cases} \partial_t^2 w_\varepsilon - \Delta w_\varepsilon = f(w_\varepsilon) + a(\varepsilon,t)w_\varepsilon + \delta(\varepsilon)\eta^\varepsilon, \\ (w_\varepsilon, \partial_t w_\varepsilon)|_{t=0} = (0,0), \end{cases} \qquad (x,t) \in (\varepsilon^{-1}\mathbb{T})^2 \times \mathbb{R}_+,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth odd, bounded function with a sufficient number of bounded derivatives, η^ε is a noise which is white in time but smooth and stationary in space, and $a(\varepsilon,t)$ and $\delta(\varepsilon)$ are parameters to be chosen. Consider the following space-time scaling:

$$u_\varepsilon(x,t) = \varepsilon^{-\gamma} w_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-1}t)$$

for some $\gamma > 0$. Namely, u_ε describes the behavior of w_ε at large scales, both in space and time. Then, by appropriately choosing parameters $\gamma = 1$, $\delta(\varepsilon) = \varepsilon^{\frac{3}{2}}$, and $a(\varepsilon, \varepsilon^{-1}t)$, we show that u_ε converges in a suitable sense to the solution u to the Wick ordered cubic SNLW:

$$\begin{cases} \partial_t^2 u - \Delta u = \lambda :u^3: + \xi, \\ (u, \partial_t u)|_{t=0} = (0,0), \end{cases}$$

for some $\lambda = \lambda(f)$. Here, we can choose $a(\varepsilon,t)$ such that it depends only on f, the noise, and $\varepsilon > 0$. See Theorem 4.1 below. We also refer readers to [12–14] for more discussion on weak universality (for stochastic parabolic equations, in particular the KPZ equation).

We conclude this introduction by stating several remarks.

Remark 1.2. The same local well-posedness result also applies to the truncated Wick ordered SNLW (1.17), uniformly in $N \in \mathbb{N}$. More precisely, given $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ and $N \in \mathbb{N}$, there exist a stopping time $T = T_\omega(\phi_0, \phi_1)$ (> 0 almost surely) and a unique solution u_N to (1.18) on $[0,T]$ such that

$$u_N \in \Psi_N + C([0,T]; H^\sigma(\mathbb{T}^2)).$$

Moreover, one can prove that the solutions u_N to (1.18) converges to the solution u to (1.16) constructed in Theorem 1.1 as $N \to \infty$.

In the discussion above, we used the Dirichlet projection $\mathbf{P}_N$ onto the spatial frequencies $\{|n| \leq N\}$ for regularization. The interpretation of the Wick ordered nonlinearity (1.14) seems to depend on this regularization procedure at this point.

One may instead use a different regularization procedure. Given a compactly supported smooth function $\rho \in L^1(\mathbb{T}^2)$ with $\int \rho dx = 1$, let $\mathbf{P}_N^\rho$ be the mollification given by $\mathbf{P}_N^\rho f = \rho_N * f$, where $\rho_N(x) = N^2\rho(Nx)$. Then, one can consider the regularized stochastic convolution $\Psi_{\rho,N} = \mathbf{P}_N^\rho \Psi$ associated to this mollification and define the Wick ordered monomials:

$$:\Psi^k(x,t): \stackrel{\text{def}}{=} \lim_{N\to\infty} H_k(\Psi_{\rho,N}(x,t); \sigma_{\rho,N}(t)),$$

where $\sigma_{\rho,N}(t) \stackrel{\text{def}}{=} \mathbb{E}[\Psi_{\rho,N}^2(x,t)]$. By proceeding as in (1.8), we have

$$\begin{aligned} \sigma_{\rho,N}(t) &= \frac{t^3}{3} + \sum_{|n|>0} |\widehat{\rho}_N(n)|^2 \bigg\{ \frac{t}{2|n|^2} - \frac{\sin(2t|n|)}{4|n|^3} \bigg\} \\ &= \frac{t^3}{3} + \sum_{|n|>0} \Big| \mathcal{F}_{\mathbb{R}^2}(\rho)\Big(\frac{n}{N}\Big) \Big|^2 \bigg\{ \frac{t}{2|n|^2} - \frac{\sin(2t|n|)}{4|n|^3} \bigg\}, \end{aligned}$$

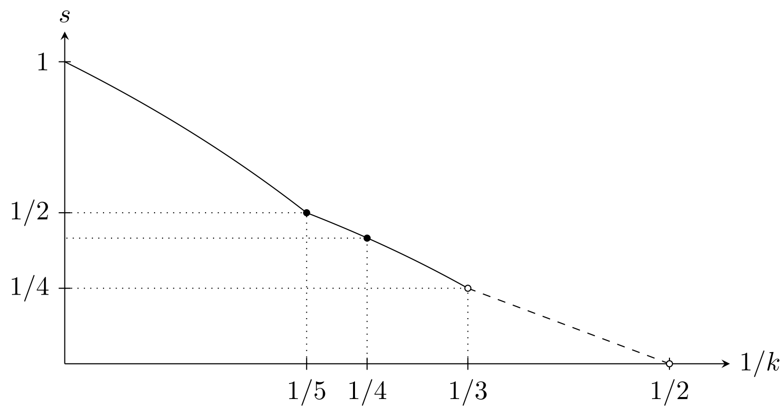

FIGURE 1. The critical regularity s_{crit} in (1.24) as a function of $\frac{1}{k}$. The deterministic NLW (1.23) is locally well-posed on and above the solid line and above the dashed line.

where $\mathcal{F}_{\mathbb{R}^2}(\rho)$ is the Fourier transform of ρ when viewed as a function on $\mathbb{R}^2$. By slightly modifying the proof of Proposition 2.1, one can prove that the Wick ordered monomials $:\Psi^k:$ do not depend on the choice of mollifiers (including the convolution kernel of the Dirichlet projection $\mathbf{P}_N$). This directly implies that the renormalized nonlinearity F_Ψ is also independent of the choice of a mollifier. Of course, the precise value of the renormalization constant will depend on ρ.

Remark 1.3. With a small modification of the proof, Theorem 1.1 also holds for the following stochastic nonlinear Klein-Gordon equation with an additive space-time white noise:

$$\partial_t^2 u + (1-\Delta)u \pm u^k = \xi. \tag{1.22}$$

On the one hand, we restrict our attention to the real-valued setting in this paper. On the other hand, it is often useful to consider complex-valued solutions to the nonlinear Klein-Gordon equation. We point out that Theorem 1.1 also holds in the complex-valued setting, provided that we adjust the white noise forcing and the renormalization procedure to the complex-valued setting. In particular, one needs to use (generalized) Laguerre polynomials instead of Hermite polynomials. See Oh-Thomann [26] for details.

Remark 1.4. In the following, we state local well-posedness of the following deterministic NLW on $\mathbb{T}^2$:

$$\partial_t^2 u - \Delta u \pm |u|^{k-1}u = 0, \tag{1.23}$$

where we allow $k \geq 2$ to take noninteger values. We extend the critical regularity s_{crit} in (1.19) to a real number $k \geq 2$ by setting

$$s_{\text{crit}} := \max\big(s_{\text{scaling}}, s_{\text{conf}}, \tfrac{3}{4} - \tfrac{3}{2k}\big) = \max\big(1 - \tfrac{2}{k-1}, \tfrac{3}{4} - \tfrac{1}{k-1}, \tfrac{3}{4} - \tfrac{3}{2k}\big). \tag{1.24}$$

This extends s_{crit} defined (1.19) to noninteger values of $k \geq 2$. As far as we know, the third regularity $\frac{3}{4} - \frac{3}{2k}$ does not correspond to any symmetry of the equation and thus it is not a critical regularity in the usual sense. It, however, imposes a regularity restriction when $2 \leq k \leq 3$.

By the standard Strichartz estimates (see Lemma 3.2) and a fixed point argument, one can easily prove local well-posedness of (1.23) in $\mathcal{H}^s(\mathbb{T}^2)$ for (i) $s \geq s_{\text{crit}}$ if $k > 3$ and (ii) $s > s_{\text{crit}}$ if $2 \leq k \leq 3$. See Subsection 3.1. Figure 1 shows the range of local well-posedness of (1.23) as a function of $\frac{1}{k}$.

2. On the stochastic convolution

In this section, we establish relevant estimates on the stochastic convolution Ψ. In particular, we prove the following regularity result on the Wick ordered monomials $:\Psi_N^\ell(x,t): = H_\ell(\Psi_N(x,t), \sigma_N(t))$ defined in (1.11).

Proposition 2.1. *Let $\ell \in \mathbb{N}$, $T > 0$ and $p \geq 1$. Then, $\{:\Psi_N^\ell:\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$. In particular, denoting the limit by $:\Psi^\ell:$, we have $:\Psi^\ell: \in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ almost surely.*

Before proceeding to the proof of Proposition 2.1, we recall some basic tools from probability theory and Euclidean quantum field theory. See [16, 21, 29]. First, recall the Hermite polynomials $H_k(x;\sigma)$ defined through the generating function:

$$F(t,x;\sigma) \stackrel{\text{def}}{=} e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma). \tag{2.1}$$

For simplicity, we set $F(t,x) := F(t,x;1)$ and $H_k(x) := H_k(x;1)$ in the following. For the reader's convenience, we write out the first few Hermite polynomials:

$$\begin{aligned} &H_0(x;\sigma) = 1, \qquad H_1(x;\sigma) = x, \qquad H_2(x;\sigma) = x^2 - \sigma, \\ &H_3(x;\sigma) = x^3 - 3\sigma x, \qquad H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2. \end{aligned} \tag{2.2}$$

Then, the monomial x^k can be expressed in term of the Hermite polynomials:

$$x^k = \sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m} \frac{(2m)!}{2^m m!} \sigma^m H_{k-2m}(x;\sigma). \tag{2.3}$$

Fix $d \in \mathbb{N}$.[7] Consider the Hilbert space $H = L^2(\mathbb{R}^d, \mu_d)$ endowed with the Gaussian measure $d\mu_d = (2\pi)^{-\frac{d}{2}} \exp(-|x|^2/2)dx$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Hermite polynomials satisfy

$$\int_{\mathbb{R}} H_k(x) H_m(x) d\mu_1(x) = \delta_{km} k! \tag{2.4}$$

for all $k, m \in \mathbb{N}$. Next, we define a *homogeneous Wiener chaos of order* k to be an element of the form $\prod_{j=1}^d H_{k_j}(x_j)$, where $k = k_1 + \cdots + k_d$ and H_{k_j} is the Hermite polynomial of degree k_j defined in (2.1). Denote the closure of homogeneous Wiener chaoses of order k under $L^2(\mathbb{R}^d, \mu_d)$ by $\mathcal{H}_k$. Let $L := \Delta - x \cdot \nabla$ be the Ornstein–Uhlenbeck operator. Then, it is known that any element in $\mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$ and that we have the Ito-Wiener decomposition:

$$L^2(\mathbb{R}^d, \mu_d) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

Moreover, we have the following hypercontractivity of the Ornstein–Uhlenbeck semigroup $U(t) := e^{tL}$ due to Nelson [20].

[7]Indeed, the discussion presented here also holds for $d = \infty$ in the context of abstract Wiener spaces. For simplicity, however, we consider only finite values for d.

Lemma 2.2. *Let $q > 1$ and $p \geq q$. Then, for every $u \in L^q(\mathbb{R}^d, \mu_d)$ and $t \geq \frac{1}{2}\log\left(\frac{p-1}{q-1}\right)$, we have*

$$\|U(t)u\|_{L^p(\mathbb{R}^d,\mu_d)} \leq \|u\|_{L^q(\mathbb{R}^d,\mu_d)}. \tag{2.5}$$

We stress that (2.5) holds, independent of the dimension d. As a consequence, we obtain the following corollary to Lemma 2.2.

Lemma 2.3. *Let $F \in \mathcal{H}_k$. Then, for $p \geq 2$, we have*

$$\|F\|_{L^p(\mathbb{R}^d,\mu_d)} \leq (p-1)^{\frac{k}{2}} \|F\|_{L^2(\mathbb{R}^d,\mu_d)}. \tag{2.6}$$

The estimate (2.6) follows immediately from noting that F is an eigenfunction of $U(t) = e^{tL}$ with eigenvalue e^{-kt} and setting $q = 2$ and $t = \frac{1}{2}\log(p-1)$ in (2.5). As a further consequence to Lemma 2.3, we obtain the following lemma [29, Theorem I.22].

Lemma 2.4. *Fix $k \in \mathbb{N}$ and $c(n_1, \dots, n_k) \in \mathbb{C}$. Given $d \in \mathbb{N}$, let $\{g_n\}_{n=1}^d$ be a sequence of independent standard complex-valued Gaussian random variables and set $g_{-n} = \overline{g_n}$. Define $S_k(\omega)$ by*

$$S_k(\omega) = \sum_{\Gamma(k,d)} c(n_1, \dots, n_k) g_{n_1}(\omega) \cdots g_{n_k}(\omega),$$

where $\Gamma(k,d)$ is defined by

$$\Gamma(k,d) = \big\{(n_1, \dots, n_k) \in \{\pm 1, \dots, \pm d\}^k\big\}.$$

Then, for $p \geq 2$, we have

$$\|S_k\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}. \tag{2.7}$$

This follows from (2.3) and Lemma 2.3. Once again, note that (2.7) is independent of $d \in \mathbb{N}$. Lemmas 2.3 and 2.4 have been very effective in the recent probabilistic study of dispersive PDEs and related areas; see, e.g., [3, 5, 8, 31, 32].

Lastly, we recall the following property of Wick products [29, Theorem I.3], extending (2.4) to a more general setting. See also [21, Lemma 1.1.1].

Lemma 2.5. *Let f and g be Gaussian random variables with variances σ_f and σ_g. Then, we have*

$$\mathbb{E}\big[H_k(f;\sigma_f)H_m(g;\sigma_g)\big] = \delta_{km} k! \big\{\mathbb{E}[fg]\big\}^k.$$

Proof of Proposition 2.1. First note that it suffices to prove the proposition for large $p \geq 1$, since $L^{p_1}(\Omega) \subset L^{p_2}(\Omega)$ for $p_1 \geq p_2$. From (1.6), we have

(2.8)
$$\mathbb{E}[\Psi_N(t_1,x)\Psi_N(t_2,y)] = \sum_{n\in\mathbb{Z}^2_N} e_n(x-y) \int_0^t \frac{\sin((t_1-\tau)|n|)}{|n|} \frac{\sin((t_2-\tau)|n|)}{|n|} d\tau,$$

where $t = \min(t_1, t_2)$. Define $\gamma(n,t)$ by

$$\gamma(n,t) \overset{\text{def}}{=} \int_0^t \left[\frac{\sin((t-\tau)|n|)}{|n|}\right]^2 d\tau.$$

By applying the Bessel potentials $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ of order ε and then setting $x=y$ (and $t_1=t_2$), we obtain

$$\mathbb{E}\big[|\langle\nabla\rangle^{-\varepsilon}\Psi_N(x,t)|^2\big]=\sum_{n\in\mathbb{Z}^2_N}\langle n\rangle^{-2\varepsilon}\gamma(n,t)\le t^3+t\sum_{n\in\mathbb{Z}^2_N}\frac{1}{\langle n\rangle^{2+2\varepsilon}}\lesssim t^3+t$$

for any $\varepsilon>0$, $x\in\mathbb{T}^2$, and $t>0$, uniformly in $N\in\mathbb{N}$. In particular, by the hypercontractivity (Lemma 2.4), we have

$$\mathbb{E}\big[|\langle\nabla\rangle^{-\varepsilon}\Psi_N(t,x)|^p\big]\lesssim_{p,t}1$$

and thus

$$\mathbb{E}\big[\|\Psi_N(\cdot,t)\|^p_{W^{-\varepsilon,p}}\big]=\mathbb{E}\big[\|\langle\nabla\rangle^{-\varepsilon}\Psi_N(\cdot,t)\|^p_{L^p(\mathbb{T}^2)}\big]<\infty$$

for any $\varepsilon>0$, $t>0$, and $p\ge1$, uniformly in $N\in\mathbb{N}$.

By Lemma 2.5 and (2.8), we have

$$\begin{aligned}\mathbb{E}\big[\,:\!\Psi^\ell_N(x,t)\!:\,:\!\Psi^\ell_N(y,t)\!:\,\big]&=\ell!\big\{\mathbb{E}[\Psi_N(x,t)\Psi_N(y,t)]\big\}^\ell\\&=\ell!\sum_{n_1,\dots,n_\ell\in\mathbb{Z}^2_N}\gamma(n_1,t)\cdots\gamma(n_\ell,t)e_{n_1}(x-y)\cdots e_{n_\ell}(x-y)\\&=\ell!\sum_{n_1,\dots,n_\ell\in\mathbb{Z}^2_N}\gamma(n_1,t)\cdots\gamma(n_\ell,t)e_{n_1+\cdots+n_\ell}(x-y).\end{aligned}$$

Proceeding as before, we obtain

$$\begin{aligned}\mathbb{E}\big[|(\langle\nabla\rangle^{-\varepsilon}:\!\Psi^\ell_N(\cdot,t)\!:)(x)|^2\big]&=\ell!\sum_{n_1,\dots,n_\ell\in\mathbb{Z}^2_N}\langle n_1+\cdots+n_\ell\rangle^{-2\varepsilon}\gamma(n_1,t)\cdots\gamma(n_\ell,t)\\&\lesssim_t\sum_{n_1,\dots,n_\ell\in\mathbb{Z}^2}\frac{1}{\langle n_1\rangle^2\cdots\langle n_\ell\rangle^2\langle n_1+\cdots+n_\ell\rangle^{2\varepsilon}}<\infty\end{aligned}$$

for any $\varepsilon>0$, $x\in\mathbb{T}^2$, and $t>0$, uniformly in N. Hence, by the hypercontractivity (Lemma 2.4), we have

$$\mathbb{E}\big[\|:\!\Psi^\ell_N(\cdot,t)\!:\|^p_{W^{-\varepsilon,p}}\big]<\infty$$

for any $\varepsilon>0$, $t>0$, and $p\ge1$, uniformly in $N\in\mathbb{N}$.

In order to analyze the time regularity, we have to estimate moments of the random field

$$\delta_h:\!\Psi^\ell_N(x,t)\!:\overset{\text{def}}{=}\,:\!\Psi^\ell_N(x,t+h)\!:-:\!\Psi^\ell_N(x,t)\!:$$

for $h\in[-1,1]$. In the following, we proceed as above and estimate

$$\mathbb{E}\big[|\delta_h(\langle\nabla\rangle^{-\varepsilon}:\!\Psi^\ell_N(\cdot,t)\!:)(x)|^2\big].$$

By applying Lemma 2.5 once again, we have

$$\begin{aligned}
&\frac{1}{\ell!}\mathbb{E}\big[\delta_h :\Psi_N^\ell(x,t): \ \delta_h :\Psi_N^\ell(y,t): \big] \\
&\quad = \big\{\mathbb{E}[\Psi_N(x,t+h)\Psi_N(y,t+h)]\big\}^\ell - \big\{\mathbb{E}[\Psi_N(x,t)\Psi_N(y,t+h)]\big\}^\ell \\
&\qquad - \big\{\mathbb{E}[\Psi_N(x,t+h)\Psi_N(y,t)]\big\}^\ell + \big\{\mathbb{E}[\Psi_N(x,t)\Psi_N(y,t)]\big\}^\ell \\
&\quad = \mathbb{E}[\delta_h\Psi_N(x,t)\Psi_N(y,t+h)] \\
&\qquad \times \sum_{j=0}^{\ell-1} \big\{\mathbb{E}[\Psi_N(x,t+h)\Psi_N(y,t+h)]\big\}^{\ell-j-1}\big\{\mathbb{E}[\Psi_N(x,t)\Psi_N(y,t+h)]\big\}^j \\
&\qquad - \mathbb{E}[\delta_h\Psi_N(x,t)\Psi_N(y,t)] \\
&\qquad \times \sum_{j=0}^{\ell-1} \big\{\mathbb{E}[\Psi_N(x,t+h)\Psi_N(y,t)]\big\}^{\ell-j-1}\big\{\mathbb{E}[\Psi_N(x,t)\Psi_N(y,t)]\big\}^j.
\end{aligned}$$

By reasoning as before, in order to estimate $\mathbb{E}\big[|\delta_h(\langle\nabla\rangle^{-\varepsilon} :\Psi^\ell(\cdot,t):)(x)|^2\big]$, we are led to bound sums of the form

$$S_{h,\varepsilon} = \sum_{n_1,\dots,n_\ell\in\mathbb{Z}_N^2} \langle n_1+\cdots+n_\ell\rangle^{-2\varepsilon} G_1(n_1,t,h)\cdots G_\ell(n_\ell,t,h), \tag{2.9}$$

where $G_i(n,t)$ is given by

$$\begin{aligned}
G_1(n,t) &= \mathbb{E}[\delta_h\widehat{\Psi}_N(n,t)\widehat{\Psi}_N(n,t_1)],\\
G_i(n,t) &= \mathbb{E}[\widehat{\Psi}_N(n,t_1)\widehat{\Psi}_N(n,t_2)], \quad i=2,\dots,\ell,
\end{aligned}$$

with $t_1,t_2\in\{t,t+h\}$. Here, $\widehat{\Psi}_N(n,t)$ denotes the spatial Fourier transform of $\Psi_N(t)$. A direct computation with (1.6) gives

(2.10)
$$\big|\mathbb{E}[\widehat{\Psi}_N(n,t_1)\widehat{\Psi}_N(n,t_2)]\big| \lesssim_t \frac{1}{\langle n\rangle^2} \qquad \text{and} \qquad \big|\mathbb{E}[\delta_h\widehat{\Psi}_N(n,t)\widehat{\Psi}_N(n,t_1)]\big| \lesssim_t \frac{|h|^\rho}{\langle n\rangle^{2-\rho}}$$

for any $\rho\in[0,1]$, where the implicit constants are independent of $h\in[-1,1]$. Note that the second estimate follows from interpolating

$$\big|\mathbb{E}[\delta_h\widehat{\Psi}_N(n,t)\widehat{\Psi}_N(n,t_1)]\big| \lesssim_t \frac{1}{\langle n\rangle^2} \qquad \text{and} \qquad \big|\mathbb{E}[\delta_h\widehat{\Psi}_N(n,t)\widehat{\Psi}_N(n,t_1)]\big| \lesssim_t \frac{|h|}{\langle n\rangle},$$

where the second bound follows from the mean value theorem. As a consequence, it follows from (2.9) and (2.10) that

$$|S_{h,\varepsilon}| \lesssim |h|^\rho$$

for any $h\in[-1,1]$, $\varepsilon>0$, and $\rho\in[0,1]$ such that $2\varepsilon-\rho>0$. This in turn implies that

$$\mathbb{E}\big[|\delta_h(\langle\nabla\rangle^{-\varepsilon} :\Psi_N^\ell(\cdot,t):)(x)|^2\big] \lesssim |h|^\rho.$$

Then, by the hypercontractivity (Lemma 2.4), this results in

$$\mathbb{E}\Big[\big\|\delta_h(:\Psi_N^\ell(\cdot,t):)\big\|_{W^{-\varepsilon,p}}^p\Big] \lesssim_{p,t} |h|^{\frac{p}{2}\rho}$$

for any $h \in [-1,1]$, $\rho \in [0,1]$, and $\varepsilon > 0$ such that $2\varepsilon > \rho$. Hence, it follows from Sobolev's embedding theorem that, given $\varepsilon > 0$, we have

$$\mathbb{E}\Big[\big\|\delta_h(:\Psi_N^\ell(\cdot,t):)\big\|_{W^{-\varepsilon,\infty}}^p\Big] \lesssim \mathbb{E}\Big[\big\|\delta_h(:\Psi_N^\ell(\cdot,t):)\big\|_{W^{-\frac{\varepsilon}{2},p}}^p\Big] \lesssim_{p,t} |h|^{\frac{p}{2}\rho}$$

for p sufficiently large such that $\varepsilon p > 4$. Moreover, for fixed $\rho \in (0, 2\varepsilon)$, we can choose $p \gg 1$ such that $\frac{p}{2}\rho > 1$, allowing us to apply Kolmogorov's continuity criterion (see [2, Theorem 8.2]) and conclude that $:\Psi_N^\ell:\in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ almost surely, for any $T > 0$ and $\varepsilon > 0$.

A similar argument also leads to the following estimate:

$$\mathbb{E}\big[|\delta_h(\langle\nabla\rangle^{-\varepsilon}(:\Psi_N^\ell(\cdot,t): - :\Psi_M^\ell(\cdot,t):))(x)|^2\big] \lesssim_t \frac{|h|^\rho}{N^{2\kappa}}$$

for all $M \geq N \geq 1$, $\kappa > 0$, $\varepsilon > 0$, and $\rho \in [0,1]$ such that $2\varepsilon - 2\kappa - \rho > 0$. By the hypercontractivity (Lemma 2.4), this results in

$$\mathbb{E}\Big[\big\|\delta_h(:\Psi_N^\ell(\cdot,t): - :\Psi_M^\ell(\cdot,t):)\big\|_{W^{-\varepsilon,p}}^p\Big] \lesssim_{p,t} \frac{|h|^{\frac{p}{2}\rho}}{N^{\kappa p}}$$

for any $\rho \in [0,1]$ and $\varepsilon, \kappa > 0$ such that

$$2\varepsilon > \rho + 2\kappa. \tag{2.11}$$

As before, by Sobolev's embedding theorem and Kolmogorov's continuity criterion, we deduce that for any $T > 0$ and $\varepsilon > 0$, there exists large $p \gg 1$ such that $\{:\Psi_N^\ell:\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$. Denoting the corresponding limit by $:\Psi^\ell:$ as in (1.13), we conclude that $:\Psi^\ell:\in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ almost surely. □

Remark 2.6. From the application of Kolmogorov's continuity criterion (see [2, Exercise 8.2]), we see that $:\Psi^\ell:\in C^\alpha([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$, $\alpha < \frac{\rho}{2} - \frac{1}{p}$, almost surely, provided that (2.11) is satisfied. In particular, by taking $p \to \infty$ and $\kappa \to 0$, we see that $\alpha + (-\varepsilon) < 0$, namely, the sum of the temporal and spatial regularities must be negative.

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. In particular, we study the fixed point problem (1.21) by constructing a pathwise contraction in a suitable function space.

3.1. **Strichartz estimates.**

We first recall the Strichartz estimates for the linear wave equation. Given $0 < s < 1$, we say that a pair (q,r) is s-admissible (a pair $(\widetilde{q},\widetilde{r})$ is dual s-admissible,[8] respectively) if $1 \leq \widetilde{q} < 2 < q \leq \infty$, $1 < \widetilde{r} \leq 2 \leq r < \infty$,

$$\frac{1}{q} + \frac{2}{r} = 1 - s = \frac{1}{\widetilde{q}} + \frac{2}{\widetilde{r}} - 2, \qquad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}, \qquad \text{and} \qquad \frac{2}{\widetilde{q}} + \frac{1}{\widetilde{r}} \geq \frac{5}{2}. \tag{3.1}$$

We refer to the first two equalities as the scaling conditions and the last two inequalities as the admissibility conditions.

Let us now state a lemma, providing a more direct description of the admissible exponents.

[8]Here, we define the notion of dual s-admissibility for the convenience of the presentation. Note that $(\widetilde{q},\widetilde{r})$ is dual s-admissible if and only if $(\widetilde{q}',\widetilde{r}')$ is $(1-s)$-admissible.

Lemma 3.1. *Let $0 < s < 1$. A pair (q,r) is s-admissible if*

$$\frac{1}{q} + \frac{2}{r} = 1 - s \qquad \text{and} \qquad 2 \le r \le \begin{cases} \frac{6}{3-4s} & \text{if } s < \frac{3}{4}, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2}$$

A pair $(\widetilde{q}, \widetilde{r})$ is dual s-admissible if

$$\frac{1}{\widetilde{q}} + \frac{2}{\widetilde{r}} = 3 - s \qquad \text{and} \qquad \max\left\{1+, \frac{6}{7-4s}\right\} \le \widetilde{r} \le \frac{2}{2-s}. \tag{3.3}$$

We say that u is a solution to the following nonhomogeneous linear wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u = f, \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1), \end{cases} \tag{3.4}$$

on a time interval containing $t = 0$, if u satisfies the following Duhamel formulation:

$$u = \cos(t|\nabla|)\phi_0 + \frac{\sin(t|\nabla|)}{|\nabla|}\phi_1 + \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} f(\tau) d\tau.$$

We now recall the Strichartz estimates for solutions to the nonhomogeneous linear wave equation (3.4).

Lemma 3.2. *Given $0 < s < 1$, let (q,r) and $(\widetilde{q}, \widetilde{r})$ be s-admissible and dual s-admissible pairs, respectively. Then, a solution u to the nonhomogeneous linear wave equation* (3.4) *satisfies*

$$\|(u, \partial_t u)\|_{L_T^\infty \mathcal{H}^s} + \|u\|_{L_T^q L_x^r} \lesssim \|(\phi_0, \phi)\|_{\mathcal{H}^s} + \|f\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}} \tag{3.5}$$

for all $0 < T \le 1$. The following estimate also holds:

$$\|(u, \partial_t u)\|_{L_T^\infty \mathcal{H}^s} + \|u\|_{L_T^q L_x^r} \lesssim \|(\phi_0, \phi)\|_{\mathcal{H}^s} + \|f\|_{L_T^1 H_x^{s-1}} \tag{3.6}$$

for all $0 < T \le 1$. Here, we used a shorthand notation $L_T^q L_x^r = L^q([0,T]; L^r(\mathbb{T}^2))$, etc.

The Strichartz estimates on $\mathbb{R}^d$ have been studied by many mathematicians. See Ginibre-Velo [11], Lindblad-Sogge [17], and Keel-Tao [15]. The first estimate (3.5) on $\mathbb{T}^2$ in Lemma 3.2 follows from Theorem 2.6 in [30] for $\mathbb{R}^2$ and the finite speed of propagation for the wave equation. The first term on the left-hand side of the second estimate (3.6) is estimated by the energy estimate (2.29) in [30] and the finite speed of propagation for the wave equation, while the second term on the left-hand side of the second estimate (3.6) is estimated by Minkowski's integral inequality and the homogeneous Strichartz estimate in (3.5):

$$\begin{aligned} \left\| \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} f(\tau) d\tau \right\|_{L_T^q L_x^r} &\le \int_0^T \left\| \mathbf{1}_{[0,t]}(\tau) \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} f(\tau) \right\|_{L_t^q([0,T]; L_x^r)} d\tau \\ &\lesssim \int_0^T \|f(\tau)\|_{H^{s-1}} d\tau. \end{aligned}$$

In the remaining part of this subsection, we consider the following deterministic wave equation with $k \ge 2$:

$$\partial_t^2 u - \Delta u \pm |u|^{k-1} u = 0. \tag{3.7}$$

Here, we allow $k \geq 2$ to take noninteger values. In particular, we prove local well-posedness of (3.7) in $\mathcal{H}^s(\mathbb{T}^2) = H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$ with (i) $s \geq s_{\text{crit}}$ if $k > 3$ and (ii) $s > s_{\text{crit}}$ if $2 \leq k \leq 3$, where s_{crit} is the regularity defined in (1.24).

Suppose that we can find an s-admissible pair (q, r) and a dual s-admissible pair $(\widetilde{q}, \widetilde{r})$ so that

$$q \geq k\widetilde{q} \qquad \text{and} \qquad r \geq k\widetilde{r}. \tag{3.8}$$

Then, Hölder's inequality with the fact that $|\mathbb{T}^2| = 1$ yields

$$\big\| |u|^{k-1} u \big\|_{L^{\widetilde{q}}_T L^{\widetilde{r}}_x} \leq T^{\frac{1}{\widetilde{q}} - \frac{k}{q}} \|u\|^k_{L^q_T L^r_x}. \tag{3.9}$$

Then, local well-posedness of (3.7) on a time interval $[0, T]$ for some $T = T(\phi_0, \phi_1) > 0$ follows from the Strichartz estimates (Lemma 3.2), (3.9), and a standard contraction argument. Note that we have a positive power of T in (3.9) when $q > k\widetilde{q}$. In this case, we can take $T = T(\|(\phi_0, \phi_1)\|_{\mathcal{H}^s}) > 0$. Indeed, this is the case when s is greater than the scaling critical regularity s_{scaling}.

Fix $0 < s < 1$. Then, in view of (3.8), we would like to maximize

$$\min\left\{\frac{q}{\widetilde{q}}, \frac{r}{\widetilde{r}}\right\}$$

under the constraints of Lemma 3.1. While this is noninspiring and can be easily done, the result gives important insights. In view of (3.1), this is essentially[9] equivalent to the following maximization problem on $J_s(r, \widetilde{r})$ defined by

$$J_s(r, \widetilde{r}) = \frac{r}{\widetilde{r}} \min\left\{1, \frac{(3-s)\widetilde{r} - 2}{(1-s)r - 2}\right\} \tag{3.10}$$

over the set

$$K(s) = \left[2, \frac{6}{(3-4s)_+}\right] \times \left[\max\left\{1, \frac{6}{7-4s}\right\}, \frac{2}{2-s}\right], \tag{3.11}$$

where $x_+ := \max(x, 0)$ with the understanding that $\frac{6}{0} = \infty$.

Lemma 3.3. *Given $0 < s < 1$, let $J_s(r, \widetilde{r})$ and $K(s)$ be as in* (3.10) *and* (3.11)*. Then, the maximum of $J_s(r, \widetilde{r})$ on $K(s)$ is given by*

$$\sup_{(r,\widetilde{r}) \in K(s)} J_s(r, \widetilde{r}) = \begin{cases} \frac{3-s}{1-s} & \text{if } \frac{1}{2} \leq s < 1, \\ \frac{7-4s}{3-4s} & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2}, \\ \frac{6}{3-4s} & \text{if } 0 < s \leq \frac{1}{4}. \end{cases}$$

Moreover, the supremum is indeed attained in each case: (i) *when $0 < s \leq \frac{1}{4}$, it is attained at $(r, \widetilde{r}) = \left(\frac{6}{3-4s}, 1\right)$,* (ii) *when $\frac{1}{4} \leq s \leq \frac{1}{2}$, it is attained at $(r, \widetilde{r}) = \left(\frac{6}{3-4s}, \frac{6}{7-4s}\right)$, and* (iii) *when $\frac{1}{2} \leq s < 1$, it is attained in the set:*

$$\frac{6}{7-4s} \cdot \frac{3-s}{1-s} \leq r \leq \begin{cases} \frac{6}{3-4s} & \text{if } \frac{1}{2} \leq s \leq 3 - \sqrt{6} \sim 0.55, \\ \frac{2}{2-s} \cdot \frac{3-s}{1-s} & \text{if } 3 - \sqrt{6} \leq s < 1, \end{cases} \quad \text{and} \quad \widetilde{r} = \frac{1-s}{3-s} r. \tag{3.12}$$

[9]Here, we allow $\widetilde{r} = 1$ that is not admissible for the Strichartz estimates.

Proof. From (3.10), we see that the maximum of $J_s(r,\widetilde{r})$ on $K(s)$ is given by $\max\{J_1(s), J_2(s)\}$, where

$$J_1(s) = \max\left\{\frac{r}{\widetilde{r}} : \frac{r}{\widetilde{r}} \le \frac{3-s}{1-s},\ (r,\widetilde{r}) \in K(s)\right\}$$

and

$$J_2(s) = \max\left\{\frac{3-s-\frac{2}{\widetilde{r}}}{1-s-\frac{2}{r}} : \frac{r}{\widetilde{r}} \ge \frac{3-s}{1-s},\ (r,\widetilde{r}) \in K(s)\right\}.$$

From (3.2) and (3.3), we have

$$\sup\left\{\frac{r}{\widetilde{r}} : (r,\widetilde{r}) \in K(s)\right\} = \begin{cases} \frac{6}{3-4s} & \text{if } 0 < s \le \frac{1}{4}, \\ \frac{7-4s}{3-4s} & \text{if } \frac{1}{4} \le s \le \frac{3}{4}, \\ \infty & \text{if } s \ge \frac{3}{4}, \end{cases}$$

and

$$\min\left\{\frac{r}{\widetilde{r}} : (r,\widetilde{r}) \in K(s)\right\} = 2-s.$$

Note that we have $\frac{6}{3-4s} \le \frac{3-s}{1-s}$ for $s \le \frac{1}{4}$ and $\frac{7-4s}{3-4s} \le \frac{3-s}{1-s}$ for $s \le \frac{1}{2}$. Hence, for $0 < s \le \frac{1}{2}$, we have

$$\sup_{(r,\widetilde{r})\in K(s)} J_s(r,\widetilde{r}) = \max\{J_1(s), J_2(s)\} = \sup\left\{\frac{r}{\widetilde{r}} : (r,\widetilde{r}) \in K(s)\right\}.$$

Next, we consider the case $\frac{1}{2} < s < 1$. On the one hand, we have $J_1(s) \le \frac{3-s}{1-s}$. On the other hand, by minimizing r and maximizing $\widetilde{r}$ under $\frac{r}{\widetilde{r}} \ge \frac{3-s}{1-s}$, we obtain

$$\begin{aligned} J_2(s) &= \max\left\{\frac{3-s-\frac{2}{\widetilde{r}}}{1-s-\frac{2}{r}} : \frac{r}{\widetilde{r}} = \frac{3-s}{1-s},\ (r,\widetilde{r}) \in K(s)\right\} \\ &= \frac{3-s}{1-s}. \end{aligned}$$

It is easy to check that this maximum is attained in the set described in (3.12). □

As a result, we can prove local well-posedness of the deterministic NLW (1.23) at the regularities stated in Remark 1.4. Indeed, it suffices to note that Lemma 3.3 guarantees the existence of an s-admissible pair (q,r) and a dual s-admissible pair $(\widetilde{q},\widetilde{r})$ satisfying (3.8), provided that (i) $s \ge s_{\text{crit}}$ if $k > 3$ and (ii) and $s > s_{\text{crit}}$ if $2 \le k \le 3$, where s_{crit} is as in (1.24). Note that when $2 \le k \le 3$, the endpoint $s = s_{\text{crit}}$ is excluded since the maximum in Lemma 3.3 is attained at $\widetilde{r} = 1$, which is not allowed for the dual s-admissibility. Then, the rest of the proof of the local well-posedness follows from the Strichartz estimates (Lemma 3.2), (3.9), and a standard fixed point argument. See also the discussion in Subsection 3.3.

3.2. **Estimating a product.** In this subsection, we state several product estimates for periodic functions on $\mathbb{T}^d$. First, recall the following fractional Leibniz rule for functions on $\mathbb{R}^d$; let $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. Then, we have

$$\big\||\nabla|^s(fg)\big\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)}\big\||\nabla|^s g\big\|_{L^{q_1}(\mathbb{R}^d)} + \big\||\nabla|^s f\big\|_{L^{p_2}(\mathbb{R}^d)}\|g\|_{L^{q_2}(\mathbb{R}^d)}. \tag{3.13}$$

This estimate is an immediate consequence of the Coifman-Meyer theorem; see [7] and the inequality (1.1) in [19]. We use (3.13) to prove the following product estimates for functions on $\mathbb{T}^d$.

Lemma 3.4. *Let* $0 \le s \le 1$.
(i) *Suppose that* $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. *Then, we have*

$$\|\langle\nabla\rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \Big(\|f\|_{L^{p_1}(\mathbb{T}^d)} \|\langle\nabla\rangle^s g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle\nabla\rangle^s f\|_{L^{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)}\Big). \tag{3.14}$$

(ii) *Suppose that* $1 < p, q, r < \infty$ *satisfy the scaling condition:* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}$. *Then, we have*

$$\big\|\langle\nabla\rangle^{-s}(fg)\big\|_{L^r(\mathbb{T}^d)} \lesssim \big\|\langle\nabla\rangle^{-s} f\big\|_{L^p(\mathbb{T}^d)} \big\|\langle\nabla\rangle^s g\big\|_{L^q(\mathbb{T}^d)}. \tag{3.15}$$

Proof. In view of the transference principle [10, Theorem 3], the first estimate (3.14) follows from the Coifman-Meyer theorem for functions on $\mathbb{R}^d$ and (3.13). The second estimate (3.15) follows from duality, the first estimate (3.14), and Sobolev's inequality

$$\begin{aligned}
\big\|\langle\nabla\rangle^{-s}(fg)\big\|_{L^r} &\le \sup_{\|\langle\nabla\rangle^s h\|_{L^{r'}} = 1} \left| \int fgh\, dx \right| \\
&\le \big\|\langle\nabla\rangle^{-s} f\big\|_{L^p} \sup_{\|\langle\nabla\rangle^s h\|_{L^{r'}} = 1} \big\|\langle\nabla\rangle^s (gh)\big\|_{L^{p'}} \\
&\lesssim \big\|\langle\nabla\rangle^{-s} f\big\|_{L^p} \sup_{\|\langle\nabla\rangle^s h\|_{L^{r'}} = 1} \Big(\|g\|_{L^{\widetilde{q}}} \big\|\langle\nabla\rangle^s h\big\|_{L^{r'}} + \big\|\langle\nabla\rangle^s g\big\|_{L^q} \|h\|_{L^{\widetilde{r}'}} \Big) \\
&\lesssim \big\|\langle\nabla\rangle^{-s} f\big\|_{L^p} \big\|\langle\nabla\rangle^s g\big\|_{L^q},
\end{aligned}$$

where the exponents satisfy the Hölder relations

$$\frac{1}{q} + \frac{1}{\widetilde{r}'} = \frac{1}{\widetilde{q}} + \frac{1}{r'} = \frac{1}{p'}, \tag{3.16}$$

and the exponents satisfy the Sobolev relations

$$\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{s}{d} \qquad \text{and} \qquad \frac{1}{\widetilde{r}'} = \frac{1}{r'} - \frac{s}{d}. \tag{3.17}$$

Altogether, (3.16) and (3.17) yield the scaling condition. □

3.3. **Local well-posedness of SNLW.** In this subsection, we present the proof of Theorem 1.1. Given an integer $k \ge 2$ and $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$, define a map Γ by

$$\begin{aligned}
v \mapsto \Gamma(v)(t) &\stackrel{\text{def}}{=} S(t)(\phi_0, \phi_1) \mp \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} F_\Psi(v(\tau)) d\tau \\
&= S(t)(\phi_0, \phi_1) \mp \sum_{\ell=0}^k \binom{k}{\ell} \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} :\Psi^\ell(\tau): v^{k-\ell}(\tau) d\tau.
\end{aligned} \tag{3.18}$$

Let s be as in Theorem 1.1. More precisely we assume that (i) $s \ge s_{\text{crit}}$ if $k \ge 4$, (ii) $s > \frac{1}{4}$ if $k = 3$, and (iii) $s > 0$ if $k = 2$. In the following, we only consider the case $s < 1$.

In view of Lemma 3.3 and (1.19), we can choose an s-admissible pair (q, r) and a dual s-admissible pair $(\widetilde{q}, \widetilde{r})$ such that

$$\min\left\{ \frac{q}{\widetilde{q}}, \frac{r}{\widetilde{r}} \right\} \ge k$$

with a strict inequality if $k = 2$ or 3.

We define $X^s(T)$ as the intersection of the energy space at level s and the Strichartz space

$$X^s(T) = C([0,T]; H^s(\mathbb{T}^2)) \cap C^1([0,T]; H^{s-1}(\mathbb{T}^2)) \cap L^q([0,T]; L^r(\mathbb{T}^2)).$$

Proposition 3.5. *Given an integer $k \geq 1$, let s, (q,r), and $(\widetilde{q}, \widetilde{r})$ be as above. Then, there exist sufficiently small $\varepsilon > 0$ and $\theta > 0$ such that*

$$\begin{aligned} \|\Gamma(v)\|_{X^s(T)} &\lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^s} + \| :\Psi^k: \|_{L^1_T H^{s-1}_x} \\ &\quad + T^\theta \sum_{\ell=1}^{k-1} \|\langle\nabla\rangle^{-\varepsilon} :\Psi^\ell: \|_{L^\infty_{T,x}} \|v\|^{k-\ell}_{X^s(T)} + T^{\frac{1}{q}-\frac{k}{q}} \|v\|^k_{X^s(T)} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \|\Gamma(v_1) - \Gamma(v_2)\|_{X^s(T)} &\lesssim T^\theta \sum_{\ell=1}^{k} \|\langle\nabla\rangle^{-\varepsilon} :\Psi^\ell: \|_{L^\infty_{T,x}} \\ &\qquad \times \Big(\|v_1\|_{X^s(T)} + \|v_2\|_{X^s(T)}\Big)^{k-\ell-1} \|v_1 - v_2\|_{X^s(T)} \\ &\qquad + T^{\frac{1}{q}-\frac{k}{q}} \Big(\|v_1\|_{X^s(T)} + \|v_2\|_{X^s(T)}\Big)^{k-1} \|v_1 - v_2\|_{X^s(T)} \end{aligned} \tag{3.20}$$

for any $T > 0$.

Proof. We only prove the first estimate (3.19) since the second estimate follows in a similar manner. As in Subsection 3.1, we can estimate the term with $\ell = 0$ in (3.18) by Lemma 3.2 and (3.9). On the other hand, we can use (3.6) in Lemma 3.2 to estimate the first term on the right-hand side of (3.19) and the term with $\ell = k$ in (3.18). Hence, it remains to prove

$$\bigg\| \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} :\Psi^\ell: \prod_{j=1}^{k-\ell} v_j d\tau \bigg\|_{X^s(T)} \lesssim T^\theta \|\langle\nabla\rangle^{-\varepsilon} :\Psi^\ell: \|_{L^\infty_{T,x}} \prod_{j=1}^{k-\ell} \|v_j\|_{X^s(T)} \tag{3.21}$$

for $1 \leq \ell \leq k-1$. To simplify the notation, we only consider the case $v_j = v$ in the following. The full estimate can be recovered by polarization or, what may be easier, by checking that the proof applies to a general product.

By interpolation between the Strichartz part of the norm and the energy part of the $X^s(T)$-norm, we have

$$\|\langle\nabla\rangle^\varepsilon v\|_{L^{q_1}_T L^{r_1}_x} \leq \|v\|^{1-\frac{\varepsilon}{s}}_{L^q_T L^r_x} \|v\|^{\frac{\varepsilon}{s}}_{L^\infty_T H^s_x} \leq \|v\|_{X^s(T)} \tag{3.22}$$

for $0 < \varepsilon < s$, where

$$\frac{1}{q_1} = \frac{1-\varepsilon/s}{q} + \frac{\varepsilon/s}{\infty} \qquad \text{and} \qquad \frac{1}{r_1} = \frac{1-\varepsilon/s}{r} + \frac{\varepsilon/s}{2}.$$

Similarly, by duality with $(L^1_T H^{s-1}_x + L^{\widetilde{q}}_T L^{\widetilde{r}}_x)^* = L^\infty_T H^{1-s}_x \cap L^{\widetilde{q}'}_T L^{\widetilde{r}'}_x$ and interpolation, we have

$$\begin{aligned} \|f\|_{L^1_T H^{s-1}_x + L^{\widetilde{q}}_T L^{\widetilde{r}}_x} &= \inf_{f=f_1+f_2} \Big(\|f_1\|_{L^1_T H^{s-1}_x} + \|f_2\|_{L^{\widetilde{q}}_T L^{\widetilde{r}}_x} \Big) \\ &= \sup_{\|g\|_{L^\infty_T H^{1-s}_x \cap L^{\widetilde{q}'}_T L^{\widetilde{r}'}_x} \le 1} \bigg| \int_0^T \int_{\mathbb{T}^2} f g dx dt \bigg| \\ &\le \sup_{\|g\|_{L^\infty_T H^{1-s}_x \cap L^{\widetilde{q}'}_T L^{\widetilde{r}'}_x} \le 1} \|\langle \nabla \rangle^\varepsilon g\|_{L^{\widetilde{q}_2'}_T L^{\widetilde{r}_2'}_x} \|\langle \nabla \rangle^{-\varepsilon} f\|_{L^{\widetilde{q}_2}_T L^{\widetilde{r}_2}_x} \\ &\lesssim \|\langle \nabla \rangle^{-\varepsilon} f\|_{L^{\widetilde{q}_2}_T L^{\widetilde{r}_2}_x} \end{aligned} \tag{3.23}$$

for $0 < \varepsilon < 1 - s$, where

$$\frac{1}{\widetilde{q}_2} = \frac{\varepsilon/(1-s)}{1} + \frac{1-\varepsilon/(1-s)}{\widetilde{q}} \qquad \text{and} \qquad \frac{1}{\widetilde{r}_2} = \frac{\varepsilon/(1-s)}{2} + \frac{1-\varepsilon/(1-s)}{\widetilde{r}}.$$

We also claim that the following estimate holds:

$$\big\| \langle \nabla \rangle^{-\varepsilon} :\Psi^\ell\colon v^{k-\ell} \big\|_{L^{\widetilde{q}_2}_T L^{\widetilde{r}_2}_x} \lesssim T^\theta \| \langle \nabla \rangle^{-\varepsilon} :\Psi^\ell\colon \|_{L^\infty_{T,x}} \| \langle \nabla \rangle^\varepsilon v\|^{k-\ell}_{L^{q_1}_T L^{r_1}} \tag{3.24}$$

for sufficiently small $\varepsilon > 0$ and $\theta > 0$. Fix $t \in [0, T]$. By applying Lemma 3.4 (ii), we have

$$\begin{aligned} \big\| \langle \nabla \rangle^{-\varepsilon} :\Psi^\ell(t)\colon v^{k-\ell}(t) \big\|_{L^{\widetilde{r}_2}_x} &\lesssim \| \langle \nabla \rangle^{-\varepsilon} :\Psi^\ell(t)\colon \|_{L^{\frac{2}{\varepsilon}}_x} \| \langle \nabla \rangle^\varepsilon v^{k-\ell}(t)\|_{L^{\widetilde{r}_2}_x} \\ &\le \| \langle \nabla \rangle^{-\varepsilon} :\Psi^\ell(t)\colon \|_{L^\infty_x} \| \langle \nabla \rangle^\varepsilon v^{k-\ell}(t)\|_{L^{\widetilde{r}_2}_x}. \end{aligned} \tag{3.25}$$

Then, by applying Lemma 3.4 (i), we have

$$\| \langle \nabla \rangle^\varepsilon v^{k-\ell}(t)\|_{L^{\widetilde{r}_2}_x} \lesssim \|v(t)\|^{k-\ell-1}_{L^{(k-\ell)\widetilde{r}_2}_x} \| \langle \nabla \rangle^\varepsilon v(t)\|_{L^{(k-\ell)\widetilde{r}_2}_x} \lesssim \| \langle \nabla \rangle^\varepsilon v(t)\|^{k-\ell}_{L^{(k-\ell)\widetilde{r}_2}_x}. \tag{3.26}$$

Note that we can choose $\varepsilon > 0$ sufficiently small such that

$$(k-1)\widetilde{q}_2 < q_1 \qquad \text{and} \qquad (k-1)\widetilde{r}_2 \le r_1. \tag{3.27}$$

This can be achieved in view of (3.8) and

$$q_1 \to q, \quad r_1 \to r, \quad \widetilde{q}_2 \to \widetilde{q}, \quad \text{and} \quad \widetilde{r}_2 \to \widetilde{r}$$

as $\varepsilon \to 0$. Hence, (3.24) follows from (3.25), (3.26), and (3.27). Note that the strict inequality in (3.27) is used to gain a factor T^θ.

Putting Lemma 3.2, (3.22), (3.23), and (3.24) together, we obtain the desired estimate (3.21). □

Proposition 3.5 with a standard fixed point argument immediately yields Theorem 1.1 in the subcritical case, i.e., $s > s_{\text{crit}}$. In this case, we have $q > k\widetilde{q}$, which provides a positive power of T on the last terms of (3.19) and (3.20). In particular, this implies that almost sure local well-posedness holds on $[-T, T]$, where $T = T_\omega(\|\phi_0, \phi_1)\|_{\mathcal{H}^s}) > 0$. Note that the mild formulation (3.18) with the continuity of the linear propagator $S(t)$, (the proof of) Proposition 3.5, and Proposition 2.1 shows that the solution v lies in $C([0, T]; H^s(\mathbb{T}^2))$.

On the other hand, in the critical case: $s = s_{\text{crit}}$ (with $k \ge 4$), we have $q = k\widetilde{q}$. Thus, the last terms of (3.19) and (3.20) do not provide any power of T. In this case, a direct application of Proposition 3.5 would yield only small data local well-posedness and thus we need to slightly modify the argument .

Let $\varepsilon > 0$ be as in Proposition 3.5. Then, in view of (3.9) and (3.22), we set $Y^s(T)$ by

$$\|v\|_{Y^s(T)} = \max\big(\|v\|_{L^q_T L^r_x}^{1-\frac{\varepsilon}{s}} \|v\|_{L^\infty_T H^s_x}^{\frac{\varepsilon}{s}}, \, \|v\|_{L^q_T L^r_x}\big).$$

Then, it follows from the proof of Proposition 3.5 that

$$\begin{aligned} \|\Gamma(v)\|_{Y^s(T)} &\lesssim \|S(t)(\phi_0, \phi_1)\|_{Y^s(T)} + \| :\Psi^k: \|_{L^1_T H^{s-1}_x} \\ &\quad + T^\theta \sum_{\ell=1}^{k-1} \|\langle\nabla\rangle^{-\varepsilon} :\Psi^\ell: \|_{L^\infty_{T,x}} \|v\|^{k-\ell}_{Y^s(T)} + T^{\frac{1}{q}-\frac{k}{q}} \|v\|^k_{Y^s(T)}. \end{aligned} \tag{3.28}$$

The difference estimate (3.20) with $X^s(T)$ replaced by $Y^s(T)$ also holds.

By the monotone convergence theorem, we have $\|v\|_{Y^s(T)} \to 0$ as $T \to 0$. Hence, together with Proposition 2.1, we can choose $T = T_\omega(\phi_0, \phi_1) > 0$ sufficiently small such that $\|S(t)(\phi_0, \phi_1)\|_{L^q_T W^{\varepsilon, r_1}_x} + \| :\Psi^k: \|_{L^1_T H^{\sigma-1}_x} \le \eta \ll 1$ almost surely, allowing us to show that Γ is a contraction on the ball of radius η in $Y^s(T)$. Lastly, noting that (3.28) holds even if we replace the $Y^s(T)$-norm on the left-hand side by the $X^s(T)$-norm, we conclude that $v \in X^s(T)$.

4. Weak universality for semilinear wave equations with random perturbation

In this section, we present an application of the local well-posedness argument presented in Section 3. In particular, we establish weak universality of the Wick ordered SNLW in the following sense. Given small $\varepsilon > 0$, we consider the following SNLW on $(\varepsilon^{-1}\mathbb{T})^2$ with a smooth noise η^ε:

$$\begin{cases} \partial_t^2 w_\varepsilon - \Delta w_\varepsilon = f(w_\varepsilon) + a(\varepsilon, t) w_\varepsilon + \delta(\varepsilon)\eta^\varepsilon, \\ (w_\varepsilon, \partial_t w_\varepsilon)|_{t=0} = (0, 0), \end{cases} \qquad (x,t) \in (\varepsilon^{-1}\mathbb{T})^2 \times \mathbb{R}_+, \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth odd function which we take bounded with a sufficient number of bounded derivatives[10] and $a(\varepsilon, t)$ and $\delta(\varepsilon)$ are parameters we will fix below. In the following, we take the noise η^ε to be white in time but smooth and stationary in space.[11] We point out that we could also work with a Gaussian noise η^ε which is regular both in space and time but, in order to fit more easily in the general framework of this paper, we prefer to stick to a noise which is white in time. Similarly, we could work with a function f with polynomial growth. For simplicity of the presentation, however, we work under the boudedness assumption on f. Indeed, we will see that the precise form of f does not matter in the limit. See also Remark 4.2 below.

Our aim is to describe the long time and large space behavior of the solution w_ε to (4.1). In order to do so, we perform a change of variables $u_\varepsilon(x,t) \overset{\text{def}}{=}$

[10]For example, in proving Theorem 4.1, it suffices to assume that $f(0) = f''(0) = 0$ and that there is a control up to the fourth derivative of f. See also Remark 4.2.

[11]Think of $\eta^\varepsilon = \psi *_x \xi$ for some smooth function ψ on $(\varepsilon^{-1}\mathbb{T})^2$. Then, with $\widetilde{\beta}_n$ as in (1.7), η^ε can be formally written as

$$\eta^\varepsilon(x,t) = \varepsilon \sum_{n \in (\varepsilon\mathbb{Z})^2} \widehat{\psi}(n) d\widetilde{\beta}_n(t) e^{2\pi i n \cdot x}. \tag{4.2}$$

$\varepsilon^{-\gamma} w_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-1}t)$ and observe that u_ε satisfies

$$\begin{cases} \partial_t^2 u_\varepsilon - \Delta u_\varepsilon = \varepsilon^{-\gamma-2}\big\{f(\varepsilon^\gamma u_\varepsilon) + \varepsilon^\gamma a(\varepsilon, \varepsilon^{-1}t) u_\varepsilon\big\} + \varepsilon^{-\gamma-\frac{1}{2}}\delta(\varepsilon)\eta_\varepsilon, \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (0,0), \end{cases} \tag{4.3}$$

where $\eta_\varepsilon(x,t) = \varepsilon^{-\frac{3}{2}}\eta^\varepsilon(\varepsilon^{-1}x, \varepsilon^{-1}t)$. The normalization for η_ε has been chosen in such a way that it converges as $\varepsilon \to 0$ to a space-time white noise ξ in law. With this normalization, we choose $\delta(\varepsilon) = \varepsilon^{\gamma+\frac{1}{2}}$ in order for the coefficient in front of η_ε to be $O_\varepsilon(1)$ as $\varepsilon \to 0$. For the sake of a simpler statement below, we apply Skorokhod's theorem[12] and introduce a new noise with the same law, still denoted by η_ε, such that it converges to the white noise ξ almost surely. We also use u_ε to denote the solution to (4.3). Then, letting Ψ_ε denote the stochastic convolution Ψ_ε given by $\Psi_\varepsilon = (\partial_t^2 - \Delta)^{-1}\eta_\varepsilon$, it follows from an argument analogous to the proof of Proposition 2.1 that Ψ_ε converges almost surely to the stochastic convolution Ψ defined in (1.4) in $C(\mathbb{R}_+; W^{\sigma,\infty}(\mathbb{T}^2))$ for any $\sigma < 0$, where we endow the space with the compact-open topology in time.

We now state the main result of this section.

Theorem 4.1. *Let $\delta(\varepsilon) = \varepsilon^{\gamma+\frac{1}{2}}$ and $\gamma = 1$. Then, there exists a choice of $a(\varepsilon,t)$ such that, as $\varepsilon \to 0$, the family of the solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to (4.3) converges almost surely to the solution u to the following Wick ordered cubic SNLW:*

$$\partial_t^2 u - \Delta u = \lambda :u^3: + \xi \tag{4.4}$$

with zero initial data, where the convergence takes place in $C([0,T_\omega]; H^\sigma(\mathbb{T}^2))$, $\sigma < 0$, for some $T = T_\omega(\Psi) > 0$. Here the constant $\lambda = \lambda(f)$ depends only on the function f.

Proof. In order to motivate the choice of γ, a, and the constant λ, let us decompose $u_\varepsilon = \Psi_\varepsilon + v_\varepsilon$ as in (1.20). Then, with our choice of $\delta(\varepsilon) = \varepsilon^{\gamma+\frac{1}{2}}$, we see that v_ε satisfies

$$\partial_t^2 v_\varepsilon - \Delta v_\varepsilon = F_\varepsilon(v_\varepsilon), \tag{4.5}$$

where $F_\varepsilon(v_\varepsilon)$ is given by

$$F_\varepsilon(v_\varepsilon) \stackrel{\text{def}}{=} \varepsilon^{-\gamma-2}\big\{f(\varepsilon^\gamma(\Psi_\varepsilon + v_\varepsilon)) + \varepsilon^\gamma a(\varepsilon, \varepsilon^{-1}t)(\Psi_\varepsilon + v_\varepsilon)\big\}.$$

Since f is chosen to be odd, we have $f(0) = f''(0) = 0$. Then, Taylor's remainder theorem gives

$$F_\varepsilon(v_\varepsilon) = \varepsilon^{-2}\big\{f'(0) + a(\varepsilon, \varepsilon^{-1}t)\big\}(\Psi_\varepsilon + v_\varepsilon) + \varepsilon^{2\gamma-2}\frac{f^{(3)}(0)}{6}(\Psi_\varepsilon + v_\varepsilon)^3 + R_\varepsilon \tag{4.6}$$

with

$$R_\varepsilon = \varepsilon^{2\gamma-2}\int_0^1 \frac{(1-\tau)^2}{2}\big\{f^{(3)}(\tau\varepsilon^\gamma(\Psi_\varepsilon + v_\varepsilon)) - f^{(3)}(0)\big\}d\tau \cdot (\Psi_\varepsilon + v_\varepsilon)^3. \tag{4.7}$$

From the explicit expression (2.2) for the Hermite polynomials, we have

$$(\Psi_\varepsilon + v_\varepsilon)^3 = H_3(\Psi_\varepsilon + v_\varepsilon; \sigma_\varepsilon) + 3\sigma_\varepsilon(\Psi_\varepsilon + v_\varepsilon),$$

[12]If we do not apply Skorokhod's theorem here, then the conclusion of Theorem 4.1 holds only along some sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to 0.

where $\sigma_\varepsilon = \sigma_\varepsilon(t) = \mathbb{E}[\Psi_\varepsilon(x,t)^2] \sim |\log \varepsilon|$.[13] Hence, from (4.6) and (4.7), we deduce that

$$\begin{aligned} F_\varepsilon(v_\varepsilon) = \varepsilon^{-2}\bigg\{ f'(0) + a(\varepsilon, \varepsilon^{-1}t) + 3\varepsilon^{2\gamma}\sigma_\varepsilon \frac{f^{(3)}(0)}{6}\bigg\}(\Psi_\varepsilon + v_\varepsilon) \\ + \varepsilon^{2\gamma-2}\frac{f^{(3)}(0)}{6} H_3(\Psi_\varepsilon + v_\varepsilon; \sigma_\varepsilon) + R_\varepsilon. \end{aligned}$$

Therefore, in order for $F_\varepsilon(v_\varepsilon)$ to have a (nontrivial) finite limit (as a space-time distribution), we must take

$$\gamma = 1 \qquad \text{and} \qquad a(\varepsilon, \varepsilon^{-1}t) = -f'(0) - \varepsilon^2\sigma_\varepsilon(t)\frac{f^{(3)}(0)}{2}.$$

With these choices and letting $\lambda = \frac{f^{(3)}(0)}{6}$, we have

$$F_\varepsilon(v_\varepsilon) = \lambda H_3(\Psi_\varepsilon + v_\varepsilon; \sigma_\varepsilon) + R_\varepsilon = \lambda :u_\varepsilon^3: + R_\varepsilon.$$

It remains to show that $R_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us analyze the behavior of R_ε. Letting

$$\Lambda_\varepsilon = \int_0^1 \frac{(1-\tau)^2}{2}\big\{ f^{(3)}(\tau\varepsilon^\gamma(\Psi_\varepsilon + v_\varepsilon)) - f^{(3)}(0)\big\} d\tau,$$

we have

$$R_\varepsilon = \Lambda_\varepsilon(\Psi_\varepsilon^3 + 3\Psi_\varepsilon^2 v_\varepsilon + 3\Psi_\varepsilon v_\varepsilon^2 + v_\varepsilon^3). \tag{4.9}$$

Moreover, by the fundamental theorem of calculus, we have

$$\Lambda_\varepsilon = \varepsilon^\gamma \int_0^1 \frac{(1-\tau)^2}{2} \int_0^\tau f^{(4)}(\alpha\varepsilon^\gamma(\Psi_\varepsilon + v_\varepsilon)) d\alpha d\tau \cdot (\Psi_\varepsilon + v_\varepsilon).$$

Thus, using the boundedness of the derivatives of f, we have

$$|\Lambda_\varepsilon(x,t)| \lesssim \varepsilon^\gamma \big\{ |\Psi_\varepsilon(x,t)| + |v_\varepsilon(x,t)| \big\}. \tag{4.10}$$

By Proposition 2.1 and (4.8) in the footnote 13, it is not difficult to see that

$$\varepsilon^\gamma \|\Psi_\varepsilon\|_{L_t^\infty([0,1]; L_x^\infty)} = o_\varepsilon(1) \tag{4.11}$$

almost surely. Hence, from (4.9), (4.10), and (4.11), we conclude that

$$|R_\varepsilon(x,t)| \le o_\varepsilon(1)\big(1 + |v_\varepsilon(x,t)|\big)^4.$$

In particular, we can write (4.5) as

$$\begin{aligned} \partial_t^2 v_\varepsilon - \Delta v_\varepsilon &= \lambda :u_\varepsilon^3: + o_\varepsilon\big(\langle v_\varepsilon\rangle^4\big) \\ &= \lambda \sum_{\ell=0}^3 \binom{3}{\ell} :\Psi_\varepsilon^\ell: v_\varepsilon^{3-\ell} + o_\varepsilon\big(\langle v_\varepsilon\rangle^4\big). \end{aligned}$$

Then, by proceeding as in Section 3 with a variant of Proposition 3.5 (with $k = 4$ in view of the fourth order error term), we obtain an a priori bound on v_ε, uniformly

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[13] For simplicity, let $\widehat{\psi}(n) = \mathbf{1}_{|n|\le 1}$ in (4.2). Then, we have

$$\begin{aligned} \eta_\varepsilon(x,t) &= \varepsilon^{-\frac{3}{2}}\eta^\varepsilon(\varepsilon^{-1}x, \varepsilon^{-1}t) \\ &= \sum_{n\in\mathbb{Z}^2} \mathbf{1}_{|n|\le\varepsilon^{-1}}\big(\varepsilon^{-\frac{1}{2}} d\widetilde{\beta}_n(\varepsilon^{-1}t)\big)e^{2\pi i n\cdot x} \stackrel{d}{=} \sum_{n\in\mathbb{Z}^2} \mathbf{1}_{|n|\le\varepsilon^{-1}} d\widetilde{\beta}_n(t) e^{2\pi i n\cdot x}, \end{aligned} \tag{4.8}$$

where we use the white noise scaling in the last equality. In view of (1.8) with (4.8), it is easy to see the logarithmic divergence of σ_ε in this case.

in $\varepsilon > 0$. Moreover, the local existence time $T = T_\omega$ depends only on Ψ and is independent of $\varepsilon > 0$.

Let u be the solution to (4.4). In an analogous manner, we can estimate the difference $v - v_\varepsilon$, where $v = u - \Psi$ as in (1.20). Together with the almost sure convergence of Ψ_ε to Ψ, we see that u_ε converges to u in $C([0, T_\omega]; H^\sigma(\mathbb{T}^2))$ for $\sigma < 0$. $\square$

Remark 4.2. If f is an odd polynomial of degree M, then we obtain the following bound on Λ_ε:

$$|\Lambda_\varepsilon(x,t)| \lesssim \max\Big(\varepsilon^\gamma\big\{|\Psi_\varepsilon(x,t)| + |v_\varepsilon(x,t)|\big\}, (\varepsilon^\gamma\big\{|\Psi_\varepsilon(x,t)| + |v_\varepsilon(x,t)|\big\})^{(M-3)_+}\Big).$$

Together with (4.9) and (4.11), we obtain

$$|R_\varepsilon(x,t)| \le o_\varepsilon(1)\big(1 + |v_\varepsilon(x,t)|\big)^{\max(4,M)}.$$

Then, by applying a variant of Proposition 3.5 (with $k = \max(4, M)$ in view of the error term), we obtain a uniform (in ε) a priori bound on v_ε and the convergence of u_ε to the solution u to (4.4) as above.

Remark 4.3. We can also consider the following SNLW on $(\varepsilon^{-1}\mathbb{T})^2$:

$$\partial_t^2 w_\varepsilon - \Delta w_\varepsilon = f(w_\varepsilon) + a(\varepsilon,t)w_\varepsilon + b(\varepsilon,t)w_\varepsilon^3 + \delta(\varepsilon)\eta^\varepsilon$$

with two parameters a, b which can be "tuned" so that, via a similar procedure, we can cancel the cubic term in the asymptotics of the nonlinear term and obtain the quintic SNLW:

$$\partial_t^2 u - \Delta u = \lambda :u^5: + \xi \tag{4.12}$$

for some $\lambda = \lambda(f)$. In this case, by choosing $\gamma = \frac{1}{2}$, the remainder takes the form

$$\widetilde{R}_\varepsilon = \widetilde{\Lambda}_\varepsilon(\Psi_\varepsilon^5 + 5\Psi_\varepsilon^4 v_\varepsilon + 10\Psi_\varepsilon^3 v_\varepsilon^2 + 10\Psi_\varepsilon^2 v_\varepsilon^3 + 5\Psi_\varepsilon v_\varepsilon^4 + v_\varepsilon^5)$$

with

$$\widetilde{\Lambda}_\varepsilon = \int_0^1 \frac{(1-\tau)^4}{4!}\big\{f^{(5)}(\tau\varepsilon^\gamma(\Psi_\varepsilon + v_\varepsilon)) - f^{(5)}(0)\big\}d\tau$$

which yields the analogous estimate

$$|\widetilde{\Lambda}_\varepsilon(x,t)| \lesssim \varepsilon^\gamma\big\{|\Psi_\varepsilon(x,t)| + |v_\varepsilon(x,t)|\big\}.$$

This implies

$$|\widetilde{R}_\varepsilon(x,t)| \le o_\varepsilon(1)\big(1 + |v_\varepsilon(x,t)|\big)^6.$$

Then, by applying a variant of Proposition 3.5 (with $k = 6$ in view of the sixth order error term), we obtain a uniform (in ε) a priori bound on v_ε and the convergence of u_ε to the solution u to (4.12). One can similarly consider SNLW with more parameters to be tuned to obtained the septic Wick ordered SNLW, etc.

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