

UNIVERSALITY OF THE NODAL LENGTH OF BIVARIATE RANDOM TRIGONOMETRIC POLYNOMIALS

JÜRGEN ANGST, VIET-HUNG PHAM, AND GUILLAUME POLY

ABSTRACT. We consider random trigonometric polynomials of the form

$$f_n(x, y) = \sum_{1 \leq k, l \leq n} a_{k,l} \cos(kx) \cos(l y),$$

where the entries $(a_{k,l})_{k,l \geq 1}$ are i.i.d. random variables that are centered with unit variance. We investigate the length $\ell_K(f_n)$ of the nodal set $Z_K(f_n)$ of the zeros of f_n that belong to a compact set $K \subset \mathbb{R}^2$. We first establish a local universality result, namely we prove that, as n goes to infinity, the sequence of random variables $n \ell_{K/n}(f_n)$ converges in distribution to a universal limit which does not depend on the particular law of the entries. We then show that at a macroscopic scale, the expectation of $\ell_{[0,\pi]^2}(f_n)/n$ also converges to an universal limit. Our approach provides two main byproducts: (i) a general result regarding the continuity of the volume of the nodal sets with respect to C^1 -convergence which refines previous findings of Rusakov and Selezniev, Iksanov, Kabluchko, and Marynuch, and Azaís, Dalmao, León, Nourdin, and Poly, and (ii) a new strategy for proving small ball estimates in random trigonometric models, providing in turn uniform local controls of the nodal volumes.

CONTENTS

1. Introduction	8331
2. Local universality	8335
3. Global universality	8342
Acknowledgment	8356
References	8356

1. INTRODUCTION

The study of nodal sets associated to various kinds of random functions is a central topic of probability theory, at the crossroad of various domains of mathematics and physics such as linear algebra, number theory, geometric measure theory, or else quantum mechanics or nuclear physics, just to name a few. In this context, universality results refer to asymptotic properties of these random nodal domains, which hold regardless of the nature of the randomness involved. Establishing such universal properties for generic zero sets allows one to manage what would otherwise be inextricable objects, which explains the tremendous importance of this

Received by the editors October 28, 2016, and, in revised form, January 30, 2017, and March 10, 2017.

2010 *Mathematics Subject Classification*. Primary 26C10; Secondary 30C15, 42A05, 60F17, 60G55.

particular area of research. As such, the literature on this topic is huge, and we refer to the introduction of [TV14] and the references therein for a general overview.

When the random functions under consideration are multivariate, the zeros are no longer isolated points but instead random curves/surfaces/manifolds whose volume is, among others, a natural quantity of interest. Ranging from algebraic manifolds to nodal lines of random eigenfunctions of Laplace–Beltrami operators on tori or spheres, this topic has very recently attracted a lot of attention. Nonexhaustively, we refer for instance to [SZ99, GW16, Let16a, Let16b] regarding random algebraic manifolds and to [RW08, ORW08, Wig10, NS10, FLL15, MPRW16] regarding random eigenfunctions. Nevertheless, in each situation considered in the above references, the underlying randomness emerges from Gaussian distribution and there actually seem to be no results dealing with the dependency of the studied phenomena on the particular nature of the randomness. One reason possibly explaining the lack of results of universality in multivariate frameworks is that most techniques successfully used in univariate settings, such as complex analysis tools or counting the changes of sign, seem hardly extendable to higher dimensions. For instance, to the best of our knowledge, there is no simple analogue in \mathbb{C}^2 of the Jensen formula which plays a central role in universality questions for univariate algebraic polynomials; see [TV14]. To the contrary, we point out the fact that whatever the dimension is, a Kac–Rice formula still holds and allows one to manage remarkably well the case of absolutely continuous random fields. In this article, we investigate the natural question of asymptotic universality of volumes in the framework of bivariate random trigonometric polynomials with random coefficients that are only assumed to be i.i.d and standardized. Let us describe our model in detail.

Let $(a_{k,l})_{k,l \geq 1}$ be a sequence of independent and identically distributed random variables whose common law satisfies $\mathbb{E}(a_{k,l}) = 0$ and $\mathbb{E}(a_{k,l}^2) = 1$. We consider the random function $f_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ and its renormalized analogue F_n defined as

$$(1.1) \quad f_n(x, y) = \sum_{1 \leq k, l \leq n} a_{k,l} \cos(kx) \cos(ly), \quad (x, y) \in \mathbb{R}^2,$$

$$(1.2) \quad F_n(x, y) := \frac{1}{n} f_n\left(\frac{x}{n}, \frac{y}{n}\right) = \frac{1}{n} \sum_{1 \leq k, \ell \leq n} a_{k,\ell} \cos\left(\frac{kx}{n}\right) \cos\left(\frac{\ell y}{n}\right).$$

We denote by $Z_K(f)$ the zeros set of a function f in a compact set $K \subset \mathbb{R}^2$ and by $\ell_K(f)$ the length or 1-dimensional Hausdorff measure of $Z_K(f)$ (provided that f is sufficiently nice and nondegenerate to ensure its existence):

$$\ell_K(f) := |Z_K(f)|, \quad \text{where } Z_K(f) := \{(x, y) \in K \subset \mathbb{R}^2, f(x, y) = 0\}.$$

Our first main result is the following local universality result which states that, at a microscopic scale, the length of the nodal set converges in distribution to a universal limit.

Theorem 1 (Local universality, Theorem 4 below). *For any fixed compact $K \subset \mathbb{R}^2$, the sequence of random variables $(\ell_K(F_n))_{n \geq 1}$ converges in distribution, as n tends to infinity, to an explicit random variable whose law is independent of the particular law of the entries $(a_{k,l})_{k,l \geq 1}$.*

In comparison with the recent work [IKM16] which uses rather complex analysis and the Hurwitz Theorem, we actually show that the sole C^1 -convergence is enough

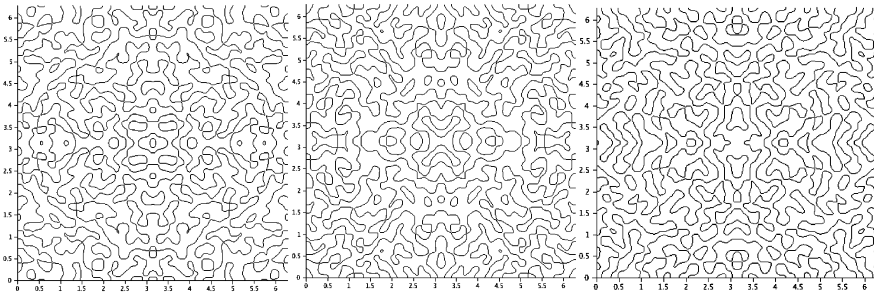


FIGURE 1. A realization of the random nodal set $Z_K(f_n)$ for $K = [0, 2\pi]^2$, $n = 20$, with, from left to right, Bernoulli, Gaussian, and centered exponential entries.

to ensure local universality. Besides, even if stated here in dimension two, our result holds in any finite dimension. Nevertheless, in [IKM16], a much wider class of distributions is considered, englobing domains of attraction of stable distributions. The article [ADL15] provides local universality for some families of absolutely continuous distributions which is an unnecessary assumption but actually entails the stronger result that all moments converge towards the corresponding moments of the (moment determined) target.

From the above local universality result and provided explicit moment controls, we can then deduce the following global universality result which states that, properly normalized, the expectation of the length of the full nodal set in the square $[0, \pi]^2$ converges to a universal constant.

Theorem 2 (Global universality, Theorem 8 below). *Whatever the law of the entries $(a_{k,l})_{k,l \geq 1}$, as n tends to infinity, we have*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} = \frac{\pi^2}{2\sqrt{3}}.$$

Remark 1. Due to the symmetry and periodicity of the trigonometric polynomials f_n , we then have $\lim_{n \rightarrow +\infty} n^{-1} \mathbb{E}[\ell_{[0,2\pi]^2}(f_n)] = 2\pi^2/\sqrt{3}$, and our proof actually establishes that for any compact set K being a finite union of rectangles:

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\ell_K(f_n)]}{n} = \frac{\text{Vol}(K)}{2\sqrt{3}}.$$

With a standard approximation procedure, one can then deduce that the latter convergence holds for any compact set K with nonempty interior and smooth boundary.

Remark 2. By choosing the trigonometric polynomials f_n of the form given by equation (1.1), we deliberately choose to work in a nonstationary framework. Let us stress here that our methods and results naturally extend to stationary cases, for instance when the trigonometric polynomials are of the form

$$\sum_{1 \leq k,l \leq n} a_{k,l} \cos(kx + ly) + b_{k,l} \sin(kx + ly),$$

where $a_{k,l}$ and $b_{k,l}$ are independent i.i.d. sequences and where the computations are actually simpler than the ones considered here.

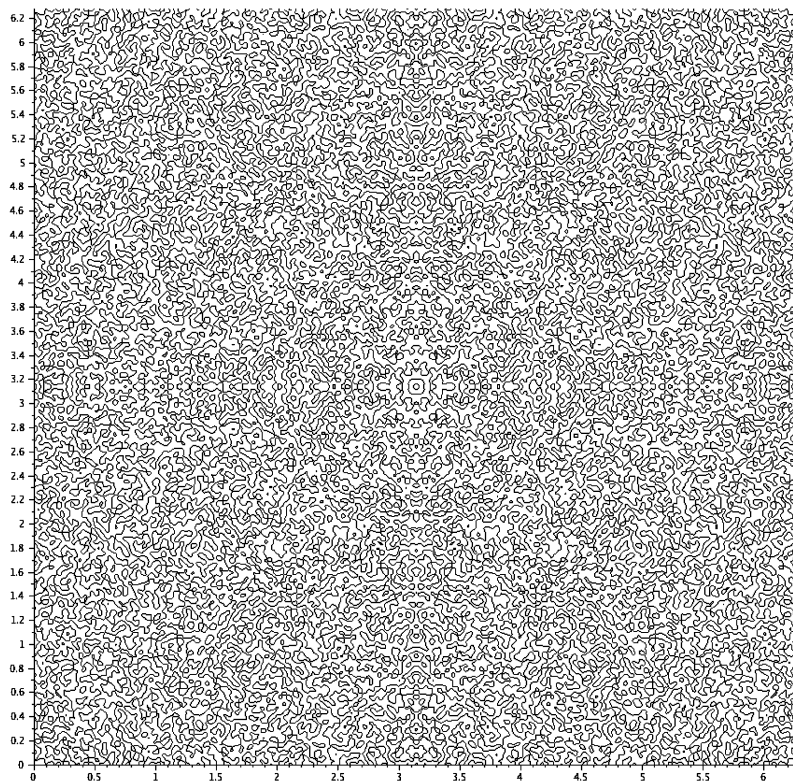


FIGURE 2. A realization of the nodal set $Z_{[0,2\pi]^2}(f_n)$ for a trigonometric polynomial of degree $n = 100$ and with symmetric Bernoulli coefficients.

Before giving the plan of the paper, let us say a few words concerning the universality of the mean number of real roots of univariate random trigonometric polynomials. It has been recently established in full generality in [Fla16], and in [AP15] under more restrictive conditions on the coefficients but with some possible control of the remainder in terms of Edgeworth expansions. The strategy of the proof in [Fla16] artfully combines a careful investigation of the number of changes of signs together with accurate small ball estimates obtained by adapting to this framework the method of Ibragimov and Maslova [IM71]. Nevertheless, such a strategy faces intricate obstructions in higher dimensions, first of all, investigating the number of changes of sign is no longer suitable. Secondly, relying on the celebrated Crofton formula, one might try to get back to the univariate case by studying only the zeros of our bivariate polynomials when restricted to random lines. However, such projections are no longer polynomials when the lines have an irrational slope. In order to avoid such heavy complications, here we follow a completely different path which consists of first establishing the local universality and next extending it to global universality via accurate controls of moments of local nodal lengths. These controls rely on suitable small ball estimates which do not follow the Ibragimov–Maslova

method, which seemed to us hard to adapt here, but instead exploit the particular ergodic properties of sequences of type $\{kx\}_{k \geq 1} \bmod(\pi)$.

The plan of the paper is the following. The next section, Section 2, is devoted to the proof of Theorem 1 concerning local universality. Its first subsection, Subsection 2.1, is dedicated to the C^1 -convergence of the rescaled trigonometric polynomials F_n towards a nondegenerate Gaussian field, whereas Subsection 2.2 deals with the (deterministic) continuity of the volumes of nodal domains with respect to C^1 -convergence on compact sets. The last two results are combined in Subsection 2.3 to deduce the announced microscopic universality. The proof of Theorem 2 on global universality is then given in Section 3. More precisely, Subsection 3.1 deals with the Gaussian case, where an exact computation of the nodal length can be performed thanks to the celebrated Kac–Rice formula. Then, in Subsection 3.2, we derive a small ball estimate, from which we deduce a uniform moment control of the local lengths. Together with the local universality, this moment control allow us to conclude in Subsection 3.3. For the sake of clarity, we give below a concise view of our proof strategy.

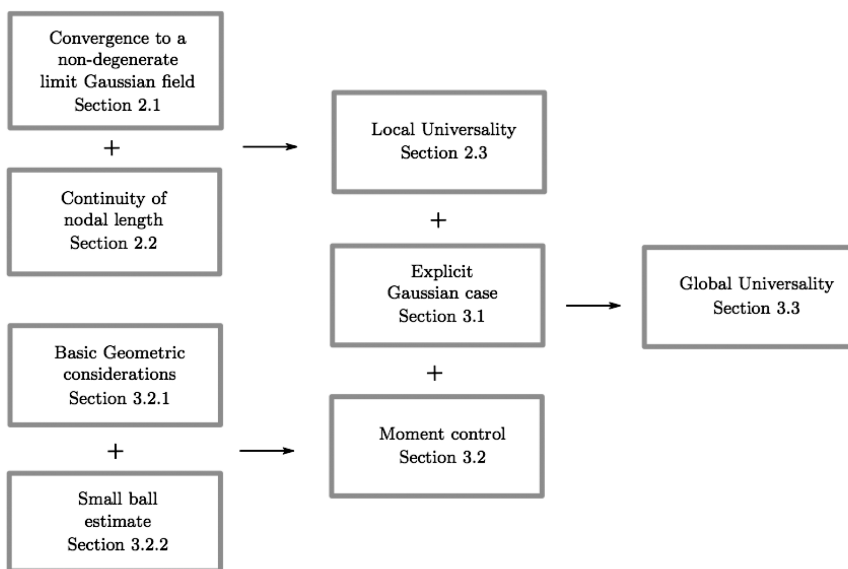


FIGURE 3. Plan of the proof of Local/Global Universality

2. LOCAL UNIVERSALITY

In this section, we give a detailed proof of Theorem 1 on the local universality of the nodal length, i.e., we show that, at the microscopic scale, the law of the nodal length of the bivariate random trigonometric polynomials converges to a universal limit as their degree tends to infinity, regardless of the particular law of their coefficients.

2.1. A limit Gaussian field. Let us first remark that, up to a scale factor, the set of zeros of the original random trigonometric polynomial f_n defined by equation (1.1) naturally identifies with the set of zeros of its rescaled analogue F_n defined

by equation (1.2). But the advantage of considering the function F_n instead of f_n is that for any fixed compact $K \subset \mathbb{R}^2$ and as n goes to infinity, the random field $(F_n(x, y))_{(x, y) \in K}$ converges in law, with respect to the C^1 topology, to an explicit smooth Gaussian field $(F_\infty(x, y))_{(x, y) \in K}$.

Proposition 1. *For any fixed compact $K \subset \mathbb{R}^2$, as n goes to infinity, the renormalized random field $(F_n(x, y))_{(x, y) \in K}$ converges with respect to the C^1 topology on K to a Gaussian field $(F_\infty(x, y))_{(x, y) \in K}$ whose covariance is given by $\rho_\infty((x, y), (x', y')) := \mathbb{E}[F_\infty(x, y)F_\infty(x', y')]$:*

$$\begin{aligned} \rho_\infty((x, y), (x', y')) &= \int_0^1 \int_0^1 \cos(sx) \cos(sx') \cos(ty) \cos(ty') ds dt \\ &= \frac{1}{4} (\sin_c(x + x') + \sin_c(x - x')) (\sin_c(y + y') + \sin_c(y - y')), \end{aligned}$$

where $\sin_c(x) := \sin(x)/x$ if $x \neq 0$ and $\sin_c(0) := 1$ by convention.

Proof. Here we use the characterization of the C^1 -convergence given in Theorem 2 and Remarks 2 and 3 of [RS01]. The convergence of finite-dimensional marginals is a direct consequence of the standard central limit theorem for independent, non-identically distributed random variables. The covariance function of the limit is obtained as the limit of the two-dimensional Riemann sums

$$\mathbb{E}[F_n(x, y)F_n(x', y')] = \frac{1}{n^2} \sum_{1 \leq k, \ell \leq n} \cos\left(\frac{kx}{n}\right) \cos\left(\frac{\ell y}{n}\right) \cos\left(\frac{kx'}{n}\right) \cos\left(\frac{\ell y'}{n}\right).$$

Moreover, if ∂_1 and ∂_2 denote the partial derivatives in the x and y components, and if we set $D_n := \mathbb{E}[|F_n(x, y) - F_n(x', y')|^2]$, $D_n^1 := \mathbb{E}[|\partial_1 F_n(x, y) - \partial_1 F_n(x', y')|^2]$, and $D_n^2 := \mathbb{E}[|\partial_2 F_n(x, y) - \partial_2 F_n(x', y')|^2]$, for all $(x, y), (x', y') \in \mathbb{R}^2$ we have

$$\begin{aligned} D_n &= \frac{1}{n^2} \sum_{1 \leq k, \ell \leq n} \left| \cos\left(\frac{kx}{n}\right) \cos\left(\frac{\ell y}{n}\right) - \cos\left(\frac{kx'}{n}\right) \cos\left(\frac{\ell y'}{n}\right) \right|^2 \\ &\leq \frac{2}{n^2} \sum_{1 \leq k, \ell \leq n} \left| \cos\left(\frac{kx}{n}\right) - \cos\left(\frac{kx'}{n}\right) \right|^2 + \left| \cos\left(\frac{\ell y}{n}\right) - \cos\left(\frac{\ell y'}{n}\right) \right|^2 \\ &\leq \left(\frac{2}{n} \sum_{1 \leq k \leq n} \left(\frac{k}{n}\right)^2 \right) \|(x, y) - (x', y')\|^2 \leq 2 \|(x, y) - (x', y')\|^2. \end{aligned}$$

In the same way, we have

$$\begin{aligned} D_n^1 &= \frac{1}{n^2} \sum_{1 \leq k, \ell \leq n} \frac{k^2}{n^2} \left| \sin\left(\frac{kx}{n}\right) \cos\left(\frac{\ell y}{n}\right) - \sin\left(\frac{kx'}{n}\right) \cos\left(\frac{\ell y'}{n}\right) \right|^2 \\ &\leq \frac{2}{n^2} \sum_{1 \leq k, \ell \leq n} \frac{k^2}{n^2} \left(\left| \sin\left(\frac{kx}{n}\right) - \sin\left(\frac{kx'}{n}\right) \right|^2 + \left| \cos\left(\frac{\ell y}{n}\right) - \cos\left(\frac{\ell y'}{n}\right) \right|^2 \right) \\ &\leq \left(\frac{2}{n} \sum_{1 \leq k \leq n} \frac{k^4}{n^4} \right) \|(x, y) - (x', y')\|^2 \leq 2 \|(x, y) - (x', y')\|^2, \end{aligned}$$

and the exact same computation yields $D_n^2 \leq 2\|(x, y) - (x', y')\|^2$. Together with the convergence of finite-dimensional marginals, the three last estimates provide the desired tightness criterion ensuring the convergence in the C^1 topology. \square

As noticed in Remark 2 in the introduction, here we consider random trigonometric polynomials in a nonstationary framework. To be able to deal with this nonstationarity in our approach of global universality at the end of the paper, we need to slightly reinforce the above convergence result by establishing a kind of uniformity in space. This is the object of the next proposition.

Proposition 2. *For any $0 < a < b < 1$ and any sequence of couples of integers (p_n, q_n) in the square $[an, bn]^2$, the stochastic process $(G_n(x, y))_{(x,y) \in [0,\pi]^2}$ defined by*

$$G_n(x, y) := F_n(p_n\pi + x, q_n\pi + y), \quad (x, y) \in [0, \pi]^2,$$

converges in distribution, as n goes to infinity, in the space $C^1([0, \pi]^2)$ towards a stationary Gaussian field G_∞ of covariance

$$\rho((x, y), (x', y')) := \frac{1}{4} \sin_c(x - x') \sin_c(y - y').$$

Proof. First of all, the tightness criterion used in the proof of Proposition 1 applies in the same way since the final bound is expressed only in terms of $\|(x, y) - (x', y')\|_2^2$ so that p_n and q_n play no role here. Thus, one is only left to consider the convergence of the covariances. Setting

$$\rho_n(x, x', p) := \frac{1}{n} \sum_{1 \leq k \leq n} \cos\left(\frac{k}{n}(x + p\pi)\right) \cos\left(\frac{k}{n}(x' + p\pi)\right),$$

we have $\mathbb{E}[F_n(p_n\pi + x, q_n\pi + y)F_n(p_n\pi + x', q_n\pi + y')] = \rho_n(x, x', p_n)\rho_n(y, y', q_n)$. By symmetry, it is enough to investigate the first factor, which can be rewritten as

$$\rho_n(x, x', p_n) = \frac{1}{2n} \sum_{1 \leq k \leq n} \cos\left(\frac{k}{n}(x + x' + 2p_n\pi)\right) + \frac{1}{2n} \sum_{1 \leq k \leq n} \cos\left(\frac{k}{n}(x - x')\right).$$

The second term is a Riemann sum converging to the desired sine cardinal, whereas the first sum is managed by a direct computation to obtain the inequality

$$\frac{1}{n} \left| \sum_{1 \leq k \leq n} \cos\left(\frac{k}{n}(x + x' + 2p_n\pi)\right) \right| \leq \frac{1}{n} \frac{1}{\left| \sin\left(\frac{x+x'}{2n} + \frac{p_n\pi}{n}\right) \right|}.$$

The right-hand side of this last equation goes to zero as n goes to infinity. Indeed, on the one hand, $(x + x')/2n$ goes to zero as n goes to infinity, whereas on the other hand, $\text{dist}(p_n/n, \mathbb{Z}) = \min_{k \in \mathbb{Z}} |k - p_n/n|$ remains uniformly bounded from below, hence the result. \square

Using the same arguments, one can moreover establish the following convergence result which will also be used at the end of proof of the global universality.

Proposition 3. *Let (p_n, q_n) be a couple of integers as in Proposition 2; then the random field $F_\infty(p_n\pi + \cdot, q_n\pi + \cdot)$ converges in distribution in the C^1 topology towards G_∞ .*

Let us go back to the convergence of the random field $(F_n(x, y))_{(x,y) \in K}$ in a fixed compact $K \subset \mathbb{R}^2$ and establish that the limit Gaussian field $(F_\infty(x, y))_{(x,y) \in K}$ is nondegenerate in the following sense.

Lemma 1. *The limit Gaussian field F_∞ obtained in Proposition 1 is nondegenerate in the sense that, almost surely, we have*

$$\nabla_{(x,y)} F_\infty \neq 0 \text{ whenever } F_\infty(x, y) = 0.$$

Proof. Let us denote by $A := \{x = 0\} \cup \{y = 0\}$ the axes of \mathbb{R}^2 and consider a compact set $K \subset \mathbb{R}^2 \setminus A$. The fact that the field F_∞ is nondegenerate on K is a consequence of the Bulinskaya Lemma; see, e.g., Proposition 6.11 of [AW09]. The only delicate point to check is that the Gaussian vector $V = (F_\infty, \partial_1 F_\infty, \partial_2 F_\infty)$ admits a uniformly bounded density on K . A necessary and sufficient condition ensuring this fact is that the determinant of the covariance matrix Γ_V of V is strictly positive on the compact K , and thus uniformly bounded from below. The covariance matrix Γ_V of V is a Gram matrix; namely, if $\langle \cdot, \cdot \rangle$ denotes the standard Hilbert scalar product in $L^2([0, 1])$, we have

$$\Gamma_V = \begin{pmatrix} \langle f, f \rangle & \langle f, g \rangle & \langle f, h \rangle \\ \langle f, g \rangle & \langle g, g \rangle & \langle g, h \rangle \\ \langle f, h \rangle & \langle g, h \rangle & \langle h, h \rangle \end{pmatrix},$$

where

$$f(s) := \cos(sx) \cos(sy), \quad g(s) := -s \sin(sx) \cos(sy), \quad h(s) := -s \cos(sx) \sin(sy).$$

The determinant of this Gram matrix vanishes if and only if the above functions of s are proportional, which only occurs on the axes $\{x = 0\}$ or $\{y = 0\}$, and hence the result. Let us now consider the case of the axes. Let us first remark that the random variable $F_\infty(0, 0)$ is a standard Gaussian variable so that $F_\infty(0, 0) \neq 0$ almost surely. Next, on the axis $\{x = 0, y \neq 0\}$, the limit process $(F_\infty(0, y))_{y \in \mathbb{R}}$ is nothing but the limit Gaussian process associated to the univariate trigonometric polynomials

$$F_n(0, y) = \frac{1}{\sqrt{n}} \sum_{1 \leq \ell \leq n} b_\ell \cos\left(\frac{\ell y}{n}\right),$$

where the variables $b_\ell = \frac{1}{\sqrt{n}} \sum_{k=1}^n a_{k,\ell}$ are independent and identically distributed, their common law being centered and with unit variance. As above, the covariance matrix of $(F_\infty(0, y), \partial_y F_\infty(0, y))$ is also a Gram matrix whose determinant only vanishes at the origin, and hence is uniformly bounded from below on any compact set of $\{x=0, y \neq 0\}$. Naturally the same reasoning holds on the set $\{y=0, x \neq 0\}$. \square

Remark 3. Note that the above arguments also actually ensure the nondegeneracy of the stationary limit field G_∞ appearing in Propositions 2 and 3.

2.2. Continuity of the nodal length. In this section, we establish that the functional that associates to a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ the length of its nodal set, or more generally its $d - 1$ -dimensional volume if $f : \mathbb{R}^d \rightarrow \mathbb{R}$, is continuous with respect to the C^1 topology on compact sets. Let us be more precise and consider the space $E := C^1(\mathbb{R}^d, \mathbb{R})$ endowed with the C^1 topology associated to the family of seminorms $\|\cdot\|_K$:

$$\|f\|_K := \sup_K \left(|f| + \sum_{i=1}^d |\partial_i f| \right), \quad K \text{ a compact subset of } \mathbb{R}^d.$$

Given such a compact $K \subset \mathbb{R}^d$, we will say that $f \in E$ is nondegenerate on K if

$$\nabla_x f \neq 0 \text{ whenever } x \in Z_K(f).$$

If $A \subset \mathbb{R}^d$ is a measurable set, we will denote by $H_{d-1}(A)$ with values in $[0, +\infty]$ its $(d - 1)$ -dimensional Hausdorff measure so that the object of interest here is the continuity in f of the nodal volume $v_K(f) := H_{d-1}(Z_K(f))$.

Theorem 3. *Let $K \subset \mathbb{R}^d$ be a compact set and let $(f_n)_{n \geq 1}$ be a sequence of functions in E which converges to a function $f \in E$ in the C^1 topology on K . If f is nondegenerate on K , then the volumes $v_K(f)$ and $v_K(f_n)$, n sufficiently large, are finite and we have*

$$\lim_{n \rightarrow +\infty} v_K(f_n) = v_K(f).$$

Proof of Theorem 3. We first need to introduce some notation. For a nondegenerate function f , we denote by $\sigma(x) = \sigma_f(x)$ the index of the first nonvanishing component of the gradient at x , namely,

$$\sigma(x) = \sigma_f(x) := \inf\{1 \leq i \leq d, \partial_i f(x) \neq 0\}.$$

If $x = (x^1, \dots, x^d)$ and $1 \leq i \leq d$, we will write

$$\pi_i(x) = \check{x}^i := (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^d).$$

Finally, if $y \in \mathbb{R}^d$ and $\delta, \varepsilon > 0$, $R^i(y, \delta, \varepsilon)$ will denote the following open rectangle:

$$R^i(y, \delta, \varepsilon) := \{x \in \mathbb{R}^d, |x^i - y^i| < \delta, |x^\ell - y^\ell| < \varepsilon, 1 \leq \ell \leq d, \ell \neq i\}.$$

Let us first prove the following lemma, which ensures that under the hypotheses of Theorem 3 and for n sufficiently large, the zeros of f_n are located in a neighborhood of the zeros of f . Here and below, $d(x, Z)$ denotes the Euclidean distance between a point $x \in \mathbb{R}^d$ and a set $Z \subset \mathbb{R}^d$.

Lemma 2. *Let $(f_n)_{n \geq 1}$ be a sequence of functions in E which converges to a function $f \in E$ with respect to the C^1 topology on the compact K . For all $\varepsilon > 0$ and for n sufficiently large, we have*

$$Z_K(f_n) \subset Z_K(f, \varepsilon) := \{x \in \mathbb{R}^d, d(x, Z_K(f)) \leq \varepsilon\}.$$

Proof of Lemma 2. By contradiction, let us suppose that there exists $\varepsilon > 0$ such that for all $N \geq 1$, there exists $n \geq N$ and $x_n \in Z_K(f_n)$ such that $d(x_n, Z_K(f)) > \varepsilon$. Since the sequence $(x_n)_{n \geq 1}$ takes values in the compact set K , one could then extract a converging subsequence $(x_{n_k})_{k \geq 1}$, converging to some $x_\infty \in K$ with $d(x_\infty, Z_K(f)) \geq \varepsilon$. But

$$\begin{aligned} |f(x_\infty)| &= |f(x_\infty) - f_{n_k}(x_{n_k})| = |f(x_\infty) - f_{n_k}(x_\infty) + f_{n_k}(x_\infty) - f_{n_k}(x_{n_k})| \\ &\leq \sup_{x \in K} |f(x) - f_{n_k}(x)| + \sup_{x \in K} |f'_{n_k}(x)| \times |x_\infty - x_{n_k}|, \end{aligned}$$

which would go to zero as k goes to infinity because f_n converges to f in the C^1 topology on K , and hence the contradiction between the two assertions $f(x_\infty) = 0$ and $d(x_\infty, Z_K(f)) \geq \varepsilon$. □

Let us go back to the proof of Theorem 3 and consider the evaluation mapping from $E \times \mathbb{R}^d$ to \mathbb{R} defined by

$$F(h, x) := h(x).$$

Being linear in h , the function F is naturally continuously Fréchet differentiable in h , and since the space E is composed of C^1 functions, F is also continuously Fréchet differentiable in the variable x . The partial derivatives in both h and x being

continuous, the function F is then actually continuously Fréchet differentiable in (h, x) ; see, e.g., Theorem 10, p. 144 of [Che01]. By hypothesis, since the function f is nondegenerate on K , if $x_0 = (x_0^1, \dots, x_0^d) \in Z_K(f)$, we have $F(f, x_0) = 0$ and there exists an index $1 \leq i = \sigma_f(x_0) \leq d$ such that $\partial_{x^i} f(x_0) \neq 0$. In other words, since $\partial_{x^i} F(f, x_0) = \partial_{x^i} f(x_0)$, the inverse $(\partial_{x^i} F(f, x_0))^{-1}$ is well defined. By the C^1 version of the implicit function theorem in Banach spaces (see, e.g., Theorems 1 and 2, pp. 315 and 317 of [LS71] or Theorems 3 and 4, pp. 138 and 139 of [Che01]), there exists $\varepsilon_0 > 0$, $\delta_0 > 0$, and a function $X_0 : E \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ of class C^1 such that

$$(2.1) \quad h(x) = 0 \iff x^i = X_0(h, \check{x}^i) \text{ for all } \begin{cases} x \in R^i(x_0, 2\delta_0, 2\varepsilon_0), \\ \|h - f\| < 2\varepsilon_0. \end{cases}$$

From the covering of the compact nodal set $Z_K(f)$ by the union of open sets of the type $R^i(x_0, \delta_0, \varepsilon_0)$, one can extract a finite covering. Namely, there exists a positive integer m , and for all $1 \leq j \leq m$, there exists $x_j \in Z_K(f)$ as well as $\varepsilon_j > 0$ and $\delta_j > 0$ such that

$$(2.2) \quad Z_K(f) \subset \bigcup_{j=1}^m V_j, \quad \text{where } V_j := R^{\sigma(x_j)}(x_j, \delta_j, \varepsilon_j).$$

For all $1 \leq j \leq m$, if $k = \sigma_f(x_j)$, we have a similar identification to the one given by equation (2.1), namely in a neighborhood of (f, x_j) ,

$$(2.3) \quad h(x) = 0 \iff x^k = X_j(h, \check{x}^k), \text{ for all } \begin{cases} x \in B^k(x_j, 2\delta_j, 2\varepsilon_j), \\ \|h - f\| < 2\varepsilon_j, \end{cases}$$

where the application $X_j : E \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is of class C^1 . In particular, setting $h = f$, we get that if $J = \{j_1, \dots, j_r\} \subset \{1, \dots, m\}$ and $\bigcap_{j \in J} V_j \neq \emptyset$, the intersection

$$\Gamma_J = Z_K(f) \cap \left(\bigcap_{j \in J} \bar{V}_j \right)$$

identifies with a parametrized hypersurface whose finite volume is given by the classical formula

$$(2.4) \quad H_{d-1}(\Gamma_J) = \int_{E_J} \sqrt{1 + |\nabla X_{j_1}(f, y)|^2} dy,$$

where the integration is performed on the compact rectangle

$$E_J := \pi_{j_1} \left(\bigcap_{j \in J} \bar{V}_j \right).$$

Taking care of the overlapping, the finite total volume of the nodal set is then given by the celebrated Poincaré formula

$$(2.5) \quad v_K(f) = \sum_{\emptyset \neq J \subset \{1, \dots, m\}} (-1)^{|J|} H_{d-1}(\Gamma_J).$$

Let us now emphasize the fact that in equation (2.2), the union not only contains the nodal set $Z_K(f)$, but there exists $\varepsilon > 0$ small enough such that this union contains a ε -neighborhood of the latter:

$$Z_K(f, \varepsilon) \subset \bigcup_{j=1}^m V_j.$$

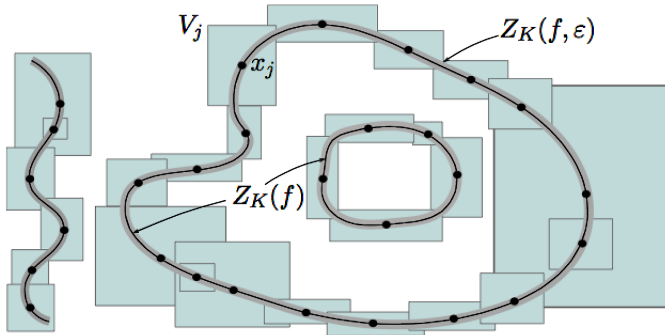


FIGURE 4. Finite covering of the compact nodal set

By Lemma 2, we get that for n large enough $Z_K(f_n) \subset Z_K(f, \epsilon) \subset \bigcup_{i=1}^m V_j$, and thus

$$Z_K(f_n) = \bigcup_{j=1}^m [Z_K(f_n) \cap \bar{V}_j].$$

From the equivalence (2.3) given by the implicit function theorem, as above, we get that if $J = \{j_1, \dots, j_r\} \subset \{1, \dots, m\}$ and $\bigcap_{j \in J} V_j \neq \emptyset$, the intersection

$$\Gamma_J^n = Z_K(f_n) \cap \left(\bigcap_{j \in J} \bar{V}_j \right)$$

also identifies with a parametrized hypersurface whose volume is given by

$$(2.6) \quad H_{d-1}(\Gamma_J^n) = \int_{E_J} \sqrt{1 + |\nabla X_{j_1}(f_n, y)|^2} dy.$$

By the Poincaré formula, we similarly have

$$(2.7) \quad v_K(f_n) = \sum_{\emptyset \neq J \subset \{1, \dots, m\}} (-1)^{|J|} H_{d-1}(\Gamma_J^n)$$

so that, comparing it to equation (2.5), we get

$$|v_K(f) - v_K(f_n)| \leq \sum_{\emptyset \neq J \subset \{1, \dots, m\}} |H_{d-1}(\Gamma_J) - H_{d-1}(\Gamma_J^n)|.$$

The right-hand side of the last equation goes to zero as n goes to infinity because, from equations (2.4) and (2.6), for any nonempty subset $J = (j_1, \dots, j_r)$ of $\{1, \dots, m\}$, we have

$$|H_{d-1}(\Gamma_J) - H_{d-1}(\Gamma_J^n)| \leq \int_{E_J} \left| \sqrt{1 + |\nabla X_{j_1}(f, y)|^2} - \sqrt{1 + |\nabla X_{j_1}(f_n, y)|^2} \right| dy,$$

and the difference $\nabla X_{j_1}(f, y) - \nabla X_{j_1}(f_n, y)$ goes to zero uniformly on E_J , since the function X_{j_1} is C^1 and since the sequence f_n converges to f in the C^1 topology on K .

□

2.3. Local universality. Let $K \subset \mathbb{R}^2$ be a compact set. Combining Proposition 1 and Lemma 1, we get that as n goes to infinity, the field $(F_n(x, y))_{x \in K}$ converges with respect to the C^1 topology on K to the nondegenerate limit field $(F_\infty(x, y))_{x \in K}$. The announced local universality result is then a direct consequence of the continuous mapping theorem together with the continuity of the nodal length established in Theorem 3.

Theorem 4. *Let $K \subset \mathbb{R}^2$ be a compact set. Then as n goes to infinity, the length $\ell_K(F_n)$ of the nodal set converges in distribution to $\ell_K(F_\infty)$.*

3. GLOBAL UNIVERSALITY

We now turn to the proof of Theorem 2 on the universality of the mean nodal length at the macroscopic level.

3.1. The Gaussian case. In this section, we consider the Gaussian case, namely we assume that all the entries $a_{k,l}$ are independent standard Gaussian variables. In this situation, the expectation of the nodal length $\ell_K(f_n)$ can be explicitly computed thanks to celebrated Kac–Rice formula, since both f_n and its derivative have explicit densities.

Lemma 3. *For $(x, y) \in \mathbb{R}^2$, the Gaussian vector $(f_n(x, y), \frac{\partial f_n}{\partial x}(x, y), \frac{\partial f_n}{\partial y}(x, y))$ is centered with explicit covariance $\Sigma = (\Sigma_{ij})_{1 \leq i, j \leq 3}$ given by*

$$\begin{aligned} \Sigma_{11} &= A_n(x)A_n(y), & \Sigma_{22} &= C_n(x)A_n(y), & \Sigma_{33} &= A_n(x)C_n(y), \\ \Sigma_{12} &= -B_n(x)A_n(y), & \Sigma_{13} &= -A_n(x)B_n(y), & \Sigma_{23} &= B_n(x)B_n(y), \end{aligned}$$

where

$$A_n(\cdot) := \sum_{1 \leq k \leq n} \cos^2(k \cdot), \quad B_n(\cdot) := \sum_{1 \leq k \leq n} k \sin(k \cdot) \cos(k \cdot), \quad C_n(\cdot) := \sum_{1 \leq k \leq n} k^2 \sin^2(k \cdot).$$

Note that the sums A_n , B_n , and C_n appearing in Lemma 3 can actually be written as simple combinations of trigonometric functions. For example, the next lemma can be found in [Wil91].

Lemma 4. *We have*

$$\begin{aligned} 4A_n(x) &= (2n + 1)g_0 + g_1, \\ 8B_n(x) &= (2n + 1)^2h_0 + (2n + 1)h_1 + h_2, \\ 48C_n(x) &= (2n + 1)^3k_0 + (2n + 1)^2k_1 + (2n + 1)k_2 + k_3, \end{aligned}$$

where, setting $z := (2n + 1)x$, and $f(x) := \csc(x) - x^{-1}$, the functions g_i , h_i , and k_i are defined as

$$\begin{aligned} g_0(x) &:= 1 + z^{-1} \sin z, & g_1(x) &= -2 + f(x) \sin z, \\ h_0(x) &:= -z^{-1} \cos z + z^{-2} \sin z, & h_1(x) &= -f(x) \cos z, & h_2(x) &= -f'(x) \sin z, \\ k_0(x) &:= 1 - 3z^{-1} \sin z - 6z^{-2} \cos z + 6z^{-3} \sin z, \\ k_1(x) &:= -3f(x) \sin z, & k_2(x) &= 6f'(x) \cos z - 1, & k_3(x) &= 3f''(x) \sin z. \end{aligned}$$

It is remarkable that, conditional to the event $f_n = 0$, the partial derivatives of f_n are independent Gaussian random variables.

Lemma 5. *Given $f_n = 0$, the conditional distribution of $\left(\frac{\partial f_n}{\partial x}, \frac{\partial f_n}{\partial y}\right)$ is centered normal with covariance matrix*

$$\Sigma_{11} \begin{pmatrix} \sigma_n^2(x) & 0 \\ 0 & \sigma_n^2(y) \end{pmatrix},$$

where we have set, for all $t \in \mathbb{R}$,

$$\sigma_n^2(t) := \frac{C_n(t)}{A_n(t)} - \left(\frac{B_n(t)}{A_n(t)}\right)^2.$$

Proof. Conditional to the event $f_n = 0$, the conditional covariance matrix $\Sigma_{f_n=0}$ of the gradient vector $\nabla f_n = \left(\frac{\partial f_n}{\partial x}, \frac{\partial f_n}{\partial y}\right)$ is given by

$$\begin{aligned} \Sigma_{f_n=0} &= \text{Var}(\nabla f_n) - \text{Cov}(\nabla f_n, f_n)[\text{Var}(\nabla f_n)]^{-1}[\text{Cov}(\nabla f_n, f_n)]^T \\ &= \frac{1}{\Sigma_{11}} \begin{pmatrix} \Sigma_{11}\Sigma_{22} - \Sigma_{12}^2 & \Sigma_{11}\Sigma_{23} - \Sigma_{12}\Sigma_{13} \\ \Sigma_{11}\Sigma_{23} - \Sigma_{12}\Sigma_{13} & \Sigma_{11}\Sigma_{33} - \Sigma_{13}^2 \end{pmatrix}. \end{aligned}$$

The result thus follows from Lemma 3; in particular, the independence of the marginals follows from the relation $\Sigma_{11}\Sigma_{23} - \Sigma_{12}\Sigma_{13} = 0$. □

Let us now describe the asymptotic behavior, as n goes to infinity, of the function $\sigma_n^2(t)$ appearing in the covariance matrix of Lemma 5.

Lemma 6. *For $n \geq 3$, uniformly in $t \in \mathbb{R}$, we have*

$$0 \leq \frac{\sigma_n^2(t)}{n^2} \leq 48,$$

and for all $t \not\equiv 0 \pmod{\pi}$, we have

$$\lim_{n \rightarrow +\infty} \frac{\sigma_n^2(t)}{n^2} = \frac{1}{3}.$$

Proof. From Lemma 3, we have

$$\frac{\sigma_n^2(t)}{n^2} = \frac{\frac{1}{n} \sum_{k=1}^n \frac{k^2}{n^2} \sin^2(kt)}{\frac{1}{n} \sum_{k=1}^n \cos^2(kt)} - \left(\frac{\frac{1}{n} \sum_{k=1}^n \frac{k}{n} \sin(kt) \cos(kt)}{\frac{1}{n} \sum_{k=1}^n \cos^2(kt)} \right)^2.$$

Thus we have, for all $t \in \mathbb{R}$,

$$0 \leq \frac{\sigma_n^2(t)}{n^2} \leq \left(\frac{1}{n} \sum_{k=1}^n \cos^2(kt) \right)^{-1}.$$

Now, using the relation $\cos(2a) = 2 \cos^2(a) - 1$, one easily gets that if $\cos(kt)^2 \leq 1/8$, then necessarily $\cos^2(2kt) \geq 1/2 \geq 1/8$ so that for $n \geq 3$ and for all $t \in \mathbb{R}$ we have

$$\frac{1}{n} \sum_{k=1}^n \cos^2(kt) \geq \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} [\cos^2(kt) + \cos^2(2kt)] \geq \frac{1}{8n} \lfloor n/2 \rfloor \geq \frac{1}{48}.$$

Now, using the explicit forms of A_n , B_n , and C_n given by Lemma 4, one deduces that for all $t \not\equiv 0 \pmod{\pi}$, we have

$$\lim_{n \rightarrow +\infty} \frac{A_n(t)}{n} = \frac{1}{2}, \quad \lim_{n \rightarrow +\infty} \frac{B_n(t)}{n^2} = 0, \quad \lim_{n \rightarrow +\infty} \frac{C_n(t)}{n^3} = \frac{1}{6}$$

so that

$$\lim_{n \rightarrow +\infty} \frac{\sigma_n^2(t)}{n^2} = \lim_{n \rightarrow +\infty} \left(\frac{C_n(t)}{n^3} \times \frac{n}{A_n(t)} - \left(\frac{B_n(t)}{n^2} \frac{n}{A_n(t)} \right)^2 \right) = \frac{1}{3}.$$

□

We are now in a position to explicitly compute the expectation of the length of a nodal curve associated to the random trigonometric polynomial $f_n(x, y)$.

Theorem 5. *Let $(a_{k,l})_{k,l \geq 1}$ be a sequence of independent standard, centered, Gaussian variables and consider the associated random trigonometric polynomial $f_n(x, y)$ defined by equation (1.1). Then, as n tends to infinity, we have*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} = \frac{\pi^2}{2\sqrt{3}}.$$

Proof. By the Kac–Rice formula, if $p_{f_n(x,y)}$ denotes the density function of $f_n(x, y)$, the rescaled expectation of the nodal length is equal to

$$\begin{aligned} \frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} &= \frac{1}{n} \iint_{[0,\pi]^2} \mathbb{E} \left(\sqrt{\left(\frac{\partial f_n}{\partial x} \right)^2 + \left(\frac{\partial f_n}{\partial y} \right)^2} \mid f_n(x, y) = 0 \right) p_{f_n(x,y)}(0) dx dy \\ &= \frac{1}{\sqrt{2\pi}} \iint_{[0,\pi]^2} \frac{1}{n \Sigma_{11}^{1/2}} \mathbb{E} \left(\sqrt{\left(\frac{\partial f_n}{\partial x} \right)^2 + \left(\frac{\partial f_n}{\partial y} \right)^2} \mid f_n(x, y) = 0 \right) dx dy. \end{aligned}$$

In other words, by Lemma 5, we have the representation

$$(3.1) \quad \frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} = \frac{1}{\sqrt{2\pi}} \iint_{[0,\pi]^2} \mathbb{E} \left[\sqrt{X_n^2(x) + Y_n^2(y)} \right] dx dy,$$

where $X_n(x)$ and $Y_n(y)$ are independent centered Gaussian variables with variance $\sigma_n^2(x)/n^2$ and $\sigma_n^2(y)/n^2$, respectively. By the Cauchy–Schwarz inequality, using the upper bound of Lemma 6, we have uniformly in $(x, y) \in [0, \pi]^2$ that

$$(3.2) \quad \mathbb{E} \left[\sqrt{X_n^2(x) + Y_n^2(y)} \right] \leq \sqrt{\mathbb{E} [X_n^2(x) + Y_n^2(y)]} \leq \sqrt{\frac{\sigma_n^2(x) + \sigma_n^2(y)}{n^2}} \leq \sqrt{96}.$$

From Lemma 6 again, as n goes to infinity, for all $x \neq 0 \pmod{\pi}$ and $y \neq 0 \pmod{\pi}$, the Gaussian vector $(X_n(x), Y_n(y))$ converges in distribution to (X_∞, Y_∞) , where (X_∞, Y_∞) is a two-dimensional centered Gaussian vector with covariance matrix $1/3 \times \text{Id}$. Since the variables are Gaussian, we have a uniform control on their moments so that for all $x \neq 0 \pmod{\pi}$ and $y \neq 0 \pmod{\pi}$,

$$(3.3) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left[\sqrt{X_n^2(x) + Y_n^2(y)} \right] = \mathbb{E} \left[\sqrt{X_\infty^2 + Y_\infty^2} \right] = \frac{1}{\sqrt{3}} \sqrt{\frac{\pi}{2}}.$$

Indeed, if $(X, Y) \sim \mathcal{N}(0, \text{Id})$, then $\sqrt{X^2 + Y^2}$ has the standard Rayleigh distribution with expectation $\sqrt{\pi/2}$. From equations (3.1), (3.2), and (3.3), by the dominated convergence theorem, one then concludes that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} = \frac{1}{\sqrt{2\pi}} \iint_{[0,\pi]^2} \mathbb{E} \left[\sqrt{X_\infty^2 + Y_\infty^2} \right] dx dy = \frac{\pi^2}{2\sqrt{3}}.$$

□

3.2. Moment control. In the above Theorem 4, we proved that given a compact set $K \subset \mathbb{R}^2$ and as n goes to infinity, the microscopic length $\ell_K(F_n)$ of the nodal set of the normalized trigonometric polynomial converges in distribution to $\ell_K(F_\infty)$. The object of this subsection is to establish a uniform upper bound for the expectation of this microscopic length, uniform in both the degree n and in the compact K . More precisely, taking care of the change of scale on the length of the nodal set, the mean macroscopic nodal length can be written as the sum

$$(3.4) \quad \frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} = \frac{\mathbb{E}[\ell_{[0,n\pi]^2}(F_n)]}{n^2} = \frac{1}{n^2} \sum_{0 \leq k, l \leq n-1} \mathbb{E}[\mathcal{L}_{n,k,l}],$$

where $\mathcal{L}_{n,k,l}$ denotes the length of the nodal set associated to $F_n(x, y)$ inside the square $[k\pi, (k + 1)\pi] \times [l\pi, (l + 1)\pi]$. We shall prove the following uniform upper bound.

Proposition 4. *There exists $\alpha > 0$ and $C > 0$ such that*

$$(3.5) \quad \sup_{n \geq 1} \sup_{0 \leq k, l \leq n-1} \mathbb{E}[\mathcal{L}_{n,k,l}^{1+\alpha}] \leq C.$$

3.2.1. Geometric considerations. In this first subsection, we prove two elementary and purely geometric results. Both results relate the length of a smooth curve drawn in a unit square to the number of its intersections with some prescribed lines. As a corollary, we derive an a priori estimate for the microscopic length of a trigonometric polynomial in a unit square. Both proofs use the so-called probabilistic method saying that a random variable X such that $\mathbb{E}(X) \geq c$ admits at least one realization ω such that $X(\omega) \geq c$.

Remark 4. At first glance, one might be tempted to use the Crofton formula in order to relate the length of the nodal domain of a trigonometric polynomial to the number of its intersections with some random lines. Nevertheless, such an approach faces two major obstructions. On the one hand, when the slope of such a line is irrational, then when restricting the bivariate trigonometric polynomial to it, the resulting random function is no longer polynomial. For this reason, in Theorem 7 we relate the length of the nodal set to its number of intersections with vertical or horizontal lines, which then allow us to derive a deterministic upper bound on the nodal length. On the other hand, since the nodal set is random, the lines intersecting it are also generically random. This randomness dependence is hard to manage when performing characteristic functions computations since we lose the structure of independent summands. This is why we prove Theorem 6 in order to “force” the lines to go through deterministic points on which the independence of summands is preserved and the characteristic functions method applicable.

Theorem 6. *There exists an absolute constant $c > 0$ such that for any unit square \mathcal{S} with corners $A, B, C,$ and D and any C^1 curve \mathcal{C} inside \mathcal{S} with length $l,$ one may find a straight line \mathcal{L} such that:*

- (i) $\{A, B, C, D\} \cap \mathcal{L} \neq \emptyset,$
- (ii) $\#\{\mathcal{L} \cap \mathcal{C}\} \geq cl.$

Proof. Using the probabilistic method, we will actually establish the above result with $c = 1/8$. On a given probability space $(\Omega, \mathcal{F}, \mathbb{P}),$ we denote by P a random

point inside the square with uniform distribution. Set

$$X_{\mathcal{C}} := \# \{ \mathcal{C} \cap ((AP) \cup (BP) \cup (CP) \cup (DP)) \}.$$

Then the result follows if one can show that $\mathbb{E}(X_{\mathcal{C}}) \geq 4cl$. Indeed, in this case, there exists one realization of the random variable such that $X_{\mathcal{C}}(\omega) \geq 4cl$, i.e., the number of intersections of one of the four lines with \mathcal{C} is at least cl . Notice that, since \mathcal{C} is assumed to be C^1 , it is rectifiable. Hence, one might try to seek for $(\mathcal{C}_p)_{p \geq 1}$ a sequence of polygonal lines \mathcal{C}_p such that:

- (i) $\forall p \geq 1, \mathbb{E}[X_{\mathcal{C}}] \geq \mathbb{E}[X_{\mathcal{C}_p}]$,
- (ii) $\ell(\mathcal{C}_p) \rightarrow \ell(\mathcal{C})$,
- (iii) $\mathbb{E}[X_{\mathcal{C}_p}] \geq \frac{1}{2}\ell(\mathcal{C}_p)$.

Assume that the curve \mathcal{C} is parametrized by two functions of class C^1 , that is to say, the curve is represented as $\mathcal{C} = \{(x(t), y(t)) \mid t \in [0, 1]\}$, and consider the polygonal line \mathcal{C}_p interpolating between the points $(x(\frac{k}{p}), y(\frac{k}{p}))$, for $0 \leq k \leq p$. At this stage, we notice that (i) is a consequence of connexity and (ii) proceeds from the fact that \mathcal{C} is rectifiable. Thus, one is only left to establish (iii). By the linearity of the expectation, without loss of generality, we may just consider the case when \mathcal{C} is the segment IJ contained in the domain OCD ; see Figure 5 below. If it is not the case, then we can always split it into two segments, respectively, contained in the domains OCD and ABC , respectively. Note that the point I is on the left of J . Assume that J is higher than I . Since for each line (AP) (or (BP) , (CP) , or (DP)) there is at most one intersection point with \mathcal{C} ,

$$\begin{aligned} \mathbb{E}[X_{\mathcal{C}}] &= \mathbb{P}\{(AP) \cap \mathcal{C} \neq \emptyset\} + \mathbb{P}\{(BP) \cap \mathcal{C} \neq \emptyset\} \\ &\quad + \mathbb{P}\{(CP) \cap \mathcal{C} \neq \emptyset\} + \mathbb{P}\{(DP) \cap \mathcal{C} \neq \emptyset\} \\ &= \frac{\lambda_2(AA_1A_2) + \lambda_2(BB_1B_2) + \lambda_2(CC_1C_2) + \lambda_2(DD_1D_2)}{\lambda_2(ABCD)} \\ &= \frac{1}{2} (A_1A_2 + B_1B_2 + C_1C_2 + D_1D_2), \end{aligned}$$

where λ_2 stands for the area or two-dimensional Lebesgue measure and A_1 and A_2 are the intersections between AI , AJ , and CD .

From I , draw a line parallel to CD which intersects AA_2 at I_1 ; similarly, draw a line parallel to (AD) which intersects CC_2 at I_2 . Now, draw the rectangle $II_2I_3I_1$. It is easy to check that the point J must lie inside this rectangle. Therefore,

$$A_1A_2 + C_1C_2 \geq II_1 + II_2 \geq II_3 \geq IJ.$$

In the last inequality, we used the simple observation that the largest distance between two points in a rectangle is the length of the diagonal. We thus have

$$\mathbb{E}[X_{\mathcal{C}}] \geq IJ/2 = \text{length}(\mathcal{C})/2.$$

Otherwise, if I is higher than J , we make an analogous reasoning by simply considering the two triangles BB_1B_2 and DD_1D_2 . □

Theorem 7. *There exists an absolute constant $c > 0$ such that for any unit square \mathcal{S} with corners A, B, C , and D and any C^1 curve \mathcal{C} inside \mathcal{S} with length l , one may find a horizontal or vertical straight line \mathcal{L} such that $\#\{\mathcal{L} \cap \mathcal{C}\} \geq cl$.*

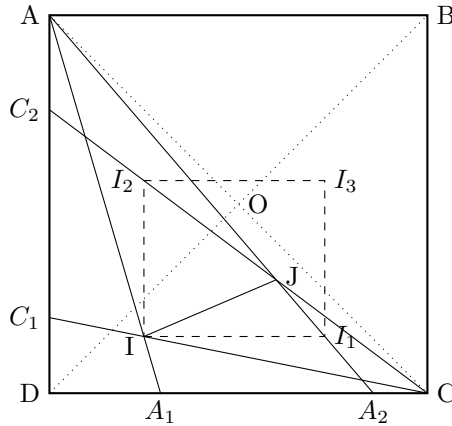


FIGURE 5. A segment case

Proof. Here, we again use the probabilistic method and the piecewise linear approximation to prove the claimed result for $c = 1/2$. Let us just consider the curve \mathcal{C} as a segment IJ . Uniformly choose a horizontal line inside the square (i.e., choose uniformly a point on AD and draw a horizontal line from this point) and define X_1 as the number of intersection points between this line and IJ . Similarly, choose uniformly a vertical line inside the square and define X_2 . Then clearly,

$$\mathbb{E}X_1 + \mathbb{E}X_2 = I_1J_1 + I_2J_2 \geq IJ,$$

where I_1J_1 and I_2J_2 are the projections of IJ on AD and CD . Therefore, there exist a horizontal line or a vertical line such that the total number of intersection points with the nodal curve is at least l . This yields the statement of the theorem. \square

We can now derive the announced a priori estimate on the microscopic nodal length.

Corollary 1. *Suppose that $Q(x, y)$ is any trigonometric polynomial of degree n and denote by $L_{n,k,l}$ the length of the nodal line of $n^{-1}Q(\frac{x}{n}, \frac{y}{n})$ in $[k\pi, (k + 1)\pi] \times [l\pi, (l + 1)\pi]$. Then we have*

$$(3.6) \quad L_{n,k,l} \leq \frac{2n}{c}.$$

Proof. Thanks to Theorem 7, there exists a vertical or horizontal line having at least $L_{n,k,l}/2$ intersection points with the nodal curve. Otherwise, restricted on this line, $Q(x, y)$ becomes a trigonometric polynomial with only one variable; so it has at most n roots over any interval length π . Then the result follows. \square

3.2.2. Small ball estimate. In this section, we show that the uniform upper bound stated in Proposition 4 actually reduces to a small ball estimate for the rescaled polynomial F_n at well-chosen lattice points. To do so, let us first recall some standard number theory considerations which will be used throughout the sequel. Let n be any positive integer, and let $p \in \mathbb{N}$. We shall denote by $\text{ord}(p)$ the order of p in the group $(\mathbb{Z}/n\mathbb{Z}, +)$, that is to say, $\text{ord}(p) = \frac{n}{\gcd(p,n)}$. Then we have the next two lemmas.

Lemma 7. $\max(\text{ord}(p), \text{ord}(p + 1)) \geq \sqrt{n}$.

Proof. Arguing by contradiction, let us assume that we have both $\text{ord}(p) < \sqrt{n}$ and $\text{ord}(p + 1) < \sqrt{n}$. We then have $\text{gcd}(p, n) > \sqrt{n}$ and $\text{gcd}(p + 1, n) > \sqrt{n}$. However, since $\text{gcd}(p, p + 1) = 1$, it holds that $\text{gcd}(\text{gcd}(p, n), \text{gcd}(p + 1, n)) = 1$, and thus $\text{gcd}(p, n)\text{gcd}(p + 1, n)$ divides n . This implies $n < \text{gcd}(p, n)\text{gcd}(p + 1, n) \leq n$, which is a contradiction. \square

Lemma 8. For any 1-periodic function f and any integer $p \geq 1$,

$$(3.7) \quad \frac{1}{n} \sum_{k=1}^n f\left(\frac{kp}{n}\right) = \frac{1}{\text{ord}(p)} \sum_{k=1}^{\text{ord}(p)} f\left(\frac{k}{\text{ord}(p)}\right).$$

Proof. It is clear that $p/n = q/\text{ord}(p)$, where $\text{gcd}(q, \text{ord}(p)) = 1$. Since the set $q \times \{1, 2, \dots, \text{ord}(p)\}$ is a complete residue system of modulo $\text{ord}(p)$ and since the function f is 1-periodic,

$$\sum_{k=1}^{\text{ord}(p)} f\left(\frac{kp}{n}\right) = \sum_{k=1}^{\text{ord}(p)} f\left(\frac{kq}{\text{ord}(p)}\right) = \sum_{k=1}^{\text{ord}(p)} f\left(\frac{k}{\text{ord}(p)}\right).$$

The result follows from the fact that one can divide the set $\{1, 2, \dots, n\}$ into $n/\text{ord}(p)$ complete residue systems. \square

Towards a small ball problem. Let us recall that $\mathcal{L}_{n,k,l}$ denotes the length of the nodal line of F_n in the square $[k\pi, (k + 1)\pi] \times [l\pi, (l + 1)\pi]$. Let us give $\alpha > 0$ to be chosen later. In virtue of Corollary 1, we have

$$\begin{aligned} \mathbb{E}(\mathcal{L}_{n,k,l}^{1+\alpha}) &= (1 + \alpha) \int_0^\infty t^\alpha \mathbb{P}(\mathcal{L}_{n,k,l} > t) dt \\ &= (1 + \alpha) \int_0^{\frac{2n}{c}} t^\alpha \mathbb{P}(\mathcal{L}_{n,k,l} > t) dt. \end{aligned}$$

Thus, one is left to estimate the term $\mathbb{P}(\mathcal{L}_{n,k,l} > t)$. To do so, we shall use the content of Theorem 6. We place ourselves on the square $[k\pi, (k + 1)\pi] \times [l\pi, (l + 1)\pi]$ and we know that there exists a straight line, say \mathcal{L} , such that:

- (i) $(k\pi, l\pi)$ or $((k + 1)\pi, l\pi)$ or $(k\pi, (l + 1)\pi)$ or $((k + 1)\pi, (l + 1)\pi)$ is on \mathcal{L} ,
- (ii) the number of roots of F_n restricted to $\mathcal{L} \cap [k\pi, (k + 1)\pi] \times [l\pi, (l + 1)\pi]$ is greater than ct .

Now, in order to fix the ideas, assume that $(k\pi, l\pi) \in \mathcal{L}$ and denote by (u, v) the unit direction vector of the straight line \mathcal{L} . We set $\phi_n(s) = F_n(k\pi + su, l\pi + sv)$ for $s \in [0, T]$, where T is the largest positive number such that $(k\pi, l\pi) + s(u, v)$ is inside the square. In particular, a simple application of Pythagoras' Theorem entails that $T \leq \pi\sqrt{2}$. As a result, we know that ϕ_n vanishes at least $r = \lfloor ct \rfloor$ times in the interval $[0, \pi\sqrt{2}]$. Let us introduce a_1 a root of ϕ' , a_2 a root of ϕ'' , a_3 a root of ϕ''' , \dots , and a_{r-1} a root of $\phi^{(r-1)}$ (which exist by a repeated application of Rolle's Theorem). We may write

$$\phi_n(x_1) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} \dots \int_{a_{r-1}}^{x_{r-1}} \phi^{(r-1)}(x_r) dx_r dx_{r-1} \dots dx_2.$$

Taking $x_1 = 0$ and using the triangle inequality, one may deduce the following inequality:

$$(3.8) \quad |\phi_n(0)| = |F_n(k\pi, l\pi)| \leq \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} \|\phi^{(r-1)}\|_\infty.$$

As a result, for any $M > 0$, we get

$$(3.9) \quad \begin{aligned} \mathbb{P}(\mathcal{L}_{n,k,l} > t) &\leq \mathbb{P}\left(|F_n(k\pi, l\pi)| \leq \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} \|\phi^{(r-1)}\|_\infty\right) \\ &\leq \mathbb{P}\left(|F_n(k\pi, l\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) + \mathbb{P}\left(\|\phi^{(r-1)}\|_\infty > M\right). \end{aligned}$$

Recall that we have assumed that $(k\pi, l\pi)$ belongs to \mathcal{L} . In the general case, we rather have

$$\begin{aligned} \mathbb{P}(\mathcal{L}_{n,k,l} > t) &\leq \mathbb{P}\left(|F_n(k\pi, l\pi)| \leq \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} \|\phi^{(r-1)}\|_\infty\right) \\ &\quad + \mathbb{P}\left(|F_n((k+1)\pi, l\pi)| \leq \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} \|\phi^{(r-1)}\|_\infty\right) \\ &\quad + \mathbb{P}\left(|F_n(k\pi, (l+1)\pi)| \leq \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} \|\phi^{(r-1)}\|_\infty\right) \\ &\quad + \mathbb{P}\left(|F_n((k+1)\pi, (l+1)\pi)| \leq \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} \|\phi^{(r-1)}\|_\infty\right), \end{aligned}$$

which yields

$$\begin{aligned} \mathbb{P}(\mathcal{L}_{n,k,l} > t) &\leq \mathbb{P}\left(|F_n(k\pi, l\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) \\ &\quad + \mathbb{P}\left(|F_n((k+1)\pi, l\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) \\ &\quad + \mathbb{P}\left(|F_n(k\pi, (l+1)\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) \\ &\quad + \mathbb{P}\left(|F_n((k+1)\pi, (l+1)\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) \\ &\quad + 4\mathbb{P}\left(\|\phi^{(r-1)}\|_\infty > M\right). \end{aligned}$$

The last estimate requires the following bound.

Lemma 9.

$$\mathbb{P}\left(\|\phi^{(r-1)}\|_\infty > M\right) \leq \frac{C}{M}.$$

Proof. Set $K = [k\pi, (k+1)\pi] \times [l\pi, (l+1)\pi]$. Relying on [Ada75, pp. 107, Lemma 5.15, inequality (25)], there exists a positive constant C (not depending on k, l)

such that for any mapping f in $\mathcal{C}^2(K)$, one gets the inequality

$$(3.10) \quad \sup_{x \in K} |f(x)| \leq C \left(\int_K (f^2(x) + \|\nabla f(x)\|^2 + \|\nabla^2 f(x)\|^2) dx \right)^{\frac{1}{2}}.$$

The fact that the above constant C does not depend on k, l can be simply obtained by a mere change of variables. Recalling that (u, v) is a unit vector, we first notice that

$$\begin{aligned} |\phi^{(r-1)}(t)| &= \left| \sum_{i+j=r-1} \partial_1^i \partial_2^j F_n(k\pi + tu, l\pi + tv) u^i v^j \right| \\ &\leq \sum_{i+j=r-1} \sup_{x \in K} \left| \sum_{i+j=r-1} \partial_1^i \partial_2^j F_n(x) \right|. \end{aligned}$$

Thus, one is left to bound from above each partial derivative $\partial_1^i \partial_2^j F_n$ on the compact set K . Here, we apply the inequality (3.10) and we get

$$\sup_{x \in K} |\partial_1^i \partial_2^j F| \leq C_K \left| \int_K \left(\sum_{0 \leq q_1 + q_2 \leq 2} |\partial_1^{i+q_1} \partial_2^{j+q_2} F_n(x)|^2 \right) dx \right|^{\frac{1}{2}}.$$

However, for any couple of indexes (i, j) , setting

$$E_{(i,j)}(x, y) := \sum_{r,s \leq n-1} \left(\frac{r}{n}\right)^i \left(\frac{s}{n}\right)^j a_{r,s} \cos^{(i)}\left(\frac{rx}{n}\right) \cos^{(j)}\left(\frac{ry}{n}\right)$$

we have by Fubini and orthogonality of the random variables $\{a_{r,s}\}$ that

$$\begin{aligned} &\mathbb{E} \left(\int_K \left(\partial_1^i \partial_2^j F_n(x) \right)^2 dx \right) \\ &= \frac{1}{n^2} \mathbb{E} \left[\int_{k\pi}^{(k+1)\pi} \int_{l\pi}^{(l+1)\pi} |E_{(i,j)}(x, y)|^2 dx dy \right] \\ &= \frac{1}{n^2} \int_{k\pi}^{(k+1)\pi} \int_{l\pi}^{(l+1)\pi} \mathbb{E} \left[|E_{(i,j)}(x, y)|^2 \right] dx dy \\ &\leq \frac{1}{n^2} \sum_{r,s \leq n-1} \left(\frac{r}{n}\right)^{2i} \left(\frac{s}{n}\right)^{2j} \leq 1. \end{aligned}$$

One is then left to employ the Markov inequality in order to conclude the following proof:

$$\begin{aligned}
 & \mathbb{P} \left(\|\phi^{(r-1)}\|_\infty \geq M \right) \\
 \leq & \mathbb{P} \left(\sum_{i+j=r-1} \sup_{x \in K} \left| \partial_1^i \partial_2^j F_n \right| \geq M \right) \\
 \leq & \frac{1}{M} \sum_{i+j=r-1} \mathbb{E} \left[\sup_{x \in K} \left| \partial_1^i \partial_2^j F_n \right| \right] \\
 \leq & \frac{C}{M} \sum_{i+j=r-1} \mathbb{E} \left[\sqrt{\int_K \left(\sum_{0 \leq q_1+q_2 \leq 2} \left| \partial_1^{i+q_1} \partial_2^{j+q_2} F_n(x) \right|^2 \right) dx} \right] \\
 \leq & \frac{C}{M} \sum_{i+j=r-1} \sqrt{\mathbb{E} \left[\int_K \left(\sum_{0 \leq q_1+q_2 \leq 2} \left| \partial_1^{i+q_1} \partial_2^{j+q_2} F_n(x) \right|^2 \right) dx \right]} \\
 \leq & \frac{rC\sqrt{6}}{M}.
 \end{aligned}$$

□

Estimation of the small ball. From equation (3.9), upper bounding the probability $\mathbb{P}(\mathcal{L}_{n,k,l} > t)$ thus reduces to establish a small ball estimate for $F_n(k\pi, l\pi)$. In this paragraph, we shall indeed establish such a small ball estimate, for any $1 < \theta < \frac{3}{2}$:

$$(3.11) \quad \mathbb{P}(|F_n(k\pi, l\pi)| \leq \epsilon) \leq C \left(\epsilon + \frac{1}{n^\theta} \right),$$

provided that $\text{ord}(k) \geq \sqrt{n}$ and $\text{ord}(l) \geq \sqrt{n}$. To proceed, we use the method of characteristic functions. First of all (see, e.g., [FGG16]) we infer that for some absolute constant $C > 0$, we have

$$(3.12) \quad \mathbb{P}(|F_n(k\pi, l\pi)| \leq \epsilon) \leq C\epsilon \int_{\mathbb{R}} \Phi_{F_n(k\pi, l\pi)}(\xi) e^{-\frac{\epsilon^2 \xi^2}{2}} d\xi,$$

where $\Phi_{F_n(k\pi, l\pi)}(\cdot)$ is the characteristic function of $F_n(k\pi, l\pi)$. Note that if X is a random variable X such that $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = 1$, then we have $|\mathbb{E}(e^{i\xi X})| \leq \exp(-\xi^2/4)$ on an interval $[-\alpha, \alpha]$ for $\alpha > 0$ small enough. As a result we may first write

$$\begin{aligned}
 \int_{|\xi| \leq \alpha n} \Phi_{F_n(k\pi, l\pi)}(\xi) e^{-\frac{\epsilon^2 \xi^2}{2}} d\xi & \leq \int_{|\xi| \leq \alpha n} \Phi_{F_n(k\pi, l\pi)}(\xi) d\xi \\
 & \leq \int_{|\xi| \leq \alpha n} \prod_{1 \leq i, j \leq n} e^{-\frac{\xi^2}{4n^2} \cos^2(i \frac{k\pi}{n}) \cos^2(j \frac{l\pi}{n})} d\xi.
 \end{aligned}$$

However, based on the following doubling formula $\cos(2x) = 2 \cos(x)^2 - 1$, we have the following dichotomy: either $|\cos(x)| \geq \frac{1}{2}$ or $|\cos(2x)| \geq \frac{1}{2}$. We may then restrict our attention to the set of indexes (i, j) such that $|\cos(i \frac{k\pi}{n})| \geq \frac{1}{2}$ and $|\cos(j \frac{l\pi}{n})| \geq \frac{1}{2}$

whose cardinality is hence necessarily larger than $\frac{n^2}{4}$. This entails that

$$(3.13) \quad \prod_{1 \leq i, j \leq n} e^{-\frac{\xi^2}{4n^2} \cos^2(i \frac{k\pi}{n}) \cos^2(j \frac{l\pi}{n})} \leq \left(e^{-\frac{\xi^2}{64n^2}} \right)^{\frac{n^2}{4}} = e^{-\frac{\xi^2}{256}}.$$

However, since $\xi \mapsto e^{-\frac{\xi^2}{256}} \in L^1(\mathbb{R})$, the bound (3.13) implies the existence of an absolute constant $C > 0$ such that

$$(3.14) \quad \sup_{n \geq 1, \epsilon > 0} \int_{|\xi| \leq \alpha n} \Phi_{F_n(k\pi, l\pi)}(\xi) e^{-\frac{\epsilon^2 \xi^2}{2}} d\xi \leq C.$$

As a result, bounding the right-hand side of (3.12) requires the control of the integral

$$(3.15) \quad I_2 := \epsilon \int_{|\xi| \geq \alpha n} \Phi_{F_n(k\pi, l\pi)}(\xi) e^{-\frac{\epsilon^2 \xi^2}{2}} d\xi.$$

Now, relying on Lemma 8, we get

$$\Phi_{F_n(k\pi, l\pi)}(\xi) = \prod_{\substack{1 \leq i \leq \text{ord}(k) \\ 1 \leq j \leq \text{ord}(l)}} \Phi_a \left(\frac{\xi}{n} \cos \left(\frac{i\pi}{\text{ord}(k)} \right) \cos \left(\frac{l\pi}{\text{ord}(l)} \right) \right)^{\frac{n^2}{\text{ord}(k)\text{ord}(l)}},$$

where Φ_a naturally stands for the characteristic function of the common law of the coefficients. By writing $u = \frac{\xi}{n}$, the integral (3.15) becomes

$$(3.16) \quad I_2 := n\epsilon \int_{|u| > \alpha} \prod_{\substack{1 \leq i \leq \text{ord}(k) \\ 1 \leq j \leq \text{ord}(l)}} \Phi_a \left(u \cos \left(\frac{i\pi}{\text{ord}(k)} \right) \cos \left(\frac{l\pi}{\text{ord}(l)} \right) \right)^{\frac{n^2}{\text{ord}(k)\text{ord}(l)}} e^{-\frac{u^2 \epsilon^2 n^2}{2}} du.$$

Now, for fixed $A < B < 1$ and $u \in \mathbb{R}/\{0\}$, we denote by $\phi = \phi_{A,B,u} : [-1, 1] \rightarrow [0, 1]$ the Lipschitz function such that $\phi(x) = 1$ when $|\Phi_a(ux)| \leq A$, $\phi(x) = 0$ when $|\Phi_a(ux)| \geq B$, and ϕ is linear on $|\Phi_a(ux)| \in [A, B]$. Note that, for any $(x, y) \in \mathbb{R}^2$, if $\phi(x) = 1$ and $\phi(y) = 0$, then necessarily (since Φ_a is 1-Lipschitz)

$$|ux - uy| \geq |\Phi_a(ux) - \Phi_a(uy)| \geq B - A.$$

Besides, if $|\Phi_a(uz)| \in]A, B[$ one may always find an interval (x, y) containing z such that (i) $|\Phi_a(ux)| = A$ and $|\Phi_a(uy)| = B$, (ii) for all $w \in (x, y)$ it holds that $|\Phi_a(uw)| \in [A, B]$. Since by definition ϕ is linear on (x, y) , we may deduce that

$$(3.17) \quad |\phi'(z)| = \left| \frac{\phi(x) - \phi(y)}{x - y} \right| \leq \frac{|u|}{B - A}.$$

As a result, setting $\Psi(x, y) := \phi(\cos(\pi x) \cos(\pi y))$, recognizing a two-dimensional Riemann sum, we get

$$(3.18) \quad \left| \frac{1}{\text{ord}(k)\text{ord}(l)} \sum_{i=1}^{\text{ord}(k)} \sum_{j=1}^{\text{ord}(l)} \Psi \left(\frac{i}{\text{ord}(k)}, \frac{j}{\text{ord}(l)} \right) - \int_{[0,1]^2} \Psi(x, y) dx dy \right| \leq \frac{\|\nabla \Psi\|_\infty}{\min(\text{ord}(k), \text{ord}(l))} \leq C_{A,B} \frac{|u|}{\sqrt{n}}.$$

Note that, by construction, ϕ implicitly depends on u, A , and B . For the sake of clarity, we will not carry this dependency in our notation. Now denote by ρ the

density of the image measure of Lebesgue on $[0, 1]^2$ by the functional $(x, y) \mapsto \cos(\pi x)\cos(\pi y)$ so that we have $\int_{[0,1]^2} \Psi(x, y) dx dy = \int_{\mathbb{R}} \phi(t)\rho(t) dt$. Since $\rho \in L^1(\mathbb{R})$, it is a well-known fact that

$$\lim_{\delta \rightarrow 0} \sup_{\lambda(A) \leq \delta} \int_A \rho(t) dt = 0.$$

Let us fix $\delta_0 > 0$ such that $\sup_{\lambda(A) \leq \delta_0} \int_A \rho(t) dt < \frac{1}{2}$. Nevertheless, one may fix $A, B > 0$ (eventually close to 1) such that $\sup_{|u| > \alpha} \lambda(\{\phi \neq 1\}) < \delta_0$. Let us detail this assertion a bit. First of all, we notice that

$$\begin{aligned} \lambda(\{\phi \neq 1\}) &= \frac{1}{u} \int_0^u \mathbf{1}_{\{|\Phi_a(t)| > A\}} dt = \frac{1}{u} \int_0^u \mathbf{1}_{\{|\Phi_a(t)|^2 > A^2\}} dt \\ &\leq \frac{1}{uA^2} \int_0^u |\Phi_a(t)|^2 dt = \frac{1}{A^2} \mathbb{E}(\sin_c(u(a_1 - a_2))). \end{aligned}$$

Assuming first that $a_1 - a_2$ does not have an atom at zero, the dominated convergence theorem ensures that $\mathbb{E}(\sin_c(u(a_1 - a_2)))$ goes to zero as u goes to infinity. Besides, for every fixed u , it holds that $\int_0^u \mathbf{1}_{\{|\Phi_a(t)| < A\}} dt$ goes to zero as A tends to one. Together, these two conditions ensure the desired result, namely

$$\lim_{A \rightarrow 1} \sup_{|u| > c} \frac{1}{u} \int_0^u \mathbf{1}_{\{|\Phi_a(t)| > A\}} dt = 0.$$

Assume now that $a_1 - a_2$ has an atom at zero. Note that $a_1 - a_2$ is not a constant variable since its variance is positive. Thus, for some $0 < c < 1$, one can write $\Phi_{a_1 - a_2} = |\Phi_a|^2 = c + (1 - c)\Psi$, where Ψ is the characteristic function of the law of $a_1 - a_2$ conditional to $a_1 \neq a_2$. Since, $\mathbf{1}_{\{|\Phi_a(t)|^2 > A^2\}} \leq \mathbf{1}_{\{|\Psi| > \frac{A^2 - c}{1 - c}\}}$ (with $\frac{A^2 - c}{1 - c} \rightarrow 1$ as $A \rightarrow 1$), we may apply the previous reasoning to the characteristic function Ψ which by construction does not have an atom at zero. Under these conditions we infer that

$$\int_{[0,1]^2} \Psi(x, y) dx dy = \int_{\mathbb{R}} \phi(t)\rho(t) dt \geq \int_{\{\phi=1\}} \rho(t) dt \geq \frac{1}{2}.$$

Relying on the bound (3.18), if one assumes that $|u| \leq \frac{\sqrt{n}}{4C_{A,B}}$, then we get the following crucial estimate:

$$(3.19) \quad \sum_{\substack{1 \leq i \leq \text{ord}(k) \\ 1 \leq j \leq \text{ord}(l)}} \phi\left(\cos\left(\frac{i\pi}{\text{ord}(k)}\right)\cos\left(\frac{l\pi}{\text{ord}(l)}\right)\right) \geq \frac{1}{4} \text{ord}(k)\text{ord}(l),$$

which implies that the cardinality of a couple of indexes (i, j) such that

$$\left| \Phi_a\left(u \cos\left(\frac{i\pi}{\text{ord}(k)}\right)\cos\left(\frac{l\pi}{\text{ord}(l)}\right)\right) \right| \leq B,$$

is greater than $\frac{1}{4}\text{ord}(k)\text{ord}(l)$ provided that $\frac{\sqrt{n}}{4C_{A,B}} > |u| > \alpha$. Coming back to (3.16), we may infer that

$$\begin{aligned} I_2 &\leq n\epsilon \int_{\frac{\sqrt{n}}{4C_{A,B}} > |u| > \alpha} \Phi_{F_n(k\pi, l\pi)}(nu) e^{-\frac{u^2 \epsilon^2 n^2}{2}} du \\ &+ n\epsilon \int_{\frac{\sqrt{n}}{4C_{A,B}} < |u|} \Phi_{F_n(k\pi, l\pi)}(nu) e^{-\frac{u^2 \epsilon^2 n^2}{2}} du \\ &\leq \frac{B \frac{n^2}{4}}{4C_{A,B}} n\sqrt{n} + n\epsilon \int_{\frac{\sqrt{n}}{4C_{A,B}} < |u|} e^{-\frac{u^2 \epsilon^2 n^2}{2}} du \\ &= \frac{B \frac{n^2}{4}}{4C_{A,B}} n\sqrt{n} + \int_{\frac{n\sqrt{n}\epsilon}{4C_{A,B}} < |x|} e^{-\frac{x^2}{2}} dx. \end{aligned}$$

Now let us give the final argument of this proof. If $\epsilon \geq \frac{1}{n^\theta}$, then $n\sqrt{n}\epsilon \geq n^{\frac{3}{2}-\theta}$ and $\int_{\frac{n\sqrt{n}\epsilon}{4C_{A,B}} < |x|} e^{-\frac{x^2}{2}} dx = o\left(\frac{1}{n^\theta}\right)$. Otherwise, if $\epsilon < \frac{1}{n^\theta}$, then

$$\begin{aligned} \mathbb{P}(|F_n(k\pi, l\pi)| \leq \epsilon) &\leq \mathbb{P}\left(|F_n(k\pi, l\pi)| \leq \frac{1}{n^\theta}\right) \\ &\leq C\left(\epsilon + \frac{1}{n^\theta}\right). \end{aligned}$$

Synthesis. This paragraph makes the synthesis of the two previous subsections. Note that, in the sequel, C stands for some universal constant which may change from line to line. Up to using Lemma 7 and doubling the size of the square on which we consider the nodal line, we will assume that $\text{ord}(k), \text{ord}(l), \text{ord}(k+1), \text{ord}(l+1) \geq \sqrt{n}$. As a matter of fact, relying on the main estimate (3.11) and Lemma 9, we get that

$$\mathbb{P}\left(|F_n(k\pi, l\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) \leq C\left(M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!} + \frac{1}{n^\theta} + \frac{C}{M}\right).$$

Making an optimization on M , we get

$$\mathbb{P}\left(|F_n(k\pi, l\pi)| \leq M \frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}\right) \leq C\left(\sqrt{\frac{(\pi\sqrt{2})^{r-1}}{(r-1)!}} + \frac{1}{n^\theta}\right).$$

As a result, provided that $\theta > 1 + \alpha$, we get the existence of an absolute constant $C > 0$ such that

$$(3.20) \quad \sup_{n, l, k} \mathbb{E}(\mathcal{L}_{n, k, l}^{1+\alpha}) < C.$$

3.3. End of the proof. In this final subsection, we make a compilation of the content of all previous subsections to establish the global universality result stated in the introduction.

Theorem 8. *Whatever the law of the entries $(a_{k,l})_{k, l \geq 1}$, as n tends to infinity, we have*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\ell_{[0, \pi]^2}(f_n)]}{n} = \frac{\pi^2}{2\sqrt{3}}.$$

Proof. Let us recall equation (3.4) which expresses the global expectation as the sum of the microscopic contributions

$$\frac{\mathbb{E}[\ell_{[0,\pi]^2}(f_n)]}{n} = \frac{1}{n^2} \sum_{0 \leq k,l \leq n-1} \mathbb{E}(\mathcal{L}_{n,k,l}).$$

Let us fix $\epsilon > 0$, and introduce $I_\epsilon := [\epsilon, 1 - \epsilon]$ and

$$\mathcal{A}_{n,\epsilon} := (nI_\epsilon \cap \mathbb{N})^2.$$

One first notices that $\#(\mathcal{A}_{n,\epsilon}) \approx (1 - 2\epsilon)^2 n^2$. Next, using the bound (3.20), we may infer that

$$(3.21) \quad \frac{1}{n^2} \sum_{(k,l) \in \mathcal{A}_{n,\epsilon}^c} \mathbb{E}(\mathcal{L}_{n,k,l}) \leq \frac{C}{n^2} \#(\mathcal{A}_{n,\epsilon}^c) \leq C(1 - (1 - 2\epsilon)^2).$$

Let us denote by $\mathcal{L}_{\infty,k,l}$ the length of the nodal set of the limit Gaussian process F_∞ in the square $[k\pi, (k + 1)\pi] \times [l\pi, (l + 1)\pi]$. Now we shall prove that

$$(3.22) \quad \lim_{n \rightarrow +\infty} \sup_{(k,l) \in \mathcal{A}_{n,\epsilon}} |\mathbb{E}[\mathcal{L}_{n,k,l}] - \mathbb{E}[\mathcal{L}_{\infty,k,l}]| = 0.$$

To do so, we denote by $(p_n, q_n) \in \mathcal{A}_{n,\epsilon}$ one pair of integers for which the above maximum is reached. Next, thanks to Proposition 2 and Remark 3, we infer that the process

$$G_n(\cdot, \cdot) = F_n(p_n\pi + \cdot, q_n\pi + \cdot)$$

converges to the nondegenerate stationary Gaussian process G_∞ . Besides, relying on Proposition 3, the same conclusion holds for the process $F_\infty(p_n\pi + \cdot, q_n\pi + \cdot)$. Hence, via the content of Subsection 2.3, we indeed obtain that

$$(3.23) \quad \lim_{n \rightarrow +\infty} \mathbb{E}[\phi(\mathcal{L}_{n,p_n,q_n})] - \mathbb{E}[\phi(\mathcal{L}_{\infty,p_n,q_n})] = 0,$$

for any continuous bounded function ϕ . Finally, for any $M > 0$, we have

$$\begin{aligned} & \frac{1}{n^2} \sum_{0 \leq k,l \leq n-1} \mathbb{E}[\mathcal{L}_{n,k,l} \mathbf{1}_{\{\mathcal{L}_{n,k,l} > M\}}] \\ & \leq \frac{C}{n^2} \sum_{0 \leq k,l \leq n-1} \mathbb{P}(\mathcal{L}_{n,k,l} > M)^{\frac{\alpha}{1+\alpha}} \\ & \leq \frac{C'}{n^2} \sum_{0 \leq k,l \leq n-1} \frac{1}{M^{\frac{\alpha}{1+\alpha}}} = \frac{C'}{M^{\frac{\alpha}{1+\alpha}}}. \end{aligned}$$

As a result, using the limit (3.23) and taking M large enough, we indeed get the asymptotics (3.22). Finally, putting (3.21) and (3.22) together with Theorem 5, we get that

$$(3.24) \quad \lim_{n \rightarrow +\infty} \left| \frac{1}{n^2} \sum_{k,l \leq n} \mathbb{E}(\mathcal{L}_{n,k,l}) - \frac{\pi^2}{2\sqrt{3}} \right| = 0,$$

which is the desired result. □

ACKNOWLEDGMENT

Viet-Hung Pham would like to thank the Vietnamese Institute for Advanced Study in Mathematics for a postdoctoral fellowship. Guillaume Poly is grateful to the Vietnamese Institute for Advanced Study for funding a one-month research stint in Hanoi, which led to a part of this research. We are grateful to an anonymous referee for valuable comments.

REFERENCES

- [Ada75] Robert A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0450957
- [ADL15] Jean-Marc Azaïs, Federico Dalmao, José León, Ivan Nourdin, and Guillaume Poly, *Local universality of the number of zeros of random trigonometric polynomials with continuous coefficients*, arXiv preprint arXiv:1512.05583, 2015.
- [AP15] Jürgen Angst and Guillaume Poly, *Universality of the mean number of real zeros of random trigonometric polynomials under a weak cramer condition*, arXiv preprint arXiv:1511.08750, 2015.
- [AW09] Jean-Marc Azaïs and Mario Wschebor, *Level sets and extrema of random processes and fields*, John Wiley & Sons, Inc., Hoboken, NJ, 2009. MR2478201
- [Che01] Ward Cheney, *Analysis for applied mathematics*, Graduate Texts in Mathematics, vol. 208, Springer-Verlag, New York, 2001. MR1838468
- [FGG16] Omer Friedland, Ohad Giladi, and Olivier Guédon, *Small ball estimates for quasi-norms*, J. Theoret. Probab. **29** (2016), no. 4, 1624–1643, DOI 10.1007/s10959-015-0622-z. MR3571257
- [Fla16] Hendrik Flasche, *Expected number of real roots of random trigonometric polynomials*, arXiv preprint arXiv:1601.01841, 2016.
- [FLL15] Yan V. Fyodorov, Antonio Lerario, and Erik Lundberg, *On the number of connected components of random algebraic hypersurfaces*, J. Geom. Phys. **95** (2015), 1–20, DOI 10.1016/j.geomphys.2015.04.006. MR3357820
- [GW16] Damien Gayet and Jean-Yves Welschinger, *Betti numbers of random real hypersurfaces and determinants of random symmetric matrices*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 4, 733–772, DOI 10.4171/JEMS/601. MR3474455
- [IKM16] Alexander Iksanov, Zakhar Kabluchko, and Alexander Marynych, *Local universality for real roots of random trigonometric polynomials*, Electron. J. Probab. **21** (2016), Paper No. 63, 19, DOI 10.1214/16-EJP9. MR3563891
- [IM71] Il'dar A. Ibragimov and Nina B. Maslova, *On the expected number of real zeros of random polynomials i. coefficients with zero means*, Theory of Probability & Its Applications, 16(2):228–248, 1971.
- [Let16a] Thomas Letendre, *Expected volume and Euler characteristic of random submanifolds*, J. Funct. Anal. **270** (2016), no. 8, 3047–3110, DOI 10.1016/j.jfa.2016.01.007. MR3470435
- [Let16b] Thomas Letendre, *Variance of the volume of random real algebraic submanifolds*, arXiv preprint arXiv:1608.05658, 2016.
- [LS71] L. A. Lusternik and V. J. Sobolev, *Elements of functional analysis*, Revised corrected edition, translated from the Russian, International Monographs on Advanced Mathematics & Physics, Hindustan Publishing Corp., Delhi, 1971. MR0342977
- [MPRW16] Domenico Marinucci, Giovanni Peccati, Maurizia Rossi, and Igor Wigman, *Non-universality of nodal length distribution for arithmetic random waves*, Geom. Funct. Anal. **26** (2016), no. 3, 926–960, DOI 10.1007/s00039-016-0376-5. MR3540457
- [NS10] Fedor Nazarov and Mikhail Sodin, *Random complex zeroes and random nodal lines*, Proceedings of the International Congress of Mathematicians. Volume III, Hindustan Book Agency, New Delhi, 2010, pp. 1450–1484. MR2827851
- [ORW08] Ferenc Oravecz, Zeév Rudnick, and Igor Wigman, *The Leray measure of nodal sets for random eigenfunctions on the torus* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 299–335. MR2401223

- [RS01] Alexander Rusakov and Oleg Seleznev, *On weak convergence of functionals on smooth random functions*, Math. Commun. **6** (2001), no. 2, 123–134. MR1908331
- [RW08] Zeév Rudnick and Igor Wigman, *On the volume of nodal sets for eigenfunctions of the Laplacian on the torus*, Ann. Henri Poincaré **9** (2008), no. 1, 109–130, DOI 10.1007/s00023-007-0352-6. MR2389892
- [SZ99] Bernard Shiffman and Steve Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), no. 3, 661–683, DOI 10.1007/s002200050544. MR1675133
- [TV14] Terence Tao and Van Vu, *Local universality of zeroes of random polynomials*, Int. Math. Res. Not. IMRN **13** (2015), 5053–5139, DOI 10.1093/imrn/rnu084. MR3439098
- [Wig10] Igor Wigman, *Fluctuations of the nodal length of random spherical harmonics*, Comm. Math. Phys. **298** (2010), no. 3, 787–831, DOI 10.1007/s00220-010-1078-8. MR2670928
- [Wil91] J. Ernest Wilkins Jr., *Mean number of real zeros of a random trigonometric polynomial*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 851–863, DOI 10.2307/2048424. MR1039266

IRMAR, UNIVERSITY OF RENNES 1, RENNES, FRANCE
Email address: jurgen.angst@univ-rennes1.fr

VIETNAMESE INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, HA NOI, VIET NAM
Email address: pgviethung@gmail.com

IRMAR, UNIVERSITY OF RENNES 1, RENNES, FRANCE
Email address: guillaume.poly@univ-rennes1.fr