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HILBERT-KUNZ DENSITY FUNCTION AND HILBERT-KUNZ MULTIPLICITY

V. TRIVEDI

ABSTRACT. For a pair (M, I), where M is a finitely generated graded module over a standard graded ring R of dimension d, and I is a graded ideal with $\ell(R/I) < \infty$, we introduce a new invariant $HKd(M, I)$ called the *Hilbert-Kunz density function*. We relate this to the Hilbert-Kunz multiplicity $e_{HK}(M, I)$ by an integral formula.

We prove that the Hilbert-Kunz density function satisfies a multiplicative formula for a Segre product of rings. This gives a formula for e_{HK} of the Segre product of rings in terms of the HKd of the rings involved. As a corollary, e_{HK} of the Segre product of any finite number of projective curves is a rational number.

1. INTRODUCTION

Let R be a Noetherian ring of prime characteristic $p > 0$ and of dimension d and let $I \subseteq R$ be an ideal of finite colength. Then we recall that the Hilbert-Kunz multiplicity of R with respect to I is defined as

$$e_{HK}(R, I) = \lim_{n\to\infty} \frac{\ell(R/I^{[p^n]})}{p^{nd}},$$

where $I^{[p^n]}$ = nth Frobenius power of I = the ideal generated by p^nth powers of elements of I. This is an ideal of finite colength and $\ell(R/I^{[p^n]})$ denotes the length of the R-module $R/I^{[p^n]}$. Existence of the limit was proved by Monsky [M]. Though this invariant has been extensively studied, over the years (see the survey article [Hu]), it has been difficult to handle (even in the graded case) as various standard techniques, used for studying multiplicities, are not applicable for the invariant e_{HK}.

Here we introduce a new invariant for a pair (M, I), where R is a Noetherian standard graded ring of dimension d over a perfect field k of char $p > 0$, I is a homogeneous ideal of R such that $\ell(R/I) < \infty$, and M is a finitely generated non-negatively graded R-module.

This invariant for a pair (M, I), which we call the *Hilbert-Kunz density function* of (M, I), is a compactly supported function

$$HKd(M, I) : \mathbb{R} \longrightarrow \mathbb{R},$$

given by

$$HKd(M, I)(x) = f(x) = \lim_{n\to\infty} g_n(x),$$

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where $g_n : \mathbb{R} \to \mathbb{R}$ is given in Notation 2.1. We show that this limit makes sense and in fact

$$HKd(M,I)(x) = f(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x),$$

where $f_n(x) = (1/q^{d-1})\ell(M/I^{[q]}M)_{\lfloor xq \rfloor}$. More precisely we prove the following theorem, which also relates the Hilbert-Kunz multiplicity with the Hilbert-Kunz density function.

Theorem 1.1. *If R is of dimension ≥ 2, then each $g_n : \mathbb{R} \longrightarrow \mathbb{R}$ is a compactly supported, piecewise linear continuous function such that $\{g_n\}_{n\in\mathbb{N}}$ is a uniformly convergent sequence. If $\lim_{n\to\infty} g_n(x) = f(x)$, then $f(x)$ is a compactly supported continuous function, and*

$$e_{HK}(M,I) = \int_{\mathbb{R}} f(x)dx.$$

We note that the HK density function plays the same role as e_{HK} vis-a-vis tight closure, in the graded setup (see Remark 2.15). Also like the e_{HK} multiplicity, the HK density function is additive (Proposition 2.14).

One of the remarkable properties of the HK density function (which also makes computations of e_{HK} in various cases simpler, and makes them possible in many new cases) is that it is "multiplicative" for Segre products.

In Proposition 2.17 we state and prove this *multiplicative formula*. In particular, we prove the following.

Proposition 1.2. *If (R,I) and (S,J) are two pairs as above, and if $HKd(R,I) = f$ and $HKd(S,J) = g$ with* dim $R = d_1$ *and* dim $S = d_2$, *then their Segre product satisfies:*

$$HKd(R\#S, I\#J)(x) = \frac{e_0(R)}{(d_1-1)!}x^{d_1-1}g(x) + \frac{e_0(S)}{(d_2-1)!}x^{d_2-1}f(x) - f(x)g(x).$$

Here $e_0(R)$ denotes the Hilbert-Samuel multiplicity of R with respect to its irrelevant maximal ideal.

This implies that e_{HK} of any finite Segre product of rings can be written in terms of the HKd functions of the rings involved, whereas Example 3.7 suggests that any such "multiplicative formula" does not hold for HK multiplicities.

In Section 3 we compute $HKd(R,I)$, for projective spaces and non-singular projective curves (and hence of arbitrary Segre products of these). Theorem 1.1 then yields formulas for HK multiplicities. We note that the HK multiplicity of a product of $\mathbb{P}^n \times \mathbb{P}^m$ was known earlier ([EY]).

In the case of a non-singular projective curve $X = \text{Proj } R$ of degree d, we can associate its *HN data* of a set of rational numbers $(d, \{r_i\}_i, \{a_i\}_i)$, where (see [B], [T1], for the corresponding study of the HK multiplicity in this context) $\{r_i\}_i$ and $\{a_i\}_i$ denote, respectively, the ranks and normalized strong Harder-Narasimhan slopes of the associated syzygy bundle V on X see (3.1) for details). Then it turns out that the density function $HKd(R,\mathbf{m})$, is a piecewise linear polynomial with rational coefficients, and with points of singularites (i.e., non-smoothness) precisely at the points $\{1-(a_i/d)\}_i$. Moreover d and and the set $\{r_i\}_i$ can also be easily recovered from the density function (see Example 3.3).

This implies that (since $HKd(R,\mathbf{m})$ and hence) the numbers $\{r_i\}$, $\{a_i\}$ are *intrinsic invariants* of the pair $(R,\mathbf{m})$.

Now, by Proposition 2.17, the HK density function of a Segre product of n projective curves $\{X_j\}_j$, corresponding to the pairs $(R_j, I_j)_j$, is a piecewise degree n-polynomial with rational coefficients, with the set of singular points $\subseteq \{1 - a_{ij}/\widetilde{d}_j, d_{ij}\}_i$, where $\{a_{ij}\}_i$ are the normalized strong HN slopes and $\widetilde{d}_j$ is the degree of the curves X_j and d_{ij} are the degrees of the chosen generators of the ideals I_j. Hence, by Theorem 1.1, we deduce, as a corollary,

The HK multiplicity of the Segre product of any finite number of projective curves is a rational number.

In Example 3.7 we write down the Hilbert-Kunz density function for the Segre product of two dimensional rings $(R, \mathbf{m}_1)$ and $(S, \mathbf{m}_2)$. If $(d, \{r_i\}_i, \{a_i\}_i)$ and $(g, \{s_j\}_j, \{b_j\}_j)$ are the datum associated to the pairs $(R, \mathbf{m}_1)$ and $(S, \mathbf{m}_2)$, respectively, then we deduce that $e_{HK}(R\#S, \mathbf{m}_1\#\mathbf{m}_2)$ is a polynomial in $\{r_i, a_i/d\}_i$ and $\{s_j, b_j/g\}_j$ but the formula for it depends on the relative positions of the a_i/d and b_j/g on the real line. On the other hand, we know (see [B], [T1]) that $e_{HK}(R, \mathbf{m}_1) = d + \sum_i r_i a_i^2/d$ and $e_{HK}(S, \mathbf{m}_2) = g + \sum_j s_j b_j^2/g$.

This suggests that unlike the functions such as multiplicity and HKd function, *e_{HK} of a Segre product of rings cannot be determined in terms of the e_{HK} of the individual rings alone.*

Overall it seems that HKd is relatively easier to calculate (as one is computing a "limit" of each graded piece rather than computing a limit of a sum of graded pieces). On the other hand, it carries more information (e.g., in the case of projective curves, the normalized slopes $\{a_i/d\}$ are precisely the points of singularities of the HKd, and $\{r_i\}$ also are recoverable from the density function).

In [T2], we give another application of HK density functions to give an approach to e_{HK} in characteristic 0.

We expect the techniques introduced in this paper to have several other interesting applications as well.

For example, in a forthcoming paper [Ma], it is shown that the HK density function of a tensor product of standard graded rings equals the *convolution* of the HK density of the factors.

Recall that the set of compactly supported continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ are in bijective correspondence with the set of their holomorphic Fourier transforms $\hat{f}$, where $\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}dx$ for $t \in \mathbb{C}$. Since HK density functions are compactly supported functions on $\mathbb{R}$, for a pair (M, I), the HK density function $f = HKd(M, I)$ corresponds to its Fourier transform $\hat{f}$ and moreover $\hat{f}(0) = e_{HK}(M, I)$. We also know that the Fourier transform of the convolution of two such functions is the pointwise product of their Fourier transforms.

In particular, the results in the present paper suggest possible applications of techniques from harmonic analysis in the study of HK multiplicities; we hope to return to this later.

One can also ask the following question.

Question 1.3. Can this notion of HK density function be generalized to a Noetherian local ring $(R, \mathbf{m})$, with respect to the $\mathbf{m}$-adic filtration?

The paper is organized as follows. In the second section we prove the main existence theorem, namely Theorem 1.1. In Lemma 2.6 (which is the heart of the theorem), we prove that the cohomologies of nth Frobenius pull back of a locally free sheaf (as is given in equation (2.3)) twisted by $\mathcal{Q}(m)$ ($\mathcal{Q}$ is a coherent sheaf of dimension $\bar{d}$) is bounded by a polynomial in m, p^n of degree $\bar{d}$ with invariants of $\mathcal{Q}$ as the coefficients.

The main theorem is inspired by the usual philosophy that *the map from R to $R^{1/p}$ is essentially the map from $R^{1/q}$ to $R^{1/qp}$* (for this we state and prove a sheaf theoretic version in Lemma 2.9).

We also look at the case of dimension 1 in Theorem 2.19, and note that, for each x, the sequence of functions $g_n(x)$ converges pointwise to $f(x)$ but need not converge uniformly. However, $\int_{\mathbb{R}} f(x)dx$ still gives the HK multiplicity.

2. Main existence theorem

Throughout the paper, R is a Noetherian standard graded ring of dimension d over a perfect field k of char $p > 0$, I is a homogeneous ideal of R such that $\ell(R/I) < \infty$, and M is a finitely generated non-negatively graded R-module.

Notation 2.1. For the pair (M, I) we define a sequence of functions $\{g_n : \mathbb{R} \to \mathbb{R}\}$, as follows: Fix $n \in \mathbb{N}$ and denote $q = p^n$. Let $x \in \mathbb{R}$; then $x \in [m/q, (m+1)/q)$ for some integer m. If $x = m/q$, then define

$$g_n(x) = 1/q^{d-1}\ell(M/I^{[q]}M)_m.$$

Otherwise, we can write $x = (1-t)m/q + t(m+1)/q$, for some unique $t \in [0,1)$, and then we define

$$g_n(x) = (1-t)g_n(m/q) + tg_n((m+1)/q).$$

Let $\mu \geq \mu(I)$ be a fixed number, where $\mu(I)$ is the minimal number of generators of the ideal I.

Lemma 2.2. *Each g_n is a compactly supported continuous function. Moreover, there is a fixed compact set containing $\bigcup_n \operatorname{supp} g_n$.*

Proof. The continuity property is obvious. Let $n_0 \in \mathbb{N}$ such that $\mathbf{m}^{n_0} \subseteq I$, where $\mathbf{m}$ is the graded maximal ideal. Therefore, for $m \geq n_0\mu q$, we have $R_m \subseteq (\mathbf{m}^{n_0})^{\mu q} \subseteq I^{\mu q} \subseteq I^{[q]}$. Let l be a positive integer such that $R_m M_l = M_{m+l}$ for $m \geq 0$. Then for $m \geq n_0\mu q + l$, we have

$$M_m = R_{m-l}M_l \subseteq (\mathbf{m}^{n_0})^{\mu q}M_l \subseteq I^{\mu q}M_l \subseteq I^{[q]}M_l.$$

This implies $\ell(M/I^{[q]}M)_m = 0$, if $m \geq l + n_0\mu q$ and, therefore, support of $g_n \subseteq [0, (n_0\mu) + l/q]$. $\square$

Remark 2.3. Since replacing R and M by $R \otimes_k \bar{k}$ and $M \otimes_k \bar{k}$, the function $g_n : \mathbb{R} \longrightarrow \mathbb{R}$ remains unchanged. We can assume without loss of generality that k is algebraically closed.

Henceforth we assume that R is a standard graded ring of dimension ≥ 2 (unless otherwise stated). Let I be generated by homogeneous elements, say $h_1, \ldots, h_\mu$ of positive degrees $d_1, \ldots, d_\mu$, respectively. Let $X = \operatorname{Proj} R$; then we have an associated canonical exact sequence of locally free sheaves of $\mathcal{O}_X$-modules (moreover the sequence is locally split exact). Due to Remark 2.3, we can also assume k is an algebraically closed field

$$0 \longrightarrow V \longrightarrow \oplus_i \mathcal{O}_X(1-d_i) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0, \tag{2.1}$$

where $\mathcal{O}_X(1-d_i) \longrightarrow \mathcal{O}_X(1)$ is given by the multiplication by the element h_i.

For a coherent sheaf $\mathcal{Q}$ of $\mathcal{O}_X$-modules we have a long exact sequence of cohomologies

(2.2)

$$0 \longrightarrow H^0(X, F^{n*}V \otimes \mathcal{Q}(m)) \longrightarrow \oplus_i H^0(X, \mathcal{Q}(q-qd_i+m)) \stackrel{\phi_{m,q}(\mathcal{Q})}{\rightarrow} H^0(X, \mathcal{Q}(q+m))$$
$$\longrightarrow H^1(X, F^{n*}V \otimes \mathcal{Q}(m)) \longrightarrow \cdots$$

for $m \geq 0$, $n \geq 0$, and $q = p^n$. (Here $F^n : X \longrightarrow X$ is the nth iterated Frobenius map.)

We fix a set of notation used throughout the paper.

Notation 2.4. Let $Q = \bigoplus_{m \geq 0} Q_m$ be a non-negatively graded finitely generated R-module and let $\mathcal{Q}$ be the associated coherent sheaf of $\mathcal{O}_X$-modules. Therefore, $Q_m = H^0(X, \mathcal{Q}(m))$ for $m \gg 0$.

(1) $\widetilde{m} \geq 1$ is the least integer such that,
$$Q_{m+1} = R_1 Q_m \text{ and } Q_m = H^0(X, \mathcal{Q}(m)) \text{ and } h^i(X, \mathcal{Q}(m-i)) = 0$$
for all $m \geq \widetilde{m}$ and for all $i \geq 1$.

(2) $\bar{d} =$ the dimension of the support of $\mathcal{Q}$ as a sheaf of $\mathcal{O}_X$-modules.

(3) Let
$$m_Q(q) = \widetilde{m} + n_0(\sum_i d_i)q,$$
where $h_1, \ldots, h_\mu$ are generators of the ideal I of degrees $d_1, \ldots, d_\mu \geq 1$, respectively, and $n_0 \geq 1$ such that $\mathbf{m}^{n_0} \subseteq I$.

(4) We also denote $\dim_k \operatorname{Coker} \phi_{m,q}(\mathcal{Q})$ by $\operatorname{coker} \phi_{m,q}(\mathcal{Q})$ (see the exact sequence (2.2) above).

(5) Let $a_1, \ldots, a_{\bar{d}} \in H^0(X, \mathcal{O}_X(1))$ be such that we have a short exact sequence of $\mathcal{O}_X$-modules
$$0 \to \mathcal{Q}_i(-1) \stackrel{a_i}{\to} \mathcal{Q}_i \to \mathcal{Q}_{i-1} \to 0 \quad \text{for} \quad 0 < i \leq \bar{d},$$
where $\mathcal{Q}_{\bar{d}} = \mathcal{Q}$ and $\mathcal{Q}_i = \mathcal{Q}/(a_{\bar{d}}, \ldots, a_{i+1})\mathcal{Q}$, for $0 \leq i < \bar{d}$, with $\dim\ \mathcal{Q}_i = i$. (Such a sequence of $\{a_i\}_i$ exists, because k is an infinite field, and since any coherent sheaf on X has only finitely many associated points.) We define

(a) $C_0(\mathcal{Q}) = h^0(X, \mathcal{Q})$ if $\bar{d} = 0$. If $\bar{d} > 0$, then
$$C_0(\mathcal{Q}) = \min\{\sum_{i=0}^{\bar{d}} h^0(X, \mathcal{Q}_i) \mid a_1, \ldots, a_{\bar{d}} \text{ is a } \mathcal{Q}-\text{sequence as above}\},$$

(b)
$$C_Q = (\mu)\left(h^0(X, \mathcal{Q}(\widetilde{m}-1)) + \max\{\ell(Q_0), \ell(Q_1), \ldots, \ell(Q_{\widetilde{m}-1})\}\right).$$

(c)
$$D_Q = C_0(\mathcal{Q}) \left[2\bar{d}(\widetilde{m} + n_0(\sum_1^\mu d_i))\right]^{\bar{d}},$$
where n_0 , μ, d_i are given as in (3) above.

(d)
$$D_1(\mathcal{Q}) = \max\{h^1(X, \mathcal{Q}), h^1(X, \mathcal{Q}(1)), \ldots, h^1(X, \mathcal{Q}(\widetilde{m}-1))\}.$$

(e)

$$D_0(\mathcal{Q}) = h^0(X, \mathcal{Q}(\widetilde{m})) + 2(\bar{d}+1)\left(\max\{q_0, |q_1|, \ldots, |q_{\bar{d}}|\}\right),$$

where

$$\chi(X, \mathcal{Q}(m)) = q_0\binom{m+\bar{d}}{\bar{d}} + q_1\binom{m+\bar{d}-1}{\bar{d}-1} + \cdots + q_{\bar{d}}$$

is the Hilbert polynomial of $\mathcal{Q}$.

The following lemma allows us to reduce our various assertions about a graded module to assertions about cohomologies of the sheaf associated to the graded module.

Lemma 2.5.

(1) *For $m+q \geq m_Q(q)$, we have* $\operatorname{coker}\phi_{m,q}(\mathcal{Q}) = \ell(Q/I^{[q]}Q)_{m+q} = 0$.
(2) *For all $n \geq 0$ and $m \in \mathbb{Z}$ (where we define $Q_m = 0$, for $m < 0$),*

$$|\operatorname{coker}\phi_{m,q}(\mathcal{Q}) - \ell(Q/I^{[q]}Q)_{m+q}| \leq C_Q.$$

Proof. For given $q = p^n$ and $m \geq 0$, let $\phi_{m,q}(Q) : \bigoplus_i Q_{q-qd_i+m} \longrightarrow Q_{m+q}$ be the map such that $Q_{q-qd_i+m} \to Q_{m+q}$ is given by multiplication by the element h_i^q.

(1) To prove the first assertion note that

$$m_Q(q) = \widetilde{m} + n_0(\sum_{i=1}^{\mu} d_i q) \geq \widetilde{m} + d_i q \implies q - qd_i + m \geq \widetilde{m}$$

for all i. Hence the map $\phi_{m,q}(Q)$ is the map $\phi_{m,q}(\mathcal{Q})$ and, therefore, $\operatorname{coker}\phi_{m,q}(\mathcal{Q}) = \ell(Q/I^{[q]}Q)_{m+q}$. Now, by the proof of Lemma 2.2, we have $\ell(Q/I^{[q]}Q)_{m+q} = 0$, as $m+q \geq \widetilde{m} + n_0\mu q$, since $\sum_i d_i \geq \mu$.

(2) Note that $h^0(X, \mathcal{Q}(t)) \leq h^0(X, \mathcal{Q}(\widetilde{m}-1))$ for all $t \leq \widetilde{m}-1$.

If $m+q < \widetilde{m}$, then

$$|\operatorname{coker}\phi_{m,q}(\mathcal{Q}) - \ell(Q/I^{[q]}Q)_{m+q}| \leq h^0(X, \mathcal{Q}(m+q)) + \ell(Q_{m+q})$$
$$\leq h^0(X, \mathcal{Q}(\widetilde{m}-1)) + \max\{\ell(Q_0), \ell(Q_1), \ldots, \ell(Q_{\widetilde{m}-1})\}.$$

If $m+q \geq \widetilde{m}$, then $h^0(X, \mathcal{Q}(m+q)) = \ell(Q_{m+q})$ and therefore

$$|\operatorname{coker}\phi_{m,q}(\mathcal{Q}) - \ell(Q/I^{[q]}Q)_{m+q}|$$
$$\leq |\sum_i \ell(\phi_{m,q}(Q)(Q_{q-qd_i+m})) - \ell(\phi_{m,q}(\mathcal{Q})(H^0(X, \mathcal{Q}(q-qd_i+m)))|.$$

Now, if $q - qd_i + m < 0$, then $Q_{q-qd_i+m} = 0$ and $h^0(X, \mathcal{Q}(q-qd_i+m)) \leq h^0(X, \mathcal{Q})$. If $q - qd_i + m \geq \widetilde{m}$, then $Q_{q-qd_i+m} = H^0(X, \mathcal{Q}(q-qd_i+m))$. This implies that

$$|\operatorname{coker}\phi_{m,q}(\mathcal{Q}) - \ell(Q/I^{[q]}Q)_{m+q}|$$
$$\leq (\mu)\left(h^0(X, \mathcal{Q}(\widetilde{m}-1)) + \max\{\ell(Q_0), \ell(Q_1), \ldots, \ell(Q_{\widetilde{m}-1})\}\right).$$

Therefore $|\operatorname{coker}\phi_{m,q}(\mathcal{Q}) - \ell(Q/I^{[q]}Q)_{m+q}| \leq C_Q$. This proves the second assertion. $\square$

Lemma 2.6. *Let $\mathcal{Q}$ be a coherent sheaf of $\mathcal{O}_X$-modules of dimension $\bar{d}$. Let P and P'' be locally-free sheaves of $\mathcal{O}_X$-modules which fit into a short exact sequence of locally-free sheaves of $\mathcal{O}_X$-modules, where $b_i \geq 0$,*

$$0 \longrightarrow P \longrightarrow \oplus_i \mathcal{O}_X(-b_i) \longrightarrow P'' \longrightarrow 0, \text{ where } b_i \geq 0. \tag{2.3}$$

Then, for $\widetilde{\mu} = \mathrm{rk}(P) + \mathrm{rk}(P'')$, the following hold:

(1)
$$h^0(F^{n*}P \otimes \mathcal{Q}(m)) \leq (\widetilde{\mu})C_0(\mathcal{Q})(m^{\bar{d}} + 1) \text{ for all } n, m \geq 0.$$

(2) *For each $q = p^n$, let $m_n \geq 0$ be an integer with the property that, for all $i \geq 1$ and $m \geq m_n$, we have $h^i(X, F^{n*}P \otimes \mathcal{Q}(m)) = 0$; then*
$$h^1(X, F^{n*}P \otimes \mathcal{Q}(m)) \leq (\widetilde{\mu})C_0(\mathcal{Q})(2m_n\bar{d})^{\bar{d}} \text{ for all } n, m \geq 0.$$

(3) *Moreover, for all $n, m \geq 0$, we have*
$$h^0(X, \mathcal{Q}(m+q)) \leq D_0(\mathcal{Q})(m+q)^{\bar{d}} \text{ and } h^1(X, \mathcal{Q}(m)) \leq D_1(\mathcal{Q}).$$

Proof. Assertion (3) is obvious from the definition of $D_1(\mathcal{Q})$ and $D_0(\mathcal{Q})$ given in Notation 2.4.

Let $\mathcal{Q}_{\bar{d}} = \mathcal{Q}$. Let $a_{\bar{d}}, \ldots, a_1 \in H^0(X, \mathcal{O}_X(1))$ with the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{Q}_i(-1) \xrightarrow{a_i} \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0,$$

where $\mathcal{Q}_i = \mathcal{Q}_{\bar{d}}/(a_{\bar{d}}, \ldots, a_{i+1})\mathcal{Q}_{\bar{d}}$, for $0 \leq i \leq \bar{d}$, and realizing the minimal value $C_0(\mathcal{Q})$. Now, by the exact sequence (2.3), we have the following short exact sequence of $\mathcal{O}_X$-sheaves:

$$0 \longrightarrow F^{n*}P \otimes \mathcal{Q}_i \longrightarrow \bigoplus_j \mathcal{Q}_i(-qb_j) \longrightarrow F^{n*}P'' \otimes \mathcal{Q}_i \longrightarrow 0.$$

This implies $H^0(X, F^{n*}P \otimes \mathcal{Q}_i) \hookrightarrow \bigoplus_j H^0(X, \mathcal{Q}_i(-qb_j))$. Therefore,

$$h^0(X, F^{n*}P \otimes \mathcal{Q}_i) \leq \sum_j h^0(X, \mathcal{Q}_i(-qb_j)) \leq (\widetilde{\mu})h^0(X, \mathcal{Q}_i), \tag{2.4}$$

as $-b_j \leq 0$. Since $F^{n*}P$ is a locally-free sheaf of $\mathcal{O}_X$-modules, we have

$$0 \longrightarrow F^{n*}P \otimes \mathcal{Q}_i(m-1) \xrightarrow{a_i} F^{n*}P \otimes \mathcal{Q}_i(m) \longrightarrow F^{n*}P \otimes \mathcal{Q}_{i-1}(m) \longrightarrow 0, \tag{2.5}$$

which is a short exact sequence of $\mathcal{O}_X$-sheaves.

Claim. For $m \geq 1$,

$$h^0(X, F^{n*}P \otimes \mathcal{Q}_i(m)) \leq (\widetilde{\mu}) \left[h^0(X, \mathcal{Q}_i) + \cdots + h^0(X, \mathcal{Q}_0)\right](m^i).$$

Proof of the claim. We prove the claim, by induction on i. For $i = 0$, the inequality holds as $h^0(X, F^{n*}P \otimes \mathcal{Q}_0(m)) \leq (\widetilde{\mu})h^0(X, \mathcal{Q}_0)$ (as dim $\mathcal{Q}_0 = 0$).

Now, for $m \geq 1$, by the exact sequence (2.5) and by induction on i, we have

$$\begin{aligned} &h^0(X, F^{n*}P \otimes \mathcal{Q}_i(m)) \\ &\quad \leq h^0(X, F^{n*}P \otimes \mathcal{Q}_i) + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(1)) + \cdots + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(m)) \\ &\quad \leq (\widetilde{\mu})h^0(X, \mathcal{Q}_i) + \widetilde{\mu}\left[h^0(X, \mathcal{Q}_{i-1}) + \cdots + h^0(X, \mathcal{Q}_0)\right](1 + 2^{i-1} + \cdots + m^{i-1}) \\ &\quad \leq (\widetilde{\mu})\left[h^0(X, \mathcal{Q}_i) + \cdots + h^0(X, \mathcal{Q}_0)\right] m^i. \end{aligned}$$

This proves the claim. □

This implies $h^0(X, F^{n*}P \otimes \mathcal{Q}(m)) = h^0(X, F^{n*}P \otimes \mathcal{Q}_{\bar{d}}(m)) \leq \widetilde{\mu}C_0(\mathcal{Q})m^{\bar{d}}$ for all $m \geq 1$.

Therefore, and for all $m \geq 0$, we have

$$h^0(X, F^{n*}P \otimes \mathcal{Q}(m)) \leq \widetilde{\mu}C_0(\mathcal{Q})(m^{\bar{d}} + 1).$$

This proves assertion (1).

Let $h^j(X, F^{n*}P \otimes \mathcal{Q}(m)) = 0$ for $m \geq m_n$, $j \geq 1$. If $m_n = 0$, then the assertion (2) is obvious. So we assume $m_n \geq 1$. Then, by the exact sequence (2.5) and descending induction on i, we have $h^j(X, F^{n*}P \otimes \mathcal{Q}_i(m)) = 0$ for all $m \geq m_n + \bar{d}$ and for $j \geq 1$. Now, for $0 \leq m < m_n + \bar{d}$,

$$\begin{aligned} &h^1(X, F^{n*}P \otimes \mathcal{Q}_i(m)) \\ &\quad \leq h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(m_n + \bar{d})) + \cdots + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(m+1)) \\ &\quad \leq (\widetilde{\mu}) \left[h^0(X, \mathcal{Q}_{i-1}) + \cdots + h^0(X, \mathcal{Q}_0)\right] [(m_n + \bar{d})^{i-1} + \cdots + ((m+1))^{i-1}] \\ &\quad \leq (\widetilde{\mu}) \left[h^0(X, \mathcal{Q}_{i-1}) + \cdots + h^0(X, \mathcal{Q}_0)\right] (m_n + \bar{d})^i, \end{aligned}$$

where the second inequality follows from the above claim. This implies, for all $0 \leq m < m_n + \bar{d}$,

$$\begin{aligned} &h^1(X, F^{n*}P \otimes \mathcal{Q}(m)) \\ &\quad \leq (\widetilde{\mu}) \left[h^0(X, \mathcal{Q}_{d-1}) + \cdots + h^0(X, \mathcal{Q}_0)\right] (m_n + \bar{d})^{\bar{d}} \leq (\widetilde{\mu})C_0(\mathcal{Q})(m_n + \bar{d})^{\bar{d}}. \end{aligned}$$

Therefore,

$$h^1(X, F^{n*}P \otimes \mathcal{Q}(m)) \leq (\widetilde{\mu})C_0(\mathcal{Q})(2m_n\bar{d})^{\bar{d}}$$

for all m, $n \geq 0$. This completes the proof. □

In the following lemma we write down a list of bounds on the cohomologies of the sheaves relevant to Theorem 1.1.

Lemma 2.7. *Let $Q = \bigoplus_{m\geq 0} Q_m$ be a non-negatively graded Noetherian R-module and let $\mathcal{Q}$ be the coherent sheaf of $\mathcal{O}_X$-modules associated to Q. Then*

1. $h^0(X, F^{n*}V \otimes \mathcal{Q}(m)) \leq (\mu)C_0(\mathcal{Q})(m^{\bar{d}} + 1)$ *for all* m, $n \geq 0$.
2. $h^1(X, F^{n*}V \otimes \mathcal{Q}(m)) \leq (\mu)(D_Q)(q^{\bar{d}})$ *and* $\sum_1^{\mu} h^1(X, \mathcal{Q}(q - qd_i + m)) \leq (\mu)(D_Q)(q^{\bar{d}})$ *for all* m, $n \geq 0$.
3. $h^0(X, \mathcal{Q}(m + q)) \leq D_0(\mathcal{Q})(m + q)^{\bar{d}}$ *for all* m, $n \geq 0$.
4. $h^1(X, \mathcal{Q}(m)) \leq D_1(\mathcal{Q})$ *for all* $m \geq 0$.
5. $|\operatorname{coker} \phi_{m,q}(\mathcal{Q}) - \ell(Q/I^{[q]}Q)_{m+q}| \leq C_Q$ *for all* $n \geq 0$ *and* $m \in \mathbb{Z}$ *(where we define* $Q_m = 0$*, for* $m < 0$*).*

Proof. Assertions (1), (3), and (4) follow from Lemma 2.6 and Assertion (5) follows from Lemma 2.5 (2).

To prove Assertion (2), let $m_Q(q) = \widetilde{m} + n_0(\sum_i d_i)q$. Note that, for $j \geq 1$ and $m + q \geq m_Q(q)$,

$$\sum_{i=1}^{\mu} H^j(X, F^{n*}\mathcal{O}_X(1 - d_i) \otimes \mathcal{Q}(m)) = \sum_{i=1}^{\mu} H^j(X, \mathcal{Q}(q - qd_i + m)) = 0,$$

as $q - qd_i + m \geq \widetilde{m}$. By Lemma 2.5, $\operatorname{coker} \phi_{m,q}(\mathcal{Q}) = 0$. Therefore, by the long exact sequence (2.2),

$$m + q \geq m_Q(q) \implies h^j(X, F^{n*}V \otimes \mathcal{Q}(m)) = 0 \text{ for all } j \geq 1.$$

Hence, by Lemma 2.6 (2), for all $m \geq 0$ and $q = p^n$, where $P = \bigoplus_i \mathcal{O}_X(1-d_i)$ or V, we have

$$h^1(X, F^{n*}P \otimes \mathcal{Q}(m)) \leq (\mu)C_0(\mathcal{Q})(2m_Q(q)\bar{d})^{\bar{d}} \leq \mu D_Q q^{\bar{d}}, \tag{2.6}$$

where $\bar{d}$ is the dimension of the support of $\mathcal{Q}$ and

$$D_Q = C_0(\mathcal{Q})(2\bar{d})^{\bar{d}}(\widetilde{m} + n_0(\sum d_i))^{\bar{d}} = C_0(\mathcal{Q})\left[2\bar{d}(\widetilde{m} + n_0(\sum d_i))\right]^{\bar{d}}. \tag{2.7}$$

This proves Assertion (2) and hence the lemma. □

Lemma 2.8. *Let $X = \text{Proj R}$ be a projective k-scheme of dimension $d-1$ with a very ample invertible sheaf $\mathcal{O}_X(1)$. Let*

$$0 \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{M}' \xrightarrow{f} \mathcal{M}'' \longrightarrow \mathcal{Q}'' \longrightarrow 0$$

be an exact sequence of sheaves of coherent $\mathcal{O}_X$-modules such that $\mathcal{Q}'$ and $\mathcal{Q}''$ are coherent sheaves of $\mathcal{O}_X$-modules with support of dimensions $< d-1$. Then, for all m, $n \geq 0$,

(1)
$$|\operatorname{coker} \phi_{m,q}(\mathcal{M}') - \operatorname{coker} \phi_{m,q}(\mathcal{M}'')| \leq C(f)(m+q)^{d-2},$$

where

$$C(f) = \mu\left[2C_0(\mathcal{Q}'') + D_0\mathcal{Q}'' + 2C_0(Q') + D_0(Q') + 2D_{\mathcal{Q}'} + D_1(\mathcal{Q}')\right].$$

Moreover,

(2) *if M' and M'' are two non-negatively graded R-modules associated to $\mathcal{M}'$ and $\mathcal{M}''$, respectively, then*

$$|\ell(M'/I^{[q]}M')_{m+q} - \ell(M''/I^{[q]}M'')_{m+q}| \leq C(f)(m+q)^{d-2} + C_{M'} + C_{M''}.$$

Proof. The above exact sequence we can break into the following two short exact sequences of $\mathcal{O}_X$-sheaves

$$0 \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{M}' \longrightarrow K \longrightarrow 0,$$
$$0 \longrightarrow K \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{Q}'' \longrightarrow 0.$$

For a locally-free sheaf P of $\mathcal{O}_X$-modules, both the above short exact sequences remain exact after tensoring with $(F^{n*}P)(m)$ for all $m \geq 0$ and $n \geq 0$. Therefore, we have long exact sequence of cohomologies

$$0 \longrightarrow H^0(X, F^{n*}P \otimes \mathcal{Q}'(m)) \longrightarrow H^0(X, F^{n*}P \otimes \mathcal{M}')$$
$$\longrightarrow H^0(X, F^{n*}P \otimes K(m)) \longrightarrow H^1(X, F^{n*}P \otimes \mathcal{Q}'(m)) \longrightarrow \cdots$$

and

(2.8)
$$0 \longrightarrow H^0(X, F^{n*}P \otimes K(m)) \longrightarrow H^0(X, F^{n*}P \otimes \mathcal{M}''(m)) \longrightarrow H^0(X, F^{n*}P \otimes Q''(m)).$$

For a coherent sheaf L of $\mathcal{O}_X$-modules

$$\operatorname{coker} \phi_{m,q}(L) = h^0(X, F^{n*}\mathcal{O}_X(1) \otimes L(m))$$
$$- \sum_{i=1}^{s} h^0(X, F^{n*}\mathcal{O}_X(1-d_i) \otimes L(m)) + h^0(X, F^{n*}V \otimes L(m)).$$

By (2.8),

$$|h^0(X, F^{n*}P \otimes K(m)) - h^0(X, F^{n*}P \otimes \mathcal{M}''(m))| \leq h^0(X, F^{n*}P \otimes Q''(m)).$$

Therefore, by Lemma 2.7, we have

$$\begin{aligned}
&|\operatorname{coker}\phi_{m,q}(K) - \operatorname{coker}\phi_{m,q}(\mathcal{M}'')| \\
&\leq h^0(X, \mathcal{Q}''(m+q)) + \sum_{i=1}^{s} h^0(X, \mathcal{Q}''(m+q-qd_i)) + h^0(X, F^{n*}V \otimes \mathcal{Q}''(m)) \\
&\leq D_0(Q'')(m+q)^{d-2} + \mu C_0(Q'')(m^{d-2}+1) + \mu C_0(Q'')(m^{d-2}+1),
\end{aligned}$$

as $h^0(X, \mathcal{Q}''(m+q-qd_i)) \leq h^0(X, \mathcal{Q}''(m+q))$.

Therefore,

$$|\operatorname{coker}\phi_{m,q}(K) - \operatorname{coker}\phi_{m,q}(\mathcal{M}'')| \leq \mu\,[2C_0(Q'') + D_0(Q'')]\,(m+q)^{d-2}. \tag{2.9}$$

Similarly, since for a locally-free sheaf P and for $m, n \geq 0$, we have

$$\begin{aligned}
&|h^0(X, F^{n*}P \otimes \mathcal{M}'(m)) - h^0(X, F^{n*}P \otimes K(m))| \\
&\qquad \leq h^0(X, F^{n*}P \otimes \mathcal{Q}'(m)) + h^1(X, F^{n*}P \otimes \mathcal{Q}'(m)),
\end{aligned}$$

we deduce, by Lemma 2.7,

$$\begin{aligned}
&|\operatorname{coker}\phi_{m,q}(\mathcal{M}') - \operatorname{coker}\phi_{m,q}(K)| \\
&\leq \mu\,[2C_0(\mathcal{Q}') + D_0(\mathcal{Q}')]\,(m+q)^{d-2} + 2\mu D_{Q'} q^{d-2} + D_1(\mathcal{Q}').
\end{aligned} \tag{2.10}$$

Therefore, by (2.9) and (2.10), for all m, $n \geq 0$, we have

$$\begin{aligned}
&|\operatorname{coker}\phi_{m,q}(\mathcal{M}') - \operatorname{coker}\phi_{m,q}(\mathcal{M}'')| \\
&\leq \mu\,[2C_0(\mathcal{Q}'') + D_0\mathcal{Q}'' + 2C_0(Q') + D_0(Q') + 2D_{\mathcal{Q}'} + D_1(\mathcal{Q}')]\,(m+q)^{d-2} \\
&= C(f)(m+q)^{d-2}.
\end{aligned} \tag{2.11}$$

Now Assertion (2) follows from Lemma 2.7 (5). □

Lemma 2.9. *Let $Y = X_{red}$, which is a reduced projective k-scheme of dimension $d-1$ with a very ample invertible sheaf $\mathcal{O}_Y(1)$. Then, for a coherent sheaf $\mathcal{N}$ of $\mathcal{O}_Y$-modules, there exists an integer $m_2 \geq 1$, depending on $\mathcal{N}$, such that we have an exact sequence of sheaves of $\mathcal{O}_Y$-modules*

$$0 \longrightarrow Q' \longrightarrow \bigoplus^{p^{d-1}} \mathcal{N}(-m_2) \longrightarrow F_*\mathcal{N} \longrightarrow Q'' \longrightarrow 0,$$

where Q' and Q'' are coherent sheaves of $\mathcal{O}_Y$-modules with support of dimensions $< d-1$.

Proof. Let $x_1, \ldots, x_{s_1}$ be the generic points of the maximal components $Y_1, \ldots, Y_{s_1}$ of Y, where $\dim\ Y_i = \dim\ Y$. We choose $f \in H^0(Y, \mathcal{O}_Y(1))$ such that f does not vanish on $\mathcal{O}_{Y,x_i}$, for all i. Note that $\mathcal{O}_{Y,x_i}$ is the function field of Y_i. In particular, $x_1, \ldots, x_{s_1} \in D_+(f)$, where $D_+(f)$ is a reduced affine variety. Let us denote $D_+(f)$ by U_f. Let $\Gamma(U_f, \mathcal{O}_Y) = A$. Let $p_1, \ldots, p_{s_1} \in \operatorname{Spec} A$ be the prime ideals corresponding to the points $x_1, \ldots, x_{s_1}$ and let $S = A \setminus p_1 \cup \cdots \cup p_{s_1}$. Then, by the Chinese Remainder Theorem

$$S^{-1}A \simeq \mathcal{O}_{Y,x_1} \times \cdots \times \mathcal{O}_{Y,x_{s_1}} \quad \text{and } S^{-1}\Gamma(U_f, \mathcal{N}) \simeq \mathcal{N}_{x_1} \times \cdots \times \mathcal{N}_{x_{s_1}}$$

and

$$S^{-1}\Gamma(U_f, F_*\mathcal{N}) \simeq (F_*\mathcal{N})_{x_1} \times \cdots \times (F_*\mathcal{N})_{x_{s_1}} = F_*(\mathcal{N}_{x_1}) \times \cdots \times F_*(\mathcal{N}_{x_{s_1}}).$$

Now if $\mathcal{N}_{x_i}$ is of rank m_i as an A_{x_i}-module, then $F_*\mathcal{N}_{x_i}$ is of rank $p^{d-1}m_i$ as an A_{x_i}-module, as F_*A is of rank p^{d-1} over A and $F_*\mathcal{N}_{x_i}$ is of rank m_i over $F_*A_{x_i}$.

This implies that there is an $\mathcal{O}_{Y,x_i}$-linear isomorphism $\phi_i : \bigoplus^{p^{d-1}} \mathcal{N}_{x_i} \longrightarrow (F_*\mathcal{N})_{x_i}$, which gives an $S^{-1}A$-linear isomorphism

$$\phi : \bigoplus^{p^{d-1}} S^{-1}\Gamma(U_f, \mathcal{N}) \longrightarrow S^{-1}\Gamma(U_f, F_*\mathcal{N}).$$

Since $\mathcal{N}$ is a coherent $\mathcal{O}_Y$-sheaf, one can choose $\widetilde{s} \in S$ and $\widetilde{\phi} : \bigoplus^{p^{d-1}} \Gamma(U_f, \mathcal{N}) \longrightarrow \Gamma(U_f, F_*\mathcal{N})$ such that $\widetilde{\phi}$ maps to $\widetilde{s} \cdot \phi$ under the localization map

$$\begin{aligned} &\mathrm{Hom}_A\left(\Gamma(U_f, \bigoplus^{p^{d-1}} \mathcal{N}), \Gamma(U_f, F_*\mathcal{N})\right) \\ &\qquad \longrightarrow \mathrm{Hom}_{S^{-1}A}\left(S^{-1}\Gamma(U_f, \bigoplus^{p^{d-1}} \mathcal{N}), S^{-1}\Gamma(U_f, F_*\mathcal{N})\right). \end{aligned}$$

Therefore, there exists $n \geq 1$ and $\psi \in \Gamma(Y, \mathrm{Hom}_{\mathcal{O}_Y}(\bigoplus^{p^{d-1}} \mathcal{N}, F_*\mathcal{N}) \otimes \mathcal{O}_Y(n))$ such that ψ restricts to $f^n \cdot \widetilde{s} \cdot \phi$ on the open set U_f (see [Ha, Lemma 5.14]). This gives an exact sequence of $\mathcal{O}_Y$-linear maps

$$0 \longrightarrow \mathrm{Ker}\ \psi \longrightarrow \oplus^{p^{d-1}} \mathcal{N}(-n) \xrightarrow{\psi} F_*\mathcal{N} \longrightarrow \mathrm{Coker}\ \psi \longrightarrow 0.$$

Since ψ localizes to a unit multiple of ϕ, it is an isomorphism at the points $x_1, x_2, \ldots, x_{s_1}$, which implies that the dimensions of the support of Ker ψ and Coker ψ are $<$ dim Y. This proves the lemma. $\square$

Lemma 2.10. *Let M be a non-negatively graded finitely generated R-module and let $\mathcal{M}$ be the associated coherent sheaf of $\mathcal{O}_X$-modules. Then there exists a non-negative integer s (e.g., $s \geq 0$ such that (nilradical $R)^{p^s} = 0$) and an integer $m_2 \geq 1$(depending on $\mathcal{M}$ and $q' = p^s$) such that*

(1) *there is a long exact sequence of sheaves of $\mathcal{O}_X$-modules*

$$0 \longrightarrow Q' \longrightarrow \bigoplus^{p^{d-1}} (F_*^s\mathcal{M})(-m_2) \xrightarrow{g} F_*^{s+1}\mathcal{M} \longrightarrow Q'' \longrightarrow 0,$$

where Q' and Q'' are coherent sheaves of $\mathcal{O}_X$-modules with support of dimensions $< d-1$.

(2) *There is a constant $C(g)$ (as given in Lemma 2.8 (1)) such that, for all m, $n \geq 0$,*

$$|p^{d-1}\ell(M/I^{[qq']}M)_{(m+q-m_2)q'} - \ell(M/I^{[qq'p]}M)_{(m+q)q'p}| \leq C(g) + 2C_M.$$

Proof. Let p^s be an integer such that (nilradical $R)^{p^s} = 0$. Then $\mathcal{N} = F_*^s\mathcal{M}$ is a coherent $\mathcal{O}_X$-module annhilated by the nilradical of $\mathcal{O}_X$. Consider the canonical short exact sequence of $\mathcal{O}_X$-modules obtained from equation (2.1),

$$0 \longrightarrow F^{n*}V \otimes \mathcal{N}(m) \longrightarrow \bigoplus_i \mathcal{N}(q - qd_i + m)) \longrightarrow \mathcal{N}(q+m) \longrightarrow 0. \tag{2.12}$$

Since $\mathcal{N}$ is annihilated by the nilradical of $\mathcal{O}_X$, the action of $\mathcal{O}_X$ on $\mathcal{N}$ filters through a canonical action of $\mathcal{O}_{X_{red}}$ on $\mathcal{N}$.

Since $\mathcal{N}$ is also a sheaf of $\mathcal{O}_{X_{red}}$-modules, by Lemma 2.9, there exists constant m_2 depending on $\mathcal{N}$ and $\mathcal{O}_{X_{red}}$ such that we have a short exact sequence of $\mathcal{O}_{X_{red}}$-modules and hence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow Q' \longrightarrow \bigoplus^{p^{d-1}} \mathcal{N}(-m_2) \xrightarrow{g} F_*\mathcal{N} \longrightarrow Q'' \longrightarrow 0,$$

where Q' and Q'' are coherent sheaves of $\mathcal{O}_{X_{red}}$-modules (and hence coherent sheaves of $\mathcal{O}_X$-modules) with support of dimensions $\leq d-2$. Therefore, by Lemma 2.8 (1), there is a constant $C(g)$ for the map g such that

$$|\operatorname{coker}\phi_{m,q}\left(\bigoplus^{p^{d-1}} \mathcal{N}(-m_2)\right) - \operatorname{coker}\phi_{m,q}(F_*\mathcal{N})| \leq C(g)(m+q)^{d-2}$$

for all $m, n \geq 0$. Therefore,

$$|p^{d-1}\operatorname{coker}\phi_{m-m_2,q}(\mathcal{N}) - \operatorname{coker}\phi_{m,q}(F_*\mathcal{N})| \leq C(g)(m+q)^{d-2}. \tag{2.13}$$

We note that, for any locally-free sheaf P of $\mathcal{O}_X$-modules, using the projection formula, we have (since k is perfect)

$$\begin{aligned} h^i(X, F^{(n+1+s)*}P \otimes \mathcal{M}(mpq')) &= h^i(X, F^{s*}\left(F^{n+1*}P \otimes \mathcal{O}(mp)\right) \otimes \mathcal{M}) \\ &= h^i(X, \left(F^{n+1*}P \otimes \mathcal{O}(mp)\right) \otimes \mathcal{N}) = h^i(X, F^{n*}P \otimes \mathcal{O}(m) \otimes F_*\mathcal{N}). \end{aligned}$$

Therefore,

$$\operatorname{coker}\phi_{(mp)q',qq'p}(\mathcal{M}) = \operatorname{coker}\phi_{m,q}(F_*\mathcal{N}). \tag{2.14}$$

Similarly

$$\begin{aligned} h^i(X, F^{(n+s)*}P \otimes \mathcal{M}((m-m_2)q')) &= h^i(X, F^{n*}P \otimes \mathcal{O}(m-m_2) \otimes F^s_*\mathcal{M}) \\ &= h^i(X, F^{n*}P \otimes \mathcal{N}(m-m_2)). \end{aligned}$$

Therefore,

$$\operatorname{coker}\phi_{(m-m_2)q',qq'}(\mathcal{M}) = \operatorname{coker}\phi_{m-m_2,q}(\mathcal{N}). \tag{2.15}$$

Hence, by (2.13),

$$|p^{d-1}\operatorname{coker}\phi_{(m-m_2)q',qq'}(\mathcal{M}) - \operatorname{coker}\phi_{(mp)q',qq'p}(\mathcal{M})| \leq C(g)(m+q)^{d-2}.$$

Therefore, by Lemma 2.7 (5),

$$|p^{d-1}\ell(M/I^{[qq']}M)_{(m+q-m_2)q'} - \ell(M/I^{[qq'p]}M)_{(m+q)q'p}| \leq C(g)(m+q)^{d-2} + 2C_M$$

for all m, $n \geq 0$. $\square$

Definition 2.11. For a pair (M, I), where M is a finitely generated non-negatively graded R-module and I is a homogeneous ideal of R such that $\ell(R/I) < \infty$. We define sequences of functions $\{f_n : \mathbb{R} \longrightarrow \mathbb{R}\}_{n\in\mathbb{N}}$ and $\{g_n : \mathbb{R} \longrightarrow \mathbb{R}\}_{n\in\mathbb{N}}$ as follows:

For $n \in \mathbb{N}$, let $q = p^n$. Define

$$f_n(x) = g_n(x) = 0, \text{ if } x < 0.$$

Let $x \geq 0$; then $m/q \leq x < m + 1/q$ for some integer $m \geq 0$. We define

$$\begin{aligned} f_n(x) &= \frac{1}{q^{d-1}}\ell\left(M_m\right) \quad \text{if} \quad 0 \leq x < 1 \\ &= \frac{1}{q^{d-1}}\ell\left(\frac{M}{I^{[q]}M}\right)_m \quad \text{if} \quad 1 \leq \frac{m}{q} \leq x < \frac{m+1}{q}, \end{aligned}$$

$$\begin{aligned} g_n(x) &= f_n(x) \quad \text{if} \quad x = \frac{m}{q} \\ &= (1-t)f_n\left(\frac{m}{q}\right) + tf_n\left(\frac{m+1}{q}\right) \\ &\qquad\qquad \text{if } x = (1-t)\left(\frac{m}{q}\right) + t\left(\frac{m}{q}\right) \text{ where } t \in [0,1). \end{aligned}$$

Proposition 2.12. *For a given pair* (M, I) *as in Definition* 2.11 *above, and where* $\dim\ R = d \geq 2$*, the sequence* $\{f_n : \mathbb{R} \longrightarrow \mathbb{R}\}_{n\in\mathbb{N}}$ *is a uniformly convergent sequence of compactly supported functions.*

More precisely, there exists $n_0 \in \mathbb{N}$ *and a constant* C *depending on* M*, such that*

$$|f_n(x) - f_{n_1}(x)| \leq C/p^n \quad \text{for all } n_1 \geq n \geq n_0 \text{ and for all } x \in \mathbb{R}. \tag{2.16}$$

Proof. Note that $\dim\ R \geq 2$. Therefore, for $X = \operatorname{Proj} R$, we have $\dim\ X \geq 1$. Let $\mathcal{M}$ be the coherent sheaf of $\mathcal{O}_X$-modules associated to M.

(A) Let $x < 1$.

If $x < 0$, then $f_n(x) = f_{n+1}(x) = 0$ for all $n \geq 1$.

Let $0 \leq x < 1$. Then $m/q \leq x < (m+1)/q$ for some integer $0 \leq m < q$. Hence

$$\frac{mp+n_1}{qp} \leq x < \frac{mp+n_1+1}{qp} \quad \text{for some integer } 0 \leq n_1 < p \text{ with } mp+n_1 < qp.$$

Therefore, $f_n(x) = (1/q^{d-1})\ell(M_m)$ and $f_{n+1}(x) = (1/(qp)^{d-1})\ell(M_{mp+n_1})$.

If $m \leq \widetilde{m}$ ($\widetilde{m}$ is defined for M as in Notation 2.4), then

$$\begin{aligned} |f_n(x) - f_{n+1}(x)| &< \left|\frac{(\ell(M_0) + \cdots + \ell(M_{\widetilde{m}})}{q^{d-1}} + \frac{(\ell(M_0) + \cdots + \ell(M_{\widetilde{m}p+n_1})}{(qp)^{d-1}}\right| \\ &\leq \frac{2\sum_0^{\widetilde{m}p+(p-1)}\ell(M_i)}{q^{d-1}}. \end{aligned}$$

If $q > m > \widetilde{m}$, then (using Hilbert polynomials)

$$\ell(M_m) = \widetilde{e}_0 m^{d-1} + \widetilde{e}_1 m^{d-2} + \cdots + \widetilde{e}_{d-1},$$

$$\ell(M_{mp+n_1}) = \widetilde{e}_0(mp+n_1)^{d-1} + \widetilde{e}_1(mp+n_1)^{d-2} + \cdots + \widetilde{e}_{d-1}$$

for some rational numbers $\widetilde{e}_0, \cdots, \widetilde{e}_{d-1}$ which are invariant of $(\mathcal{M}, \mathcal{O}_X(1))$. In this case

$$|f_n(x) - f_{n+1}(x)| \leq \frac{(d-1)\widetilde{e}_0 + |\widetilde{e}_1| + \cdots + |\widetilde{e}_{d-1}|}{q}.$$

This implies that, for $\widetilde{C}_2(M) = 2\sum_0^{\widetilde{m}p+(p-1)}\ell(M_i) + (d-1)\widetilde{e}_0 + |\widetilde{e}_1| + \cdots + |\widetilde{e}_{d-1}|$,

$$|f_n(x) - f_{n+1}(x)| \leq \frac{\widetilde{C}_2(M)}{q} \quad \text{for all} \quad x < 1 \quad \text{for all} \quad n \geq 0. \tag{2.17}$$

(B) Let $x \geq 1$. We fix two integers m_2 and $q' = p^s$ (as in Lemma 2.10) such that we have an exact sequence of sheaves of $\mathcal{O}_X$-modules,

$$0 \longrightarrow Q' \longrightarrow \bigoplus^{p^{d-1}} (F_*^s \mathcal{M})(-m_2) \xrightarrow{g} F_*^{s+1}\mathcal{M} \longrightarrow Q'' \longrightarrow 0.$$

Let $s \in R_1$ which avoids all minimal primes of the ring R (note that R is a standard graded ring and k is infinite). For $0 \leq n_1 < q'$ and $0 \leq n_2 < p$, we consider the following exact sequences of graded R-modules:

$$0 \longrightarrow Q'_{n_1} \longrightarrow M(-m_2 q') \xrightarrow{f_{n_1}} M(n_1) \longrightarrow Q''_{n_1} \longrightarrow 0,$$

where f_{n_1} is the multiplication map given by $s^{n_1+m_2q'}$. This induces canonical exact sequences of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{Q}'_{n_1} \longrightarrow \mathcal{M}(-m_2 q') \xrightarrow{f_{n_1}} \mathcal{M}(n_1) \longrightarrow \mathcal{Q}''_{n_1} \longrightarrow 0,$$

Similarly we have exact sequences of graded R-modules

$$0 \longrightarrow K'_{n_2,n_1} \longrightarrow M \xrightarrow{h_{n_2,n_1}} M(n_1p+n_1) \longrightarrow K''_{n_2,n_1} \longrightarrow 0,$$

where h_{n_2,n_1} is the multiplication map given by $s^{n_1p+n_2}$. This induces exact sequences of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{K}'_{n_2,n_1} \longrightarrow \mathcal{M} \xrightarrow{h_{n_2,n_1}} \mathcal{M}(n_1p+n_1) \longrightarrow \mathcal{K}''_{n_2,n_1} \longrightarrow 0.$$

By construction, each of the sheaves $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{Q}'_{n_1}$, $\mathcal{Q}''_{n_1}$, $\mathcal{K}'_{n_2,n_1}$, and $\mathcal{K}''_{n_2,n_1}$, has support of dimension $< d-1$.

Let

$$\widetilde{C}_0(M) = \max_{0\leq n_1 < q',\ 0 \leq n_2 < p} \Big\{ C(f_{n_1}) + C_{Q'_{n_1}} + C_{Q''_{n_1}}, C(g) + 2C_M, C(h_{n_2,n_1}) + C_{K'_{n_2,n_1}} + C_{K''_{n_2,n_1}} \Big\},$$

where $C(f_{n_1})$, $C(g)$ and $C(h_{n_2,n_1})$ are the constants (see Lemma 2.8) associated to the maps f_{n_1}, g and h_{n_2,n_1} respectively.

Since $x \geq 1$, for given $q = p^n$, there exists a unique integer $m \geq 0$, such that $(m+q)/q \leq x < (m+q+1)/q$. Therefore, for $q' = p^s$ we have

$$\frac{(m+q)q'+n_1}{qq'} \leq x < \frac{(m+q)q'+n_1+1}{qq'} \text{ for some } n_1 < q'$$

and

$$\frac{(m+q)q'p+n_1p+n_2}{qq'p} \leq x < \frac{(m+q)q'p+n_1p+n_2+1}{qq'p} \text{ for some } n_2 < p.$$

Hence, by definition

$$f_{n+s}(x) = \frac{1}{(qq')^{d-1}} \ell\left(\frac{M}{I^{[qq']}M}\right)_{(m+q)q'+n_1} \text{ and}$$

$$f_{n+s+1}(x) = \frac{1}{(qq'p)^{d-1}} \ell\left(\frac{M}{I^{[qq'p]}M}\right)_{(m+q)q'p+n_1p+n_2}.$$

Let $m_M(q) = \widetilde{m} + n_0(\sum_i d_i)q$ (defined as in Notation 2.4). If $m \geq m_M(q)$, then we have $mq' \geq m_M(qq')$ and $mq'p \geq m_M(qq'p)$. Therefore, by Lemma 2.5 (1), for $m \geq m_M(q)$,

$$\ell\left(\frac{M}{I^{[qq']}M}\right)_{(m+q)q'+n_1} = \ell\left(\frac{M}{I^{[qq'p]}M}\right)_{(m+q)q'p+n_1p+n_2} = 0,$$

which implies $|f_{n+s}(x) - f_{n+s+1}(x)| = 0$.

Therefore, we can assume $m \leq m_M(q)$ and hence can assume that $(m+q)^{d-2} \leq L_0 q^{d-2}$, where $L_0 = (\widetilde{m} + n_0(\sum_i d_i) + 1)^{d-2}$.

We have

$$|f_{n+s}(x) - f_{n+s+1}(x)| = |f_{n+s}(\frac{(m+q)q'+n_1}{qq'}) - f_{n+s+1}(\frac{(m+q)q'p+n_1p+n_2}{qq'p})|.$$

Hence we have

$$|f_{n+s}(x) - f_{n+s+1}(x)| \leq A_1(x) + A_2(x) + A_3(x),$$

where

$$A_1(x) = |f_{n+s}(\frac{(m+q)q'+n_1}{qq'}) - f_{n+s}(\frac{(m+q-m_2)q'}{qq'})|,$$
$$A_2(x) = |f_{n+s}(\frac{(m+q-m_2)q'}{qq'}) - f_{n+s+1}(\frac{(m+q)q'p}{qq'p})|,$$
$$A_3(x) = |f_{n+s+1}(\frac{(m+q)q'p}{qq'p}) - f_{n+s+1}(\frac{(m+q)q'p+n_1p+n_2}{qq'p}).$$

Now

$$A_1(x) = \frac{1}{(qq')^{d-1}}\left|\ell\left(\frac{M}{I^{[qq']}M}\right)_{(m+q)q'+n_1} - \ell\left(\frac{M}{I^{[qq']}M}\right)_{(m-m_2+q)q'}\right|,$$
$$A_1(x) \leq \frac{C(f_{n_1})(m+q)^{d-2} + C(Q'_{n_1}) + C(Q''_{n_1})}{(qq')^{d-1}} \leq \frac{1}{qq'}\frac{\widetilde{C}_0(M)L_0}{(q')^{d-2}},$$
$$A_2(x) = \frac{1}{(qq'p)^{d-1}}\left|p^{d-1}\ell\left(\frac{M}{I^{[qq']}M}\right)_{(m+q-m_2)q'} - \ell\left(\frac{M}{I^{[qq'p]}M}\right)_{(m+q)q'p}\right|,$$
$$A_2(x) \leq \frac{C(g)(m+q)^{d-2} + 2C_M}{(qq'p)^{d-1}} \leq \frac{\widetilde{C}_0(M)L_0q^{d-2}}{(qq'p)^{d-1}} \leq \frac{1}{qq'}\frac{\widetilde{C}_0(M)L_0}{q'^{d-2}p^{d-1}},$$
$$A_3(x) = \frac{1}{(qq'p)^{d-1}}\left|\ell\left(\frac{M}{I^{[qq'p]}M}\right)_{(m+q)q'p} - \ell\left(\frac{M}{I^{[qq'p]}M}\right)_{(m+q)q'p+n_1p+n_2}\right|,$$
$$A_3(x) \leq \frac{C(h_{n_2,n_1})(mq'p+qq'p)^{d-2} + C_{K'_{n_2,n_1}} + C_{K''_{n_2,n_1}}}{(qq'p)^{d-1}} \leq \frac{1}{qq'}\frac{\widetilde{C}_0(M)L_0}{p}.$$

Therefore,

$$\begin{aligned}|f_{n+s}(x) - f_{n+s+1}(x)| &\leq A_1(x) + A_2(x) + A_3(x)\\ &\leq \frac{L_0}{qq'}\left[\frac{\widetilde{C}_0(M)}{(q')^{d-2}} + \frac{\widetilde{C}_0(M)}{q'^{d-2}p^{d-1}} + \frac{\widetilde{C}_0(M)}{p}\right].\end{aligned}$$

Let $\widetilde{C}_1(M) = 3L_0\widetilde{C}_0(M)$. In particular, $\widetilde{C}_1(M)$ is a constant (which depends only on M) such that

(2.18)

$$|f_{n+s}(x) - f_{n+s+1}(x)| \leq \widetilde{C}_1(M)/(qq') = \widetilde{C}_1(M)/p^{n+s} \text{ for all } n \geq 0 \text{ and } x \geq 1.$$

Since

$$|\widetilde{C}_1(M)/p^{n_0} + \widetilde{C}_1(M)/p^{n_0+1} + \cdots + | \leq 2\widetilde{C}_1(M)/p^{n_0}$$

for $C \geq 2\widetilde{C}_1(M)$ we get

$$|f_n(x) - f_{n_1}(x)| \leq C/p^n \quad \text{for all } n_1 \geq n \geq n_0 \text{ and for all } x \geq 1.$$

Combining this with (2.17), we get that for any $C \geq 2\widetilde{C}_2(M) + 2\widetilde{C}_1(M)$

$$|f_n(x) - f_{n_1}(x)| \leq C/p^n \quad \text{for all } n_1 \geq n \geq n_0 \text{ and for all } x \in \mathbb{R}.$$

This proves the proposition. □

Proof of Theorem 1.1. By Remark 2.3, we may assume $k = \bar{k}$. For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ and $g_n : \mathbb{R} \longrightarrow \mathbb{R}$ be functions as given in Definition 2.11.

Claim. Both the sequences $\{f_n\}_n$ and $\{g_n\}_n$ converge uniformly and to the same limit function.

Proof of the claim. Let $q = p^n$ and $x \in \mathbb{R}$. If $x < 0$, then $f_n(x) = g_n(x) = 0$ for all $n \geq 0$.

Let $x \geq 0$; then $x = (1-t)\frac{\lfloor xq \rfloor}{q} + t\frac{\lfloor xq \rfloor + 1}{q}$ for some $t \in [0, 1)$. Therefore,

$$f_n(x) = \frac{1}{q^{d-1}} \ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor}$$

and

$$g_n(x) = \frac{(1-t)}{q^{d-1}} \ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor} + \frac{t}{q^{d-1}} \ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor + 1}.$$

Let

$$0 \longrightarrow Q' \longrightarrow M(-1) \xrightarrow{f} M \longrightarrow Q'' \longrightarrow 0$$

be the exact sequence of graded R-modules where the map f is given by multiplication by an element $s \in R_1$, By choosing such an s which avoids all minimal primes of M, we ensure that support of each of Q' and Q'' is of dimension $< d$. If

$$0 \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{M}(-1) \xrightarrow{f} \mathcal{M} \longrightarrow \mathcal{Q}'' \longrightarrow 0$$

is the associated exact sequence of sheaves of $\mathcal{O}_X$-modules, then by Lemma 2.10 (2) and Lemma 2.5 (1),

$$|\ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor} - \ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor + 1}| \leq (C(g) + 2C_M)L_0 q^{d-2} = C_1 q^{d-2}$$

for all $n \geq 1$ and $x \geq 0$, where $L_0 = (\widetilde{m} + n_0(\sum_i d_i) + 1)^{d-2}$.

This implies, for all $n \geq 1$ and $x \geq 0$, we have,

$$|f_n(x) - g_n(x)| = \frac{t}{q^{d-1}} \left| \ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor} - \ell\left(\frac{M}{I^{[q]}M}\right)_{\lfloor xq \rfloor + 1} \right| \leq \frac{C_1}{p^n}.$$

By Proposition 2.12, there is a constant C depending on M and $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f_{n_1}(x)| \leq C/p^n \quad \text{for all } n_1 \geq n \geq n_0 \text{ and for all } x \in \mathbb{R}.$$

This implies,

$$\begin{aligned} |g_n(x) - g_{n_1}(x)| &\leq |g_n(x) - f_n(x)| + |f_n(x) - f_{n_1}(x)| + |f_{n_1}(x) - g_{n_1}(x)| \\ &\leq \frac{C_1}{p^n} + \frac{C}{p^n} + \frac{C_1}{p^{n_1}}. \end{aligned}$$

Therefore, we have

$$|f_n(x) - f_{n_1}(x)|, |f_n(x) - g_n(x)|, |g_n(x) - g_{n_1}(x)| \leq \frac{2C_1 + C}{p^n}$$

for all $n_1 \geq n \geq n_0$ and for all $x \in \mathbb{R}$.

Hence $\{f_n\}_n$ and $\{g_n\}_n$ are uniformly convergent sequences with the same limit. This proves the claim.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the limit function given by

$$f(x) = \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} g_n(x)dx.$$

By the proof of Lemma 2.2, g_n is a continuous function with the support $g_n \subseteq [0, (n_0\mu) + l/q]$. Therefore, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous compactly supported real valued function such that supp $f \subseteq [0, n_0\mu]$. For $q = p^n$ where $n \geq 1$, we can write

$$\frac{1}{q^d}\ell(M/I^{[q]}M) = \frac{1}{q^d}\sum_{m\geq 0}\ell(M/I^{[q]}M)_m$$

$$= \int_0^{1/q} \frac{1}{q^{d-1}}\ell(M_0)dx + \cdots + \int_{1-\frac{1}{q}}^{1} \frac{1}{q^{d-1}}\ell(M_{q-1})dx + \int_1^{1+\frac{1}{q}} \frac{1}{q^{d-1}}\ell(\frac{M}{I^{[q]}M})_q dx$$

$$+ \int_{1+\frac{1}{q}}^{1+\frac{2}{q}} \frac{1}{q^{d-1}}\ell(\frac{M}{I^{[q]}M})_{q+1}dx + \cdots + \int_{n_0\mu-\frac{1}{q}}^{n_0\mu} \frac{1}{q^{d-1}}\ell(\frac{M}{I^{[q]}M})_{n_0\mu q-1}dx$$

$$= \int_0^{n_0\mu} f_n(x)dx.$$

Therefore,

$$e_{HK}(M, I) = \lim_{n\to\infty}\int_0^{n_0\mu} f_n(x)dx.$$

But, as $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, we have

$$\lim_{n\to\infty}\int_0^{n_0\mu} f_n(x)dx = \int_0^{n_0\mu} \lim_{n\to\infty} f_n(x)dx = \int_{\mathbb{R}} f(x)dx.$$

This proves the theorem. □

Having proved the existence of the Hilbert-Kunz density function we are ready to check some properties of the function.

Remark 2.13. Note that (as argued in the proof of the above theorem) the support $HKd(M, I) \subseteq [0, n_0\mu]$, where n_0 and μ are invariants depending on I and R, as given in Notation 2.4 part (3).

The first thing we note (Proposition 2.14 below) is that like HK multiplicity, the function

$$HKd(-, I) : \{\text{finitely generated graded } R \text{ modules}\} \longrightarrow \mathcal{C}_c^0(\mathbb{R})$$

is additive, where $\mathcal{C}_c^0(\mathbb{R})$ denotes the set of continous compactly supported real valued functions. Hence we can reduce various results about an HK density function of a module to an HK density function of an integral domain. Corollary 2.18, shows that the HKd function is a multiplicative functor with respect to the Segre products on the set of graded R-modules.

Proposition 2.14 (Additive property). *Let R be a standard graded ring of dimension $d \geq 2$ over a perfect field, and let $I \subset R$ be a homogeneous ideal of finite colength. Let M be a finitely generated graded R module. Let Λ be the set of minimal prime ideals P of R such that* $\dim\ R/P = \dim\ R$. *Then*

$$HKd(M, I) = \sum_{P \in \Lambda} HKd(R/P, I)\lambda(M_P).$$

Proof. As usual, there is no loss of generality in assuming that the ground field k is algebraically closed. Let $\mathcal{M}$ be the sheaf of $\mathcal{O}_X$-modules associated to M. For $q = p^n$, recall $f_n(M)(x) = \frac{1}{q^{d-1}}\ell(\frac{M}{I^{[q]}M})_{m+q}$, where $\lfloor xq \rfloor = m+q$. Let $\widetilde{f}_n(\mathcal{M})(x) = \operatorname{coker}\phi_{m,q}(\mathcal{M})/q^{d-1}$. Note, by Lemma 2.7 (5), we have

$$\lim_{n\to\infty} \widetilde{f}_n(M)(x) = \lim_{n\to\infty} f_n(M)(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Therefore, if $\mathcal{Q}$ is a coherent sheaf on X, then we can define

$$HKd(\mathcal{Q}, I) := HKd(Q, I),$$

where Q is any finitely generated graded R-module with $\mathcal{Q}$ as the associated sheaf of $\mathcal{O}_X$-modules. Note that, due to Remark 2.13, one can assume $(m+q)^{d-2} \leq (n_0\mu q)^{d-2}$. Therefore, it follows from Lemma 2.8 that if $M' \longrightarrow M''$ is a generic isomorphism of R-modules (i.e., the kernel and cokernel of the map are of dimension $<$ dim R), then $HKd(M', I) = HKd(M'', I)$. Similarly if $\mathcal{M}' \longrightarrow \mathcal{M}''$ is a generic isomorphism of coherent sheaves of $\mathcal{O}_X$-modules, then $HKd(\mathcal{M}', I) = HKd(\mathcal{M}'', I)$.

Now let $s \geq 0$ such that (nilradical $R)^{p^s} = 0$. Define $\mathcal{N} = F^s_*(\mathcal{M})$, let $q' = p^s$. Then $\mathcal{N}$ is a coherent sheave of $\mathcal{O}_{X_{red}}$-modules. Let

$$C(h) = \max_{0 \leq n_1 < q'} \left\{C(h_{n_1}) \mid h_{n_1} : \mathcal{M} \longrightarrow \mathcal{M}(n_1)\right\},$$

where h_{n_1} is a fixed generically isomorphic map of sheaves of $\mathcal{O}_X$-modules and $C(h_{n_1})$ is the constant (see Lemma 2.8) associated to the map h_{n_1} (note that since k is infinite, we can always find such a map h_{n_1} for each n_1). Moreover, (compare (2.15)

$$\operatorname{coker}\phi_{m,q}(\mathcal{N}) = \operatorname{coker}\phi_{mq,qq'}(\mathcal{M}) \quad \text{for all} \quad m \geq 0 \quad \text{and} \quad n \geq 0.$$

Therefore,

$$\begin{aligned}|\frac{1}{q'^{d-1}}\widetilde{f}_n(\mathcal{N})(x) - \widetilde{f}_{n+s}(\widetilde{\mathcal{M}})| &= \frac{1}{(qq')^{d-1}}|\operatorname{coker}\phi_{mq,qq'}(\mathcal{M}) - \operatorname{coker}\phi_{mq+n_1,qq'}(\mathcal{M})| \\ &\leq \frac{C(h_n)(m+q)^{d-2}}{(qq')^{d-1}} \leq \frac{C(h)(n_0\mu)^{d-2}}{qq'}.\end{aligned}$$

This implies $HKd(\mathcal{N}, I)/(q')^{d-1} = HKd(M, I)$.

Let $Y_1, \ldots, Y_r$ be the irreducible reduced components of $Y = X_{red}$ corresponding to the prime ideals in the set $\Lambda = \{P_1, \ldots, P_r\}$. Let $x_1, \ldots, x_r$ denote the respective generic points in Y. Now the canonical generic isomorphism $\mathcal{N} \longrightarrow \bigoplus_i \mathcal{N} \mid_{Y_i}$ of sheaves of $\mathcal{O}_Y$ (hence $\mathcal{O}_X$-modules) gives

$$HKd(\mathcal{N}, I) = \sum_{i=1}^r HKd(\mathcal{N} \mid_{Y_i}, I).$$

Since $\mathcal{N}_i = \mathcal{N} \mid_{Y_i}$ is a coherent sheaf of $\mathcal{O}_{Y_i}$-modules, there exists $a \geq 0$ such that $\mathcal{N}_i(a)$ is globally generated (Theorem 5.17, Chapter II in [Ha]) for all i. Hence, if rank $\mathcal{N}_{x_i}$ = rank $(\mathcal{N}_i)_{x_i} = r_i$ as $\mathcal{O}_{Y_i,x_i} = \mathcal{O}_{Y,x_i}$-modules, then there exists a generic isomorphism $\bigoplus^{r_i}\mathcal{O}_{Y_i} \longrightarrow \mathcal{N}_i(a)$ of $\mathcal{O}_Y$-modules. Note that $\mathcal{N}_i$ is generically isomorphic to $\mathcal{N}_i(a)$. Therefore,

$$HKd(\mathcal{N}_i, I) = HKd(\mathcal{N}_i(a), I) = HKd(\mathcal{O}_{Y_i}, I)\ell(\mathcal{N}_{x_i})$$

$$\implies HKd(\mathcal{N}, I) = \sum_{i=1}^{r} HKd(\mathcal{O}_{Y_i}, I)\ell(\mathcal{N}_{x_i}) = (p^s)^{d-1}\sum_{i=1}^{r}\ell(M_{P_i})HKd(R/P_i, I).$$

Therefore,

$$HKd(M, I) = \sum_{i=1}^{r}\ell(M_{P_i})HKd(R/P_i, I).$$

Hence the result. □

Remark 2.15. For R and I as above, in addition suppose R is an equidimensional ring and $I \subseteq J$ are two graded ideals of R. Then we claim:

$$HKd(R, I) \simeq HKd(R, J) \text{ if and only if } J \subseteq I^*,$$

where I^* denotes the *tight closure* of I in R. To see this, we use the following result by [HH] and [A]: If $(R, \mathbf{m})$ is a formally unmixed local ring with $\mathbf{m}$-primary ideals $I \subseteq J$. Then $e_{HK}(I) = e_{HK}(J)$ if and only if $J \subseteq I^*$.

Note that in the graded case, the completion $\hat{R}$ of R with respect to R_+ is an equidimensional local ring. Also it is easy to see that the tight closure of a graded ideal is a graded ideal. Now if $HKd(I) = HKd(J)$, then by Theorem 1.1 we have $e_{HK}(I) = e_{HK}(J)$, therefore $e_{HK}(\hat{I}) = e_{HK}(\hat{J})$. By [HH] and [A], we have $\hat{J} \subseteq (\hat{I})^* \subset (I^*)^\wedge$. Hence $J \subseteq I^*$. Conversely $J \subseteq I^*$ implies that $e_{HK}(I) = e_{HK}(J)$. But then $HKd(I) \geq HKd(J)$ are continuous functions with the same integrals, which implies $HKd(I) = HKd(J)$.

Definition 2.16. Similar to the HK density function for a pair $(R, \mathbf{m})$, where R is a standard graded ring R, of dim $R \geq 2$, and $\mathbf{m}$ is the graded maximal ideal, we can define the Hilbert-Samuel density function as

$$HSd(R)(x) = F(x) = \lim_{n\to\infty} F_n(x), \text{ where } F_n(x) = \frac{1}{q^{d-1}}\ell(R_{\lfloor xq\rfloor}).$$

One can check that

$$F : \mathbb{R} \to \mathbb{R} \text{ is given by } F(x) = 0 \text{ for } x < 0$$

and

$$F(x) = e_0(R, \mathbf{m})x^{d-1}/(d-1)! \text{ for } x \geq 0,$$

where $e_0(R, \mathbf{m})$ is the Hilbert-Samuel multiplicity of R with respect to $\mathbf{m}$.

Note that

$$HKd(R, I)(x) = HSd(R)(x) = e_0(R, \mathbf{m})x^{d-1}/(d-1)! \text{ for all } x < \min\{n \mid I_n \neq 0\},$$

in particular for all $x < 1$.

Proposition 2.17. *Let $R_1, \ldots, R_r$ be standard graded rings of dimensions ≥ 2, over an algebraically closed field k of* char $p > 0$, *with irrelevant maximal ideals*

$\mathbf{m}_1, \ldots, \mathbf{m}_r$ and let $I_1, \ldots, I_r$ be homogeneous ideals, respectively, such that $\ell(R_i/I_i) < \infty$. Let us denote $HSd(R_i)(x) = \widetilde{F}_i(x)$ and $HKd(R_i, I_i) = \widetilde{f}_i(x)$. Then

$$HKd(R_1\#\cdots\#R_r, I_1\#\cdots\#I_r)(x) = \prod_{i=1}^{r} \widetilde{F}_i(x) - \prod_{i=1}^{r}\left(\widetilde{F}_i(x) - \widetilde{f}_i(x)\right).$$

In particular,

$$e_{HK}(R_1\#\cdots\#R_r, I_1\#\cdots\#I_r) = \int_0^{n_0\mu}\left\{\prod_{i=1}^{r} \widetilde{F}_i(x) - \prod_{i=1}^{r}\left(\widetilde{F}_i(x) - \widetilde{f}_i(x)\right)\right\}dx,$$

where $R\#S$ denotes the Segre product of graded rings R and S, given by $(R\#S)_n = R_n \otimes S_n$.

Proof. We prove the case $r = 2$, the rest follows by induction. Let $(R, \mathbf{m}_1)$ and $(S, \mathbf{m}_2)$ be two standard graded rings of dimension d_1 and d_2, respectively, such that I and J are two homogeneous ideals of R and S, respectively, with $\ell(R/I) < \infty$ and $\ell(S/J) < \infty$. Then

$$\begin{aligned}
&\ell\left(\frac{R\#S}{(I\#J)^{[q]}}\right)_{m+q}\\
&\quad= \ell(R_{m+q})\ell(S_{m+q}) - \left[\ell(R_{m+q}) - \ell(\frac{R}{I^{[q]}})_{m+q}\right]\left[\ell(S_{m+q}) - \ell(\frac{S}{J^{[q]}})_{m+q}\right]\\
&\quad= \ell(R_{m+q})\ell(S/J^{[q]})_{m+q} + \ell(S_{m+q})\ell(R/I^{[q]})_{m+q} - \ell(R/I^{[q]})_{m+q}\ell(S/J^{[q]})_{m+q}.
\end{aligned}$$

Let $F(x)$ and $G(x)$ be HSd functions of R and S, respectively, and let $f(x)$ and $g(x)$ be HKd functions of (R, I) and (S, J), respectively. Then

$$\frac{1}{q^{d_1+d_2-2}}\ell\left(R\#S/(I\#J)^{[q]}\right)_{\lfloor xq\rfloor} = F_n(x)g_n(x) + G_n(x)f_n(x) - f_n(x)g_n(x).$$

If $n_0 \geq 1$ is such that $\mathbf{m}_1^{n_0} \subseteq I$ and $\mathbf{m}_2^{n_0} \subseteq J$, for graded maximal ideals $\mathbf{m}_1$ and $\mathbf{m}_2$ of R and S, respectively, and $\mu \geq \mu(I)$ and $\mu(J)$, then, by Lemma 2.2, $F_n(x)g_n(x)$, $G_n(x)g_n(x)$ $f_n(x)g_n(x)$ are bounded real valued functions with support in the interval $[0, n_0\mu]$. Moreover, by Theorem 1.1, $f_n(x)$ and $g_n(x)$ converge uniformly to $f(x)$ and $g(x)$, respectively. It is obvious that, on any compact interval, the functions $F_n(x)$ and $G_n(x)$ converge uniformly to $F(x)$ and $G(x)$, respectively. Therefore, $F_n(x)g_n(x) + G_n(x)f_n(x) - f_n(x)g_n(x)$ converge uniformly to $F(x)G(x) + G(x)f(x) - f(x)g(x)$ and

$$HKd\ (R\#S, I\#J) = F(x)g(x) + G(x)f(x) - f(x)g(x).$$

This implies that

$$\begin{aligned}
e_{HK}(R\#S.I\#J) = {} & \frac{e_0(R)}{(d_1-1)!}\int_0^{n_0\mu} x^{d_1-1}g(x)dx\\
&+ \frac{e_0(S)}{(d_2-1)!}\int_0^{n_0\mu} x^{d_2-1}f(x)dx - \int_0^{n_0\mu} f(x)g(x)dx.
\end{aligned}$$

This proves the proposition. $\square$

Corollary 2.18 (Multiplicative property)**.** *For pairs (R, I) and (S, J) with $\dim R = d_1$ and $\dim S = d_2$, if $F(x)$ and $G(x)$ denote HSd functions of R and S, respectively, as given in Definition 2.16, then we have*

$$F_{R\#S} - HKd(R\#S, I\#J) = [F_R - HKd(R, I)] \cdot [F_S - HKd(S, J)].$$

Proof. Follows from Proposition 2.17. □

Theorem 2.19. *Let R be a standard graded reduced ring of dimension 1 and let I be a homogeneous ideal of R such that $\ell(R/I) < \infty$. Let $f_n(x) = \ell(R/I^{[q]})_{\lfloor xq \rfloor}$. Then $\{f_n(x)\}_{n\in\mathbb{N}}$ is a convergent (but need not be uniformly convergent) sequence for every $x \in [0,\infty)$ and*

$$e_{HK}(I,R) = \int_{\mathbb{R}} f(x)dx,$$

where $f(x) = \lim_{n\to\infty} f_n(x)$.

Proof. Let $h_1, \ldots, h_\mu$ be a set of homogeneous generators of I of degree $d_1, \ldots, d_\mu$ such that $d_0 = 0 < d_1 \leq d_2 \leq \ldots \leq d_\mu$.

Since R is a one dimensional ring there exists an integer $m_0 \geq 1$ such that $\ell(R_m) = \ell(R_{m+1})$ for all $m \geq m_0$. For $n \in \mathbb{N}$, we define

$$T_n = (0, m_0/q] \cup (d_1, d_1 + m_0/q] \cup \cdots \cup (d_\mu, d_\mu + m_0/q] \subseteq (0,\infty].$$

Claim. If $x \notin T_n$ then $f_n(x) = f_{n+l}(x)$ for all $l \geq 0$.

Proof of the claim. Since $T_{n+l} \subseteq T_n$ for all $l \geq 0$, it is enough to prove that $x \notin T_n$ implies $f_n(x) = f_{n+1}(x)$. Note that $x \notin T_n$; then

$$m = \lfloor xq \rfloor \notin (0, m_0) \cup (d_1q, d_1q + m_0) \cup \cdots \cup (d_\mu q, d_\mu q + m_0).$$

By definition

$$f_n(x) = \ell(R/I^{[q]})_m \quad \text{and} \quad f_{n+1}(x) = \ell(R/I^{[qp]})_{mp+n_1},$$

where $\lfloor xpq \rfloor = \lfloor xq \rfloor p + n_1$ for some $0 \leq n_1 < p$. Choose a non-zero divisor $a \in R_1$. Then we have the injective map $R_m \to R_{mp+n_1}$ given by $y \mapsto a^{n_1}y^p$ (this is a composition of two maps namely $R_m \to R_{mp}$, given by $y \mapsto y^p$, and $R_{mp} \to R_{mp+n_1}$, given by $x \mapsto a^{n_1}x$) which is an isomorphism (as k-vectorspaces) as $m = \lfloor xq \rfloor \geq m_0$. This gives a canonical surjective map $\phi : (R/I^{[q]})_m \longrightarrow (R/I^{[qp]})_{mp+n_1}$. Now to prove the claim, it is enough to prove that $(I^{[qp]})_{mp+n_1} \subseteq \phi(I^{[q]}_m)$. Let $f \in (I^{[qp]})_{mp+n_1}$; then $f = h_1^{qp}r_1 + \cdots + h_\mu^{qp}r_\mu$, where deg $r_j = mp + n_1 - d_jqp$.

If $r_j \neq 0 \implies mp + n_1 - d_jqp \geq 0 \implies m - d_jq \geq -n_1/p \implies m - d_jq \geq 0 \implies xq \geq d_jq$.

(1) $xq = d_jq$; then $n_1 = 0$ and $mp - d_jqp = 0$. Therefore, $r_j \in R_0 = k$. Hence $r_j = l_j^p$ for some $l_j \in R_0$.
(2) $xq > d_jq \implies m \geq d_jq+m_0$, but then deg $r_j \geq mp-d_jqp+n_1 \geq m_0p+n_1$. Therefore, $r_j = l_j^p a^{n_1}$ for some $l_j \in d_jq + m$.

This implies $f = (h_1^q l_1 + \cdots + h_\mu^q l_\mu)^p a^{n_1} \in \phi(I^{[q]}_m)$. This proves the claim.

Define $f(x) = \lim_{n\to\infty} f_n(x)$; this makes sense because

(1) if $x = 0$, then $f_n(0) = \ell(R_0)$ for all n.
(2) If $x > 0$, then there exists $n > 0$ such that $x \notin T_n$, which implies that $f_n(x) = f_{n+1}(x) = \cdots = f(x)$.

Moreover, for $y\in\mathbb{R}$, $f_n(y)\leq L_2(R)$, where $L_2(R)=\max\{\ell(R_0),\ell(R_1),\ldots,\ell(R_{m_0})\}$. Therefore, we have

$$\begin{aligned}|\int_{\mathbb{R}} f_n(x)dx-\int_{\mathbb{R}} f(x)dx| &\leq \int_{\mathbb{R}} |f_n(x)-f(x)|dx\\ &\leq \int_{T_n} |f_n(x)-f(x)|dx \leq L_2(R)(\mu+1)m_0/q.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)dx=\int_{\mathbb{R}} f(x)dx=\int_{\mathbb{R}}\left(\lim_{n\to\infty} f_n(x)\right)dx.$$

□

Remark 2.20. It is easy to check that in the case of dimension 1, $f_n\to f$ does not converge to f uninformly.

3. Examples

3.1. Projective spaces and their Segre products.

Example 3.1. Let $X=\mathbb{P}^d_k$ and let $R=k[X_0,\ldots,X_d]=\bigoplus_m R_m$. We denote the function $HKd(R,\mathbf{m})$ by $HKd(\mathbb{P}^d_k)$ and for a fixed $q=p^n$ we denote the map $\phi_{m,q}(R)$ by ϕ_m where we recall that $\phi_{m,q}(R): R_1^{[q]}\otimes R_m\to R_{m+q}$ is the canonical multiplication map. For $A_m=\binom{m+d}{d}$, it is obvious that

$$\begin{aligned}\operatorname{coker}\phi_{tq+l} &= A_{(t+1)q+l}-A_1\operatorname{coker}\phi_{(t-1)q+l}\\ &\quad + A_2\operatorname{coker}\phi_{(t-2)q+l}+\cdots+(-1)^{t+1}A_{t+1}\operatorname{coker}\phi_{l-q}.\end{aligned}$$

Now, for $q=p^n$,

$$f_n(x)=\operatorname{coker}\phi_{tq+l},\quad\text{where } \frac{(t+1)q+l}{q}\leq x<\frac{((t+1)q+l+1)}{q}\quad\text{with } 0\leq l<q.$$

Hence

$$f_n(x)=(1/q^d)A_{(t+1)q+l}-A_1f_n(x-1)+\cdots+(-1)^{t+1}A_{t+1}f_n(x-t-1).$$

Moreover, $f_n(x)=0$ if $x\geq d+1$. Therefore,

$$\begin{aligned}&HKd(\mathbb{P}^d_k)(x)\\ &\quad = f(x)=\frac{1}{d!}\left[x^d-\widetilde{A}_1(x-1)^d+\widetilde{A}_2(x-2)^d+\cdots+(-1)^{t+1}\widetilde{A}_{t+1}(x-t-1)^d\right],\end{aligned}$$

where $\widetilde{A}_1=(d+1)$ and $\widetilde{A}_2=\binom{d+1}{2}$ and $\widetilde{A}_{i+1}$ are defined iteratively as

$$\widetilde{A}_{i+1}=A_1\widetilde{A}_i-A_2\widetilde{A}_{i-1}+\cdots+y(-1)^{i-1}A_i\widetilde{A}_1+(-1)^iA_{i+1}.$$

This implies $\widetilde{A}_i=\binom{d+1}{i}$ for $1\leq i\leq d$. In particular,

$$\begin{aligned}HKd(\mathbb{P}^d_k)(x) &= x^d/d! \text{ for } 0\leq x<1\\ &= x^d/d!-A_i^d(x) \text{ for } i\leq x<i+1 \quad\text{provided}\quad 1\leq i\leq d\\ &= 0\qquad\text{otherwise,}\end{aligned}$$

where

$$A_i^d(x)=\frac{1}{d!}\left[\binom{d+1}{1}(x-1)^d+\cdots+(-1)^{i+1}\binom{d+1}{i}(x-i)^d\right].$$

Moreover, $HSd(\mathbb{P}^d_k)(x)=x^d/d!$.

Therefore, for the Segre product $\mathbb{P}_k^d\#\mathbb{P}_k^e$, where $d \leq e$, we have

$$\begin{aligned} &HKd(\mathbb{P}_k^d\#\mathbb{P}_k^e)(x)\\ &= \frac{x^dx^e}{d!e!} - A_i^d(x)A_i^e(x) \quad \text{for} \quad i \leq x < i+1, \quad \text{provided} \quad 1 \leq i \leq d\\ &= \frac{x^dx^e}{d!e!} - \frac{x^d}{d!}A_i^e(x) \quad \text{for} \quad d \leq x < e\\ &= 0, \qquad \text{for} \quad e \leq x. \end{aligned}$$

Remark 3.2. Similar (but more complicated) formulas can be obtained for arbitrary Segre products of projective spaces. The Hilbert-Kunz multiplicity of the Segre product of $\mathbb{P}_k^n \times \mathbb{P}_k^m$ has been computed by [EY].

3.2. **Projective curves and their Segre products.**

Example 3.3. Let R be a Noetherian standard graded ring of dimension 2. Then, for a pair $(R, \mathbf{m})$, where $\mathbf{m}$ is the graded maximal ideal, $e_{HK}(R, \mathbf{m})$ has been computed in [B] and [T1], Here we compute $HKd(R, \mathbf{m}) = f : \mathbb{R} \longrightarrow \mathbb{R}$ using the similar techniques used in these two papers.

Recall that if $x \in [0,1)$, then

$$f_n(x) = \frac{1}{q}\ell(R_m), \text{ where } m/q \leq x < m+1/q.$$

This implies that $HKd(R, \mathbf{m})(x) = f(x) = \lim_{n\mapsto\infty} f_n(x) = (d)(x)$, where $d := e_0(R, \mathbf{m})$ is the Hilbert-Samuel multiplicity of R with respect to the graded maximal ideal $\mathbf{m}$.

Now let $1 \leq x$; then $(m+q)/q \leq x < (m+q+1)/q$ for some $m \geq 0$, and

$$f_n(x) = \frac{1}{q}\ell(R/\mathbf{m}^{[q]})_{m+q} = \frac{1}{q}\ell(R/\mathbf{m}^{[q]})_{\lfloor xq\rfloor}.$$

Let $h_1, \ldots, h_s \in R_1$ be a set of generators of $\mathbf{m}$ and let

$$0 \longrightarrow V \longrightarrow \bigoplus \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

be the map of locally free sheaves of $\mathcal{O}_X$-modules. By Lemma 2.5, part(2), it follows that

$$HKd(R, \mathbf{m})(x) = f(x) = \lim_{n\mapsto\infty} \frac{1}{q}h^1(X, F^{n*}V(\lfloor(x-1)q\rfloor)).$$

By Theorem 2.7 in [L], there exists $n_1 \gg 0$ such that

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} \subset F^{n_1*}V$$

is the strong Harder-Narasimhan filtration of $F^{n_1*}V$. Let

$$a_i = \mu_i(F^{n_1*}V)/p^{n_1} = \mu(E_i/E_{i-1})/p^{n_1}, \ r_i = \operatorname{rank}(E_i/E_{i-1}) \tag{3.1}$$

be the normalized HN slope of V. Note that $a_i's$ are independent of the choice of n_1 as $\mu_i(F^{n*}V) = p^{n-n_1}\mu_i(F^{n_1*}V)$ for all $n \geq n_1$. Since $V \hookrightarrow \oplus\mathcal{O}_X$, $a_i(V) \leq 0$. In fact

$$-\frac{a_1}{d} < -\frac{a_2}{d} < \cdots < -\frac{a_{l+1}}{d}.$$

Moreover, we can take $n_1 \gg 0$ such that

$$\mu_i(F^{n_1*}V) - \mu_{i+1}(F^{n_1*}V) \geq 2g-2.$$

Therefore,

$$h^1(X, F^{n*}V(m)) = \sum_{i=1}^{l+1} h^1(X, F^{n-n_1*}(E_i/E_{i-1})(m))$$

and

$$-\frac{a_1 q}{d} < -\frac{a_1 q}{d} + (d-3) < -\frac{a_2 q}{d} < -\frac{a_2 q}{d} + (d-3) < \cdots < -\frac{a_{l+1} q}{d}.$$

Hence, we have

$$\begin{aligned}
&0 \leq m < -\frac{a_1 q}{d} \\
&\quad \implies f_n(x) = -\frac{1}{q}\sum_{i=1}^{l+1}(a_i q r_i + r_i d m + r_i(g-1)) - \frac{a_i q}{d} \leq m < -\frac{a_i q}{d} + (d-3) \\
&\quad \implies f_n(x) = -\frac{1}{q}\sum_{j=i+1}^{l+1}(a_j q r_j + r_j d m + r_j(g-1)) + \frac{C_i}{q} - \frac{a_i q}{d} \leq m < -\frac{a_{i+1} q}{d} \\
&\quad \implies f_n(x) = -\frac{1}{q}\sum_{j=i+1}^{l+1}(a_j q r_j + r_j d m + r_j(g-1)),
\end{aligned}$$

where $|C_i| \leq r_i(g(X)-1)$.

Therefore,

$$\begin{array}{lcl}
1 \leq x < 1 - a_1/d & \implies & f(x) = -\sum_{i=1}^{l+1}(a_i r_i + r_i d(x-1)) \\
1 - a_1/d \leq x < 1 - a_2/d & \implies & f(x) = -\sum_{i=2}^{l+1}(a_i r_i + r_i d(x-1)) \\
1 - a_i/d \leq x < 1 - a_{i+1}/d & \implies & f(x) = -\sum_{j=i+1}^{l+1}(a_j r_j + r_j d(x-1))
\end{array}$$

This implies

$$\begin{aligned}
e_{HK}(R, \mathbf{m}) &= \int_{x=0}^{1-a_{l+1}/d} f(x)dx \\
&= \int_{x=0}^{1} f(x)dx + \int_{x=1}^{1-a_1/d} f(x)dx + \cdots + \int_{x=1-a_l/d}^{1-a_{l+1}/d} f(x)dx \\
&= d/2 - \int_{y=0}^{-a_1/d}[a_1 r_1 + (r_1 d)y]dy - \int_{y=0}^{-a_2/d}[a_2 r_2 + (r_2 d)y]dy \\
&\quad + \cdots + - \int_{y=0}^{-a_{l+1}/d}[a_{l+1} r_{l+1} + (r_{l+1} d)y]dy.
\end{aligned}$$

Therefore,

$$e_{HK}(R, \mathbf{m}) = \frac{d}{2} + \sum_{i=1}^{l+1} \frac{r_i a_i^2}{2d}.$$

Remark 3.4. As $\{a_i\}_i$ are distinct numbers, the above formula for f implies that $HKd(R, \mathbf{m})$ determines the data $(d, \{r_i\}_i, \{a_i\}_i)$.

Moreover, for a pair (R, I), where I is a graded ideal generated by homogeneous elements $h_1, \ldots, h_\mu$ of degrees $d_1 \leq \cdots \leq d_\mu$, respectively,

$$\begin{aligned} HKd(R,I)(x) &= f(x) \\ &= \lim_{n\mapsto\infty} \frac{1}{q} h^1(X, F^{n*}V(\lfloor (x-1)q \rfloor)) - \lim_{n\mapsto\infty} \frac{1}{q} \sum_{i=1}^{\mu} h^1(X, \mathcal{O}_X(\lfloor xq \rfloor - qd_i)). \end{aligned}$$

It is easy to check that the second term is a piecewise linear polynomial (with rational coefficients) with singularities at distinct points of the set $\{d_1, \ldots, d_\mu\}$ and support in $[0, d_\mu]$. In particular, there exists rational numbers $0 = x_0 < x_1 < \cdots < x_s \leq \max\{n_0\mu, d_\mu\}$ and linear polynomials $q_i(x) \in \mathbb{Q}[x]$, such that $HKd(R,I)(x) = q_i(x)$ if $x \in [x_i, x_{i+1}]$ and $HKd(R,I)(x) = 0$ otherwise.

For the following corollary, we use the notation of Proposition 2.17.

Corollary 3.5. *Any Segre product of projective curves has rational Hilbert-Kunz multiplicity. More precisely, $e_{HK}(R_1\# \cdots \# R_r, I_1 \# \cdots \# I_r)$ is a rational number, where* dim $R_i = 2$, *for each i.*

Proof. Let $n_0, \mu \geq 1$ such that $\mathbf{m}_i^{n_0} \subseteq I$ and $\mu \geq \mu(I_i)$ for all i, where $\mathbf{m}_i$ denotes the graded maximal ideal of R_i. Let $\widetilde{d}$ be the maximum of the degree of the chosen generators of I_i and $n_0\mu$. Now, the above calculation shows that one can take a finite subdivision of the interval $[0, \widetilde{d}]$ by rational points t_i, namely

$$[0, \widetilde{d}] = \bigcup_{1\leq i\leq m} [t_i, t_{i+1}], \text{ where } t_1 < t_2 < \ldots < t_m$$

such that each function $HKd(R_i, I_i)(x)$, on each such interval $[t_j, t_{j+1}]$, is a linear polynomial in $\mathbb{Q}[x]$. Note that each $HSd(R_i)(x)$ is a polynomial in $\mathbb{Q}[x]$ on the whole of $[0, \widetilde{d}]$. Therefore, the assertion follows from Proposition 2.17. $\square$

Remark 3.6. By the same reasoning as in the above corollary, one can prove that the Hilbert-Kunz multiplicities of the arbitrary Segre product of full flag varieties, $\mathbb{P}_k^n$, Hirzebruch surfaces, projective curves, etc., (over a fixed algebraically closed field k) are rational numbers.

Example 3.7. Let $(R, \mathbf{m}_1)$ and $(S, \mathbf{m}_2)$ be two standard graded rings of dimension 2 with graded maximal ideals $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively. Let V_1 and V_2 be corresponding syzygy bundles for $(R, \mathbf{m}_1)$ and $(S, \mathbf{m}_2)$ with normalized slopes $a_1, a_2, \ldots, a_{i_3}$ and $b_1, \ldots, b_{j_2}$, respectively. Let $X = \text{Proj } R$ and $Y = \text{Proj } S$ be of degree d and g, respectively. If

$$-\frac{a_1}{d} < -\frac{a_2}{d} < \cdots < -\frac{a_{i_1}}{d} \leq -\frac{b_1}{g} < -\frac{b_2}{g} < \cdots < -\frac{b_{j_1}}{d} \leq -\frac{a_{i_1+1}}{d} < \cdots < -\frac{a_{i_2}}{d}$$

$$\leq -\frac{b_{j_1+1}}{g} < -\frac{b_{j_1+2}}{g} < \cdots < -\frac{b_{j_2}}{d} \leq -\frac{a_{i_2+1}}{d} < \cdots < -\frac{a_{i_3}}{d},$$

then for $x_i = a_i/d$ and $y_i = b_i/g$ we find (after some computation) that

$$\begin{aligned} e_{HK}(R\#S, \mathbf{m}_1\#\mathbf{m}_2) = \frac{dg}{6} \Bigg[& 2 + \sum_{j\geq 1} 3s_j y_j^2 - \sum_{j\geq 1} s_j y_j^3 + \sum_{i\geq 1} 3r_i x_i^2 - \sum_{i\geq 1} r_i x_i^3 \\ & + (\sum_{j\geq 1} 3s_j y_j)(r_1 x_1^2 + \cdots + r_{i_1} x_{i_1}^2) - (\sum_{j\geq 1} s_j)(r_1 x_1^3 + \cdots + r_{i_1} x_{i_1}^3) \\ & + (\sum_{i\geq i_1+1} 3r_i x_i)(s_1 y_1^2 + \cdots + s_{j_1} y_{j_1}^2) - (\sum_{i\geq i_1+1} r_i)(s_1 y_1^3 + \cdots + s_{j_1} y_{j_1}^3) \\ & + (\sum_{j\geq j_1+1} 3s_j y_j)(r_{i_1+1} x_{i_1+1}^2 + \cdots + r_{i_2} x_{i_2}^2) - (\sum_{j\geq j_1+1} s_j)(r_{i_1+1} x_{i_1+1}^3 + \cdots + r_{i_2} x_{i_2}^3) \\ & + (\sum_{i\geq i_2+1} 3r_i x_i)(s_{j_1+1} y_{j_1+1}^2 + \cdots + s_{j_2} y_{j_2}^2) - (\sum_{i\geq i_2+1} r_i)(s_{j_1+1} y_{j_1+1}^3 + \cdots + s_{j_2} y_{j_2}^3) \Bigg]. \end{aligned}$$

Note that every term of the first row is non-negative, whereas, in the second to last row, all the first terms are non-positive and all the second terms are non-negative.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India

Email address: vija@math.tifr.res.in