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EQUILIBRIUM STATES AND ZERO TEMPERATURE LIMIT ON TOPOLOGICALLY TRANSITIVE COUNTABLE MARKOV SHIFTS

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Abstract. Consider a topologically transitive countable Markov shift and, let f be a summable potential with bounded variation and finite Gurevic pressure. We prove that there exists an equilibrium state μ_{tf} for each $t > 1$ and that there exists accumulation points for the family $(\mu_{tf})_{t>1}$ as $t \to \infty$. We also prove that the Kolmogorov-Sinai entropy is continuous at ∞ with respect to the parameter t, that is, $\lim_{t\to\infty} h(\mu_{tf}) = h(\mu_\infty)$, where μ_∞ is an accumulation point of the family $(\mu_{tf})_{t>1}$. These results do not depend on the existence of Gibbs measures and, therefore, they extend results of [Israel J. Math. 125 (2001), pp. 93–130] and [Ergodic Theory Dynam. Systems 19 (1999), pp. 1565–1593] for the existence of equilibrium states without the big images and preimages (BIP) property, [J. Stat. Phys. 119 (2005), pp. 765–776] for the existence of accumulation points in this case and, finally, we extend completely the result of [J. Stat. Phys. 126 (2007), pp. 315–324] for the entropy zero temperature limit beyond the finitely primitive case.

1. Introduction

The thermodynamic formalism is a branch of the ergodic theory that studies existence, uniqueness, and properties of equilibrium states, that is, measures that maximize the value $h(\mu) + \int f d\mu$ where $h(\mu)$ is the Kolmogorov-Sinai entropy. If, for each $t > 1$, there is a unique equilibrium state μ_{tf} associated to the potential tf, an interesting problem is to study the accumulation points of the family $(\mu_{tf})_{t>1}$, as well as the behavior of the family as $t \to \infty$, since in statistical mechanics any accumulation points are the ground states of the system.

We are interested in the case of topologically transitive countable Markov shifts. It is well known from [4] that there is at most one equilibrium state, and existence is guaranteed usually by means of a strong condition on the dynamics. It is usually assumed that the incidence matrix satisfies properties like being finitely primitive, which is equivalent to the big images and preimages (BIP) property when the shift is topologically mixing, which is equivalent to the existence of Gibbs measures [15]. In general, it is very difficult to find a simple condition without such hypothesis. Our first result goes in this direction, and therefore extends the results for the

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existence of equilibrium states beyond the finitely primitive case, as the ones in [12] and [16].

Theorem 1. *Let Σ be a topologically transitive countable Markov shift and let $f : \Sigma \to \mathbb{R}$ be a summable potential such that $V(f) < \infty$ and $P_G(f) < \infty$. Then, for any $t > 1$ there is a unique equilibrium state μ_{tf} associated to the potential tf. Also, we have that as $t \to \infty$, there exists accumulation points for the family $(\mu_{tf})_{t>1}$.*

The proof of this result is similar to one of the results in [12], that guarantees under similar conditions the existence of an eigenmeasure of the dual Ruelle operator. But it is not shown that in fact such an eigenmeasure gives birth to an equilibrium state. As is well known that is not always the case. See the example at the end of [15] for more information.

In the case of Markov shifts with finite alphabet, the existence of accumulation points is trivial. For countable Markov shifts, Jenkinson, Mauldin, and Urbański in [9] gave conditions on the potential f to guarantee the existence of accumulation points for the family $(\mu_{tf})_{t>1}$, assuming that Σ is a finitely primitive Markov shift with countable alphabet, and they also prove that these accumulation points are maximizing measures for f. Here, we do not focus on the maximizing property, since it is already known that if there exists an accumulation point for the equilibrium states as $t \to \infty$, then it is maximizing [2]. Again, we emphasize these results are in the context where there exists Gibbs measures.

We notice that existence of accumulation points does not imply the existence of the zero temperature limit of the equilibrium states, for which another conditon is usually required. One of the first works on the subject was done by Brémont in [3], where the convergence when f depends only on a finite number of coordinates is proved, that is, f is locally constant, and Σ is a topologically transitive Markov shift with finite alphabet. More recently, Leplaideur in [11] gave an explicit form for the zero temperature limit when Σ is a topologically Markov shift with finite alphabet.[1] Also, in [6], following ideas in a different context from [18], an example of a Lipschitz potential for which there is no convergence if we drop the requirement of locally constant is given. In fact, if we do not require the potential to be locally constant, the situation is much more complex, as can be seen in [7], where for any two ergodic measures with the same entropy fixed, it is possible to find a Lipschitz potential such that the equilibrium states of this potential accumulates on both ergodic measures. In the case of Markov shifts with countable alphabet and the BIP property, the existence of the limit for locally constant potentials has been proved by Kempton [10]. The question of whether the equilibrium states for a locally constant potential in the non-BIP setting, that is, without the existence of Gibbs measures, is still open.

Also, under the same conditions as [9], Morris has proved in [14] the existence of the limit $\lim_{t\to\infty} h(\mu_{tf})$ of the family of associated Kolmogorov-Sinai entropies and showed that this limit agrees with the supremum of the entropies over the set of the maximizing measures of f. We are able to give an extension of this result in the same context as of the previous theorem.

[1]The proof uses an aperiodic incidence matrix, but it is essentially the same in the present context, as can be seen also in similar results of [5].

Theorem 2. *Let Σ be a topologically transitive countable Markov shift and let $f : \Sigma \to \mathbb{R}$ be a summable Markov potential such that $V(f) < \infty$ and $P_G(f) < \infty$. Then*

$$h(\mu_\infty) = \limsup_{t\to\infty} h(\mu_{tf}) = \sup_{\mu\in\mathcal{M}_{\max}(f)} h(\mu)\,,$$

where μ_∞ is an accumulation point of the family $(\mu_{tf})_{t>1}$.

As we have stated, the main development is that we do not depend on the existence of Gibbs measures for the whole space, since we use the condition that f is a summable potential to guarantee the existence of the equilibrium state μ_{tf} for each $t > 1$. To the best of our knowledge, this is the first proof of the convergence in the zero temperature limit beyond the finitely primitive case.

Our proofs are based on an elaborate construction similar to a diagonal argument. We approximate our countable Markov shift by compact invariant subshifts Σ_k and use the results in [1] to locate the ground states on a well-determined compact subshift Σ_{k_0}. Then, as $k \to \infty$, we use the fact that the potential is summable and the variational principle, through a fine control of the entropy, to show that there exists equilibrium states for each $t > 1$ and their accumulation points as $t \to \infty$. As usual, this is not as simple as it might seem at first glance, in particular since Σ is not σ-compact, and also since in this general setting we lose some important tools in classical thermodynamic formalism, such as the Gibbsianess of the equilibrium states and some strong properties on the Ruelle operator. We give some more detail on this in the next section.

The paper is organized as follows. In the next section, we give the basic definitions and notation that we use in the proofs and results, as well as more details on the setting we are working on. In section 3 we construct the basic approximation by compacts that we have to deal with to prove our results. In section 4, we use the existence of the equilibrium states on the compact case to show the existence of a unique equilibrium state in the case of countable Markov shifts, therefore proving Theorem 1. It is in section 4 that most of our main hypothesis show their strength, since we have to make careful estimates to control the entropy and prove the existence of an equilibrium state. In section 5, we use the existence of accumulation points in zero temperature to show the existence of the entropy zero temperature limit, therefore proving Theorem 2.

2. Preliminaries

Let $\mathcal{A}$ be a countable alphabet and let $\mathbf{M}$ be a matrix of zeros and ones indexed by $\mathcal{A} \times \mathcal{A}$. Let Σ be a topological Markov shift on the alphabet $\mathcal{A}$ with incidence matrix $\mathbf{M}$, that is,

$$\Sigma = \{x \in \mathcal{A}^{\mathbb{N}\cup\{0\}} : \mathbf{M}_{x_i,x_{i+1}} = 1\}\,,$$

where the dynamics are given by the shift map $\sigma : \Sigma \to \Sigma$, that is, the map defined by $\sigma((x_n)_{n\geq 0}) = (x_n)_{n\geq 1}$. Just to ease the calculations we suppose that $\mathcal{A} = \mathbb{N}$. Recall that a word ω is admissible if ω appears in $x \in \Sigma$. It is well known that Σ is a metric space with topology compatible to the product topology. Also, the topology has a sub-base made by cylinders that are both open and closed sets defined as

$$[\omega] = \{x \in \Sigma : x \text{ starts with the word } \omega\}\,.$$

From now on, we assume that Σ is topologically transitive in the sense that σ is topologically transitive. It is well known that this is equivalent to $\mathbf{M}$ being irreducible, which is also equivalent to the fact that given any symbols $i, j \in \mathcal{A}$ there is an admissible word ω such that $i\omega j$ is also admissible.

Fix a potential $f : \Sigma \to \mathbb{R}$, for each $n \geq 1$ we define the nth variation of f as

$$V_n(f) = \sup\{|f(x) - f(y)| : x_0 \dots x_{n-1} = y_0 \dots y_{n-1}\}\,.$$

We say that f has summable (or bounded) variations if

$$V(f) = \sum_{n \in \mathbb{N}} V_n(f) < \infty\,.$$

We say that f is coercive if $\lim_{i \to \infty} \sup f|_{[i]} = -\infty$, and that f is summable if it satisfies

$$\sum_{i \in \mathbb{N}} \exp(\sup(f|_{[i]})) < \infty\,. \tag{2.1}$$

Observe that if f is summable, then f is coercive. Moreover, it is shown in [14][2] that the summability condition implies, for any $t > 1$, that

$$\sum_{i \in \mathbb{N}} \sup(-tf|_{[i]}) \exp(\sup(tf|_{[i]})) < \infty\,. \tag{2.2}$$

It follows from [1] that the summability condition allows us to guarantee existence of maximizing measures for the potential f, since it is coercive. Also, it is the key ingredient to prove in this paper the existence of the equilibrium state associated to tf for each $t > 1$.

From now on, we assume that $V(f) < \infty$ and that f is summable. Observe that in this case f is uniformly continuous and bounded above.

We use the following notation:

$$\mu(f) := \int f d\mu\,.$$

Define

$$\beta = \sup\{\mu(f) : \mu \in \mathcal{M}_\sigma(\Sigma)\}\,, \tag{2.3}$$

where $\mathcal{M}_\sigma(\Sigma)$ is the set of σ-invariant Borel probability measures on Σ. When $\mu(f) = \beta$, we say that μ is a maximizing measure, and denote the set of maximizing measures by $\mathcal{M}_{\max}(f)$. For each $a \in \mathcal{A}$, let

$$Z_n(f, a) = \sum_{\sigma^n x = x} \exp(S_n f(x)) 1_{[a]}(x)\,,$$

with $S_n f(x) = \sum_{i=0}^{n-1} f(\sigma^i(x))$. Then, the Gurevic pressure of f is defined as

$$P_G(f) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(f, a)\,.$$

Since Σ is topologically transitive, the above pressure definition is independent from the choice of $a \in \mathcal{A}$. Furthermore $-\infty < P_G(f) \leq \infty$ and satisfies the variational principle (see [16] and [8])

$$P_G(f) = \sup\{h(\mu) + \mu(f) : \mu \in \mathcal{M}_\sigma(\Sigma) \text{ and } \mu(f) > -\infty\}\,. \tag{2.4}$$

So, in this case we have that the Gurevic pressure is the same as the topological pressure, and then the hypothesis that requires finite pressure can be read in any way.

[2]One can easily notice the BIP property is not required for this.

Also, $P_G(f)$ satisfies another variational principle (see [16] and [8])

$$P_G(f) = \sup\{P_G(f|_{\Sigma'}) : \Sigma' \text{ is a compact subshift of } \Sigma\}\,. \tag{2.5}$$

A measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is called an equilibrium state associated to f when $h(\mu) + \mu(f)$ is well defined and reaches the supremum in (2.4), that is,

$$P_G(f) = h(\mu) + \mu(f)\,.$$

We say that a measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is an invariant Gibbs state associated to f if there is a constant $C > 1$ such that for any $x \in \Sigma$ and each $n \geq 1$

$$C^{-1} \leq \frac{\mu[x_0 \dots x_{n-1}]}{\exp(S_n f(x) - nP_G(f))} \leq C\,. \tag{2.6}$$

When Σ is compact the equilibrium states and the invariant Gibbs states are unique and agree; see, for example, [12]. Furthermore, when Σ is compact and the potential f has summable variations, we can choose

$$C = \exp(4V(f))\,. \tag{2.7}$$

Recall that the Ruelle operator L_f associated to f is defined as

$$(L_f g)(x) = \sum_{\sigma y = x} \exp(f(y))g(y)\,,$$

and since f is summable, $P_G(f) < \infty$ and $V(f) < \infty$, L_f is well defined. Also, one can look into the dual operator L_f^* defined as

$$(L_f^*\mu)(g) = \mu(L_f g)\,,$$

with $g : \Sigma \to \mathbb{C}$. These operators and their properties are the base for the following well-known results.

In [4] the uniqueness of the equilibrium state, whenever it exists, assuming Σ is a topologically transitive countable Markov shift and f is bounded above with summable variation and $P_G(f) < \infty$ is proved. In this case, we will denote by μ_f the unique equilibrium state associated to f.

Furthermore, it is proved in [4] that if the equilibrium state exists, then

$$\mu_f = h d\nu,$$

where h is the eigenfunction of L_f associated to the eigenvalue $\lambda = e^{P_G(f)}$ and ν is the eigenmeasure of L_f^* associated to the same eigenvalue. In [12], it is proved that assuming f is a summable potential, then there exists an eigenmeasure ν for L_f^*, but it is not proved that it will, in fact, result in an equilibrium state.

In this paper, we take a different approach. Instead of finding an eigenmeasure to L_f^* and working out its properties to prove it is in fact an equilibrium state under our hypothesis, we show that the equilibrium states on a suitable sequence of compact Markov shifts accumulate on a measure that is, in fact, an equilibrium state on the countable Markov shift, by the argument we have sketched in the previous section. This approach gives us some minor benefits, in particular the fact that we can easily realize that the equilibrium states family is tight.

We can suppose, w.l.o.g. that $f \leq 0$, since f is bounded above by the fact that it is coercive and $V(f) < \infty$, so we can consider $f - \sup f$ instead of f.

3. Compact subshifts approximation

We suppose that f is a summable potential such that $P_G(f) < \infty$ and $V(f) < \infty$ from now on. Our aim is to prove the existence of equilibrium states for tf and $t > 1$ and accumulation points for such a sequence of equilibrium states as we approach the zero temperature limit on topologically transitive countable Markov shifts. We accomplish this using an approximation by compact subshifts of Σ and its Gibbs equilibrium states in each compact subspace.

We can choose a sequence $(\Sigma_k)_{k\in\mathbb{N}}$ of compact topologically transitive subshifts of Σ such that for any $k \in \mathbb{N}$ we have $\Sigma_k \subsetneq \Sigma$ and $\Sigma_k \subsetneq \Sigma_{k+1}$ and such that the variational principle (2.5) can be resumed to

$$P_G(f) = \sup\{P_G(f|_{\Sigma_k}) : k \in \mathbb{N}\}\,. \tag{3.1}$$

In fact, for each $k \in \mathbb{N}$ we can choose $\mathcal{A}_k := \{0, \dots, m_k\} \cup \{\text{finite elements}\}$, where m_k is a strictly increasing sequence in $\mathbb{N}$ and the choice of finite elements, which depends on k, is made to allow us to connect any of the symbols in $\{0, \dots, m_k\}$. It is always possible to choose such a finite alphabet since Σ is topologically transitive. We can also choose $m_k = \max\{j : j \in \mathcal{A}_{k-1}\} + 1$ for $k \geq 1$, which assures us that $\mathcal{A}_k \subset \mathcal{A}_{k+1}$ and $\bigcup_{k\in\mathbb{N}} \mathcal{A}_k = \mathcal{A}$. This construction is classical and appears, for example, in [13].

It is not difficult to show that for all k we have $\Sigma_k \subset \Sigma$, $\sigma|_{\Sigma_k}$ is also topologically transitive and that this construction satisfies (3.1), since any compact subshift of Σ is contained in some Σ_k for large k.

Also, notice that our proof below works mainly because (3.1) is true in this sequence, and it does not require that Σ is in fact decomposed into this sequence, which would be impossible, since Σ is not σ-compact.

Let $f_k := f|_{\Sigma_k} : \Sigma_k \to \mathbb{R}$ be the restriction of f to Σ_k and

$$\beta_k := \sup\{\mu(f_k) : \mu \in \mathcal{M}_\sigma(\Sigma_k)\}\,.$$

Since f is summable, we have that f is coercive. Therefore, the main theorem in [1] says that there is a finite set $F \subset \mathcal{A}$ such that (2.3) becomes

$$\beta = \sup\{\mu(f) : \mu \in \mathcal{M}_\sigma(\Sigma_F)\}\,,$$

where Σ_F is the restriction of Σ to the alphabet F, and since Σ_F is compact, then $\mathcal{M}_{\max}(f) \neq \emptyset$. Moreover, it also implies that for any $\mu \in \mathcal{M}_{\max}(f)$ we have $\mathrm{supp}(\mu) \subset \Sigma_{\mathrm{F}}$.

Denote by $P : [1, \infty) \to \mathbb{R}$ the function $t \mapsto P(t) = P_G(tf) < \infty$, and consider the sequence $(P_k)_{k\in\mathbb{N}}$ such that $P_k : [1, \infty) \to \mathbb{R}$ is the function $t \mapsto P_k(t) = P_G(tf_k)$. Since for each $k \in \mathbb{N}$

$$\sum_{\sigma^n x = x} \exp(S_n t f(x)) 1_{[a]}(x) \geq \sum_{\sigma^n x = x} \exp(S_n t f(x)) 1_{[a]}(x) 1_{\Sigma_k}(x)\,,$$

then for any $t \geq 1$ we have $P_k(t) \leq P_{k+1}(t) \leq P(t)$, and it follows from (3.1) that

$$P(t) = \sup\{P_k(t) : k \in \mathbb{N}\}\,. \tag{3.2}$$

Below we show that the sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ in $\mathcal{M}_\sigma(\Sigma)$ has a convergent subsequence. For this, we prove that this sequence is tight. Let us recall that a subset $\mathcal{K} \subset \mathcal{M}_\sigma(\Sigma)$ is tight if for every $\epsilon > 0$ there is a compact set $K \subset \Sigma$ such that $\mu(K^c) < \epsilon$ for any $\mu \in \mathcal{K}$.

Lemma 1. *For each $t > 1$ the equilibrium states sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ is tight.*

Proof. Our proof is similar to the proof in [9]. Let $\epsilon > 0$ and

$$K = \{x \in \Sigma : 1 \leq x_m \leq n_m \text{ for each } m \in \mathbb{N}\},$$

where $(n_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ is an increasing sequence. Then the set K is compact in the product topology and satisfies

$$\begin{aligned}\mu_{tf_k}(K^c) &= \mu_{tf_k}\Big(\bigcup_{m\in\mathbb{N}} \{x \in \Sigma : x_m > n_m\}\Big) \\ &\leq \sum_{m\in\mathbb{N}} \sum_{i>n_m} \mu_{tf_k}(\{x \in \Sigma : x_m = i\}) \\ &= \sum_{m\in\mathbb{N}} \sum_{i>n_m} \mu_{tf_k}([i]) .\end{aligned} \tag{3.3}$$

We choose $(n_m)_{m\in\mathbb{N}}$ such that

$$\sum_{i>n_m} \mu_{tf_k}([i]) < \frac{\epsilon}{2^{m+1}} .$$

Let $\mu \in \mathcal{M}_\sigma(\Sigma)$ such that $S = \mu(f)$ satisfies $-\infty < S < \infty$. It is sufficient to choose $\mu = \frac{1}{p}\sum_{j=0}^{p-1} \delta_{\sigma^j \bar{x}}$ with $\bar{x} \in \mathrm{Per}_p(\Sigma_0)$. Notice that since $\Sigma_0 \subset \Sigma_k$ for all $k \in \mathbb{N}$, we can consider μ to be a measure well defined both in Σ or in Σ_k for any k. Let $S_k = \mu(f_k)$ and by the previous comment, we have that S_k is well defined, and it is also clear that $S_k = S$ for any $k \in \mathbb{N}$, since $\mu(f)$ is the ergodic average of f over a periodic orbit in Σ_0 and f and f_k are the same in Σ_0 for any k. Furthermore, we have that

$$P_k(t) - tS_k = P_G(t(f_k - S_k)) \geq h(\mu) + t(\mu(f_k) - S_k) = h(\mu) \geq 0 . \tag{3.4}$$

Since each Σ_k is compact, from (2.6), (2.7) and the fact that $\exp(4V(tf_k)) \leq \exp(4tV(f)) < \infty$, then for all $x \in [i]$ we have that

$$\exp(-4V(tf)) \leq \frac{\mu_{tf_k}[i]}{\exp(tf_k(x) - P_k(t))} \leq \exp(4V(tf)) . \tag{3.5}$$

Recall that $S_k = S$ for all k and by (3.4), we obtain that for each $x \in [i]$

$$\begin{aligned}\mu_{tf_k}([i]) &\leq \exp(4tV(f) + tf_k(x) - P_k(t)) \\ &\leq \exp(4tV(f) + t\sup f|_{[i]} - P_k(t)) \\ &= \exp(t(4V(f) + \sup f|_{[i]} - S_k))\exp(tS_k - P_k(t)) \\ &\leq \exp(t(4V(f) + \sup f|_{[i]} - S)) .\end{aligned}$$

Since the potential f is coercive, then for i large enough we have the inequality $4V(f) + \sup f|_{[i]} - S \leq 0$, and since $t > 1$ we have

$$\mu_{tf_k}([i]) \leq \exp(4V(f) + \sup f|_{[i]} - S) . \tag{3.6}$$

Moreover, from the summability condition (2.1) we can suppose that the sequence $(n_m)_{m\in\mathbb{N}}$ satisfies

$$\sum_{i>n_m} \exp(\sup f|_{[i]}) < \frac{\epsilon}{2^{m+1}} \exp(S - 4V(f)) .$$

Therefore, for any k

$$\sum_{i>n_m} \mu_{tf_k}([i]) \leq \sum_{i>n_m} \exp(4V(f) + \sup f|_{[i]} - S) < \frac{\epsilon}{2^{m+1}} . \tag{3.7}$$

From (3.3) and (3.7), we conclude that

$$\mu_{tf_k}(K^c) < \sum_{m\in\mathbb{N}} \frac{\epsilon}{2^{m+1}} = \epsilon\,.$$

$\square$

Remark 1. The last passage in the proof of Lemma 1 is the key point where the summability condition is in fact needed. It is not the only point where the summability condition is essential, but it would be interesting to know if it is possible to use a different argument assuming only that f is coercive, even if we need some additional regularity on f.

4. Proof of Theorem 1

In this section we prove our first theorem, which is a consequence of the summability condition (2.1), the finiteness of the pressure, and the tightness of the sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$.

By *Prohorov's theorem*, there exists a subsequence $(\mu_{tf_{k_m}})_{m\in\mathbb{N}}$ of the sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ and a measure $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ such that

$$\mu_t = \lim_{m\to\infty} \mu_{tf_{k_m}}\,.$$

Our aim is to prove that μ_t is, indeed, an equilibrium state associated to the potential tf. For this purpose, we have to estimate both $\mu_t(tf)$ and $h(\mu_t)$ to prove they are finite, so that $h(\mu_t) + \mu_t(tf)$ is well defined and then we can finally verify that they are also maximal. We begin with $\mu_t(tf)$.

From (3.5), for each $m \in \mathbb{N}$ we have

$$\mu_{tf_{k_m}}[i] \le \exp(4tV(f) + \sup(tf|_{[i]}) - P_0(t))\,,$$

and notice that for each $i \ge 1$ we have $\partial[i] = \emptyset$, where $\partial[i]$ is the topological boundary of the cylinder $[i]$. Therefore, the cylinder $[i]$ is a continuity set of μ_t, and taking the limit as $m \to \infty$ we obtain

$$\mu_t[i] \le \exp(4tV(f) + \sup(tf|_{[i]}) - P_0(t))\,.$$

Then, for any $t > 1$, we have

$$\begin{aligned}
\mu_t(-tf) &= \mu_t\left(\sum_{i\in\mathbb{N}} -tf|_{[i]}\right) \\
&\le \sum_{i\in\mathbb{N}} \sup(-tf|_{[i]})\mu_t[i] \qquad (4.1)\\
&\le C_t \sum_{i\in\mathbb{N}} \sup(-tf|_{[i]})\exp(\sup(tf|_{[i]})) < \infty\,.
\end{aligned}$$

So $\mu_t(tf)$ is finite for any $t > 1$ and in fact we can approximate it well through our compact sequence, as the following lemma shows.

Lemma 2. *For each $t > 1$ we have that $\mu_t(tf) = \lim_{m\to\infty} \mu_{tf_{k_m}}(tf)$.*

Proof. Observe that this result is not a direct consequence of the weak* convergence of the sequence $(\mu_{tf_{k_m}})_{m\in\mathbb{N}}$ since the potential tf is not bounded below.

For each $N \in \mathbb{N}$, denote by $f^N : \Sigma \to \mathbb{R}$ the function

$$f^N(x) := \begin{cases} f(x) & \text{if } f(x) \geq -N, \\ -N & \text{otherwise,} \end{cases}$$

and set $g^N : \Sigma \to \mathbb{R}$ such that $f = f^N + g^N$.

Notice that both f^N and g^N are continuous, and that f^N is bounded. Also, it is easy to verify that $|g^N(x)| \leq |f(x)|$ and $g^N(x) \leq 0$ for all $x \in \Sigma$.

First, let us show that given $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that $|\mu_{tf_{k_m}}(tg^N)| < \frac{\epsilon}{4}$ and $|\mu_t(tg^N)| < \frac{\epsilon}{4}$ for all $N \geq N_0$ and any m.

We prove it for μ_t since for $\mu_{tf_{k_m}}$ it is similar and, in fact, it works with the same N_0.

Given $n_0 \in \mathbb{N}$, since f is coercive and $V(f) < \infty$ we can choose $N_0 = N_0(n_0)$ large enough such that $g^N|_{[i]} = 0$ if $i \leq n_0$ for all $N \geq N_0$. In this way, we can proceed as in (4.1) and we get

$$\begin{aligned} |\mu_t(tg^N)| = \mu_t(-tg^N) &= \mu_t\left(\sum_{i>n_0} -tg^N|_{[i]}\right) \\ &\leq \sum_{i>n_0} \sup(-tg^N|_{[i]})\mu_t[i] \\ &\leq C_t \sum_{i>n_0} \sup(-tg^N|_{[i]}) \exp(\sup(tf|_{[i]})) \\ &\leq C_t \sum_{i>n_0} \sup(-tf|_{[i]}) \exp(\sup(tf|_{[i]}))\,, \end{aligned}$$

and if we choose n_0 large enough, we get $|\mu_t(tg^N)| < \frac{\epsilon}{4}$ for $N \geq N_0$ and all m as desired.

Finally, to conclude the proof, given $\epsilon > 0$, fix $N \geq N_0$ as above and we have that

$$\begin{aligned} |\mu_{tf_{k_m}}(tf) - \mu_t(tf)| &\leq |\mu_{tf_{k_m}}(tf^N) - \mu_t(tf^N)| + |\mu_{tf_{k_m}}(tg^N) - \mu_t(tg^N)| \\ &\leq |\mu_{tf_{k_m}}(tf^N) - \mu_t(tf^N)| + |\mu_{tf_{k_m}}(tg^N)| + |\mu_t(tg^N)| \\ &\leq |\mu_{tf_{k_m}}(tf^N) - \mu_t(tf^N)| + \frac{\epsilon}{2}\,, \end{aligned}$$

and recall that since f^N is continuous and bounded and $\mu_t = \lim_{m\to\infty} \mu_{tf_{k_m}}$, the result follows. □

Now we turn our attention to the estimates on $h(\mu_t)$. Let $\alpha = \{[a] : a \in \mathcal{A}\}$ be the natural partition of Σ; then for any $\mu \in \mathcal{M}_\sigma(\Sigma)$ we have its Kolmogorov-Sinai entropy defined by

$$h(\mu) = \inf_n \frac{1}{n} H(\mu \mid \alpha^n)\,. \tag{4.2}$$

Since for any $m \in \mathbb{N}$ we have

$$h(\mu_{tf_{k_m}}) = P_{k_m}(t) + \mu_{tf_{k_m}}(-tf)\,,$$

the sequence $(\mu_{tf_{k_m}}(-tf))_{m\in\mathbb{N}}$ is bounded above and $P_{k_m}(t) \leq P(t)$. Then, there is a constant $B > 0$ such that

$$h(\mu_{tf_{k_m}}) \leq P(t) + \mu_{tf_{k_m}}(-tf) \leq B\,. \tag{4.3}$$

In this way, we have the following.

Proposition 1. *For each $N \in \mathbb{N}$ there is $n_0 \geq N$ such that the sequence $(H(\mu_{tf_{k_m}} \mid \alpha^{n_0}))_{m\in\mathbb{N}}$ is bounded above.*

Proof. Suppose that the proposition is not true; then there exists $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$ the sequence $(H(\mu_{tf_{k_m}} \mid \alpha^n))_{m\in\mathbb{N}}$ is not bounded. Let $n' \geq N_0$ and $B' > 0$; then there exists a subsequence $(k_l)_{l\in\mathbb{N}}$ of the sequence $(k_m)_{m\in\mathbb{N}}$ such that $H(\mu_{tf_{k_l}} \mid \alpha^{n'}) > B'$ for any $l \in \mathbb{N}$, particularly we have $\frac{1}{n'}H(\mu_{tf_{k_l}} \mid \alpha^{n'}) > \frac{B'}{n'}$. Since B' is choosing arbitrarily, then we can choose the sequence $(k_l)_{l\in\mathbb{N}}$ such that the last inequality is true for $B' = 2Bn'$, i.e., $\frac{1}{n'}H(\mu_{tf_{k_l}} \mid \alpha^{n'}) > 2B$ for any $l \in \mathbb{N}$. On the other hand the sequence $(h(\mu_{tf_{k_m}}))_{m\in\mathbb{N}}$ is bounded above with upper bound B; then by (4.2) we obtain a contradiction. Therefore, the proposition is true. $\square$

Lemma 3. *Let $\mu_t = \lim_{m\to\infty} \mu_{tf_{k_m}}$; then for each $N \in \mathbb{N}$ there are $n_0 \geq N$ and a subsequence $(k_l)_{l\in\mathbb{N}}$ of the sequence $(k_m)_{m\in\mathbb{N}}$ such that*

$$H(\mu_t \mid \alpha^{n_0}) = \lim_{l\to\infty} H(\mu_{tf_{k_l}} \mid \alpha^{n_0}) .$$

Proof. Let $N \in \mathbb{N}$. By the above proposition there exists $n_0 \geq N$ such that $(H(\mu_{tf_{k_m}} \mid \alpha^{n_0}))_{m\in\mathbb{N}}$ is bounded above. This implies that there is a subsequence $(k_l)_{l\in\mathbb{N}}$ of the sequence $(k_m)_{m\in\mathbb{N}}$ such that $\lim_{l\to\infty} H(\mu_{tf_{k_l}} \mid \alpha^{n_0}) < \infty$.

Moreover, for $l_0 \in \mathbb{N}$ and any $j \geq l_0$ we have

$$\inf_{l\geq l_0}\{-\mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega])\} \leq -\mu_{tf_{k_j}}[\omega]\log(\mu_{tf_{k_j}}[\omega]).$$

Therefore, summing up all the $[\omega] \in \alpha^{n_0}$ we obtain that

$$\sum_{[\omega]\in\alpha^{n_0}} \inf_{l\geq l_0}\{-\mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega])\} \leq \inf_{j\geq l_0}\left\{\sum_{[\omega]\in\alpha^{n_0}} -\mu_{tf_{k_j}}[\omega]\log(\mu_{tf_{k_j}}[\omega])\right\} .$$

Moreover, taking the limit as $l_0 \to \infty$ and using the monotone convergence theorem for the integrable function $\phi_m(x) = \inf_{l\geq m}\{-\mu_{tf_{k_l}}(x)\log(\mu_{tf_{k_l}}(x))\}$ with the counting measure on the set $\{[\omega] : [\omega] \in \alpha^{n_0}\}$, we get

$$\sum_{[\omega]\in\alpha^{n_0}} \liminf_{l\to\infty}(-\mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega])) \leq \liminf_{l\to\infty} \sum_{[\omega]\in\alpha^{n_0}} -\mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega]) .$$

Since $\mu_t = \lim_{l\to\infty} \mu_{tf_{k_l}}$ and $\mu_t(\partial[\omega]) = 0$ for any cylinder $[\omega] \in \alpha^{n_0}$, then we have to $\mu_t[\omega]\log(\mu_t[\omega]) = \lim_{l\to\infty} \mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega])$.

Therefore, it follows that

$$\sum_{[\omega]\in\alpha^{n_0}} -\mu_t[\omega]\log(\mu_t[\omega]) \leq \liminf_{l\to\infty} \sum_{[\omega]\in\alpha^{n_0}} -\mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega]) .$$

By a similar proceeding we can prove that

$$\sum_{[\omega]\in\alpha^{n_0}} -\mu_t[\omega]\log(\mu_t[\omega]) \geq \limsup_{l\to\infty} \sum_{[\omega]\in\alpha^{n_0}} -\mu_{tf_{k_l}}[\omega]\log(\mu_{tf_{k_l}}[\omega]).$$

Therefore, we conclude that

$$H(\mu_t \mid \alpha^{n_0}) = \lim_{l\to\infty} H(\mu_{tf_{k_l}} \mid \alpha^{n_0}) .$$

$\square$

Proposition 2. *Let $(\mu_{tf_{k_l}})_{l\in\mathbb{N}}$ be a sequence given by the previous lemma. Then $\limsup_{l\to\infty} h(\mu_{tf_{k_l}}) \leq h(\mu_t)$.*

Proof. Suppose that $\limsup_{l\to\infty} h(\mu_{tf_{k_l}}) > h(\mu_t)$; then we can choose $\epsilon > 0$ such that $h(\mu_t) \le \limsup_{l\to\infty} h(\mu_{tf_{k_l}}) - 3\epsilon$. From (4.2), $\frac{1}{n}H(\mu_t \mid \alpha^n) \le h(\mu_t) + \epsilon$ for n large enough, and by Lemma 3, there is $n_0 \ge n$ such that

$$\lim_{l\to\infty} H(\mu_{tf_{k_l}} \mid \alpha^{n_0}) = H(\mu_t \mid \alpha^{n_0}).$$

Finally by (4.2), there is $l_0 \in \mathbb{N}$ such that for $j \ge l_0$ we have $h(\mu_{tf_{k_j}}) \le \frac{1}{n_0}H(\mu_{tf_{k_j}} \mid \alpha^{n_0})$. Therefore,

$$\begin{aligned} h(\mu_{tf_{k_j}}) &\le \frac{1}{n_0}H(\mu_t \mid \alpha^{n_0}) + \epsilon \\ &\le h(\mu_t) + 2\epsilon \\ &\le \limsup_{l\to\infty} h(\mu_{tf_{k_l}}) - \epsilon\,, \end{aligned}$$

and taking the lim sup as $j \to \infty$ in the left side of the inequality, we obtain a contradiction. In this way, we conclude that $\limsup_{l\to\infty} h(\mu_{tf_{k_l}}) \le h(\mu_t)$. □

Proof of Theorem 1. Let $(k_l)_{l\in\mathbb{N}}$ be a sequence as in Lemma 3. Since $(k_l)_{l\in\mathbb{N}}$ is a subsequence of $(k_m)_{m\in\mathbb{N}}$ and the sequence $(P_{k_l}(t))_{l\in\mathbb{N}}$ is increasing, from (3.2) and Lemma 2 we have

$$\lim_{l\to\infty} P_{k_l}(t) = P(t)$$

and

$$\lim_{l\to\infty} \mu_{tf_{k_l}}(tf) = \mu_t(tf)\,.$$

Therefore,

$$P(t) = \lim_{l\to\infty} P_{k_l}(t) \le \limsup_{l\to\infty} h(\mu_{tf_{k_l}}) + \limsup_{l\to\infty} \mu_{tf_{k_l}}(tf_{k_l}) \le h(\mu_t) + \mu_t(tf)\,.$$

The above inequality shows that μ_t is an equilibrium state associated to the potential tf. Since $P(tf) < \infty$, tf is bounded above and $V(tf) < \infty$. From [4] we have that the equilibrium state associated to the potential tf is unique and, therefore, $\mu_t = \mu_{tf}$.

To prove that the family $(\mu_{tf})_{t>1}$ is tight, notice that from equation (3.6) and that $\mu_t = \mu_{tf}$, we can take the limit as $m \to \infty$ at both sides, maybe under a convergent subsequence of μ_{tf_k} and noticing that the cylinders are continuity sets, and we get

$$\mu_{tf}([i]) \le \exp(4V(f) + \sup f|_{[i]} - S)\,.$$

Observe that the right side does not depend on t. So, following the same conclusion of Lemma 1, we can conclude that $(\mu_{tf})_{t>1}$ is tight. Taking any subsequence $t_k \to \infty$, we find an accumulation point for $(\mu_{tf})_{t>1}$ as $t \to \infty$, and Theorem 1 is proved. □

A consequence of this proof is the following corollary which states that the whole sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ is in fact convergent to the equilibrium state μ_{tf}. That is interesting in the sense that we can approximate the equilibrium states of the non-compact case by the ones in the invariant compact subshifts, which is not trivial since the space is not σ-compact.

Corollary 1. *For each $t > 1$ the equilibrium states sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ converges to μ_{tf}.*

Proof. Suppose that the sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ is not convergent. Recall $\mathcal{M}_\sigma(\Sigma)$ is a metrizable space and let d be any distance on this space. Then, there is $\epsilon_0 > 0$ and a subsequence $(\mu_{tf_{k_j}})_{j\in\mathbb{N}}$ of the sequence $(\mu_{tf_k})_{k\in\mathbb{N}}$ such that for any $j \in \mathbb{N}$

$$d(\mu_{tf_{k_j}}, \mu_{tf}) \geq \epsilon_0 .$$

In particular, notice that $(\mu_{tf_{k_j}})_{j\in\mathbb{N}}$ cannot converge to μ_{tf}.

Now, by Lemma 1, the sequence $(\mu_{tf_{k_j}})_{j\in\mathbb{N}}$ is tight, therefore the argument used to prove Theorem 1 implies that there exists $(\mu_{tf_{k_i}})_{i\in\mathbb{N}}$ as a subsequence of $(\mu_{tf_{k_j}})_{j\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \mu_{tf_{k_i}} = \mu_{tf} .$$

Since the equilibrium state is unique, this is a contradiction, which proves the result. □

Observe that, by Remark 1, the assumption that f is summable is essential for the proof of Theorem 1, and therefore, also for Theorem 2. Otherwise, there are counterexamples, like the one in [17], where it can be checked that the potential is not summable, although it is Markov.

5. Proof of Theorem 2

In this section we prove the other theorem of this work. This proof is a direct consequence of Theorem 1 and the following result.

The following proposition shows that there exist k_0 large enough such that the set of the f_k-maximizing measures is the same for each $k \geq k_0$. This is the key point where we use the results in [1] to locate the ground states.

Proposition 3. *There is k_0 such that, for each $k \geq k_0$, we have that $\beta_k = \beta$ and $\mathcal{M}_{\max}(f_k) = \mathcal{M}_{\max}(f)$.*

Proof. It is proved in [1] that there exists a finite set $F \subset \mathbb{N}$ such that

$$\beta = \sup\{\mu(f) : \mu \in \mathcal{M}_\sigma(\Sigma_F)\},$$

and every measure $\mu \in \mathcal{M}_{\max}(f)$ satisfies $\mathrm{supp}(\mu) \subset \Sigma_{\mathrm{F}}$. Moreover, there exists $k_0 \geq 0$ such that $\Sigma_F \subset \Sigma_{k_0}$. Therefore, we can suppose w.l.o.g. that $\Sigma_F = \Sigma_{k_0}$. Then

$$\begin{aligned}\beta &= \sup\{\mu(f) : \mu \in \mathcal{M}_\sigma(\Sigma_{k_0})\}\\ &= \sup\{\mu(f_{k_0}) : \mu \in \mathcal{M}_\sigma(\Sigma_{k_0})\}\\ &= \beta_{k_0}.\end{aligned}$$

Observe that for each k we have $\beta_k \leq \beta_{k+1} \leq \beta$, this is because $\mathcal{M}_\sigma(\Sigma_k) \subset \mathcal{M}_\sigma(\Sigma_{k+1})$. Besides that, $\mu(f_k) = \mu(f_{k+1})$ for any $\mu \in \mathcal{M}_\sigma(\Sigma_k)$; then $\beta_k = \beta$ for each $k \geq k_0$ and every measure $\mu \in \mathcal{M}_{\max}(f)$ satisfies $\mathrm{supp}(\mu) \subset \Sigma_{\mathrm{k_0}}$. Moreover, we have that

$$\beta = \mu(f) = \mu(f_k) ,$$

that is, $\mu \in \mathcal{M}_{\max}(f_k)$. Let $k \geq k_0$, if the probability measure $\mu_k \in \mathcal{M}_{\max}(f_k)$. Using that $\mathrm{supp}(\mu_{\mathrm{k}}) \subset \Sigma_{\mathrm{k}}$, we have

$$\beta_k = \mu_k(f_k) = \mu_k(f),$$

since $\beta_k = \beta$; then we conclude that $\mu_k \in \mathcal{M}_{\max}(f)$. □

The following proof is basically a consequence of Theorem 1 and the previous proposition. In addition to this we have that this result is a complete generalization of the theorem of Morris in [14] beyond the finitely primitive case.

Proof of Theorem 2. Since the Σ_k's are compact, then the functions $t \mapsto h(\mu_{tf_k})$ are decreasing for each $k \in \mathbb{N}$. Besides that, using (9) and Lemma 2 joined to Corollary 1, it follows immediately from the variational principle that $\lim_{k\to\infty} h(\mu_{tf_k}) = h(\mu_{tf})$. Therefore, for each $t_1 > t_0 > 1$ and any $k \in \mathbb{N}$ we have to $h(\mu_{t_0 f_k}) > h(\mu_{t_1 f_k})$. Then taking the limit as $k \to \infty$ we obtain that

$$h(\mu_{t_0 f}) = \lim_{k\to\infty} h(\mu_{t_0 f_k}) \geq \lim_{k\to\infty} h(\mu_{t_1 f_k}) = h(\mu_{tf}),$$

i.e., the family $(h(\mu_{tf}))_{t>1}$ is non-increasing and particularly it is bounded above for t large enough. Using again the variational principle we obtain that

$$\frac{h(\mu_{tf})}{t} + \mu_{tf}(f) = \sup\left\{\frac{h(\mu)}{t} + \mu(f) : \mu \in \mathcal{M}_\sigma(\Sigma) \text{ and } \mu(f) > -\infty\right\}.$$

Then taking an increasing sequence $(t_i)_{i\in\mathbb{N}}$ in $(1,\infty)$ such that $\lim_{j\to\infty} \mu_{t_j f} = \mu_\infty$ and using the convexity of the function $P(t)$, it follows immediately that this function admits an asymptote as $t \to \infty$ with slope β, i.e., $P(t) = h + t\beta + \rho(t)$ with $h \in \mathbb{R}$ and $\lim_{t\to\infty} \rho(t) = 0$.

On the other hand, for each $k \geq k_0$ we have $\mathcal{M}_{\max}(f) = \mathcal{M}_{\max}(f_k)$, therefore $\mathcal{M}_{\max}(f)$ is a compact set of $\mathcal{M}_\sigma(\Sigma_{k_0})$ and so is compact in $\mathcal{M}_\sigma(\Sigma)$. By the upper semi-continuity of the function $\mu \mapsto h(\mu)$ restricted to the compact set $\mathcal{M}_{\max}(f)$, there exists a maximal element $\hat{\mu} \in \mathcal{M}_{\max}(f)$, i.e., a probability measure such that $h(\mu) \leq h(\hat{\mu})$ for any $\mu \in \mathcal{M}_{\max}(f)$, particularly $h(\mu_\infty) \leq h(\hat{\mu})$.

Since $\hat{\mu}(f) = \beta$, by the variational principle it follows that

$$h(\hat{\mu}) + t\beta(f) \leq P(t) = h + t\beta(f) + \rho(t). \tag{5.1}$$

This inequality is valid for each $t > 1$, therefore $h(\hat{\mu}) \leq h + \rho(t)$, taking the limit as $t \to \infty$ it follows that $h(\hat{\mu}) \leq h$. Besides that, μ_{tf} is an equilibrium state associated to the potential tf; then

$$h + t\beta(f) + \rho(t) = h(\mu_{tf}) + t\mu_{tf}(f) \leq h(\mu_{tf}) + t\beta(f), \tag{5.2}$$

i.e., $h(\mu_\infty) \leq \limsup_{t\to\infty} h(\mu_{tf})$.

Now we just need to prove that $\limsup_{t\to\infty} h(\mu_{tf}) \leq h(\mu_\infty)$. In fact, observe that for any increasing sequence $(t_j)_{j\in\mathbb{N}}$ in $(1,\infty)$ and each $t' \geq t_j$ we have

$$\inf_{t\geq t_j}\{-\mu_{tf}[\omega]\log(\mu_{tf}[\omega])\} \leq -\mu_{t'f}[\omega]\log(\mu_{t'f}[\omega]).$$

Summing up all the $[\omega] \in \alpha^{n_0}$ we obtain that for each $t' \geq t_j$

$$\sum_{[\omega]\in\alpha^{n_0}} \inf_{t\geq t_j}\{-\mu_{tf}[\omega]\log(\mu_{tf}[\omega])\} \leq \sum_{[\omega]\in\alpha^{n_0}} -\mu_{t'f}[\omega]\log(\mu_{t'f}[\omega]),$$

it follows immediately from the above inequality that

$$\sum_{[\omega]\in\alpha^{n_0}} \inf_{t\geq t_j}\{-\mu_{tf}[\omega]\log(\mu_{tf}[\omega])\} \leq \inf_{t'\geq t_j}\left\{\sum_{[\omega]\in\alpha^{n_0}} -\mu_{t'f}[\omega]\log(\mu_{t'f}[\omega]\right\}. \tag{5.3}$$

Using the monotone convergence theorem for $\phi_j(x) = \inf_{t\geq t_j}\{-\mu_{tf}(x)\log(\mu_{tf}(x))\}$ with the counting measure on the set $\{[\omega] : [\omega] \in \alpha^{n_0}\}$, it follows that

$$\lim_{j\to\infty} \sum_{[\omega]\in\alpha^{n_0}} \inf_{t\geq t_j}\{-\mu_{tf}[\omega]\log(\mu_{tf}[\omega])\} = \sum_{[\omega]\in\alpha^{n_0}} \lim_{j\to\infty}\inf_{t\geq t_j}\{-\mu_{tf}[\omega]\log(\mu_{tf}[\omega])\}.$$

On the other hand $\mu_\infty(\partial[\omega]) = 0$ for each $[\omega] \in \alpha^{n_0}$ and $\lim_{j\to\infty}\mu_{t_jf} = \mu_\infty$; then $\lim_{j\to\infty}\mu_{t_jf}[\omega] = \mu_\infty[\omega]$. Therefore, taking the limit as $j \to \infty$ in both sides of (5.3) we conclude that

$$\sum_{[\omega]\in\alpha^{n_0}} -\mu_\infty[\omega]\log(\mu_\infty[\omega]) \leq \liminf_{t\to\infty} \sum_{[\omega]\in\alpha^{n_0}} -\mu_{tf}[\omega]\log(\mu_{tf}[\omega]).$$

An analogous proof shows that $\limsup_{t\to\infty} H(\mu_{tf}|\alpha^{n_0}) \leq H(\mu_\infty|\alpha^{n_0})$, therefore $\lim_{t\to\infty} H(\mu_{tf}|\alpha^{n_0}) = H(\mu_\infty|\alpha^{n_0})$.

Then following the same proof as of Proposition 2 we obtain that

$$\limsup_{t\to\infty} h(\mu_{tf}) \leq h(\mu_\infty).$$

Finally using the inequalities (5.1) and (5.2), it follows that

$$h(\hat{\mu}) \geq h(\mu_\infty) \geq \limsup_{t\to\infty} h(\mu_{tf}) \geq h \geq h(\hat{\mu}),$$

and this concludes our proof. ☐

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