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NON-ERGODIC BANACH SPACES ARE NEAR HILBERT

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Abstract. We prove that a non-ergodic Banach space must be near Hilbert. In particular, ℓ_p $(2 < p < \infty)$ is ergodic. This reinforces the conjecture that ℓ_2 is the only non-ergodic Banach space. As an application of our criterion for ergodicity, we prove that there is no separable Banach space which is complementably universal for the class of all subspaces of ℓ_p, for $1 \leq p < 2$. This solves a question left open by W. B. Johnson and A. Szankowski in 1976.

1. Introduction

The solution of Gowers [21] and Komorowski–Tomczak-Jaegermann [31] to the homogeneous Banach space problem provides that every Banach space having only one equivalence class for the relation of isomorphism between its infinite dimensional subspaces must be isomorphic to ℓ_2. G. Godefroy formulated the question about the number of non-isomorphic subspaces of a Banach space X not isomorphic to ℓ_2. This question was studied, in the context of descriptive set theory, by V. Ferenczi and C. Rosendal [17], who introduced the notion of *ergodic Banach space* to study the classification of the relative complexity of the isomorphism relation between the subspaces of a separable Banach space.

Our general references for descriptive set theory will be [4, 30]. A *Polish space* is a separable topological space which admits a compatible complete metric. The Borel sets of a Polish space comprise the σ-algebra generated by the open sets. A set X equipped with a σ-algebra is called a *Borel standard space* if there exists a Polish topology on X for which that σ-algebra arises as the collection of Borel subsets of X. A function between two Borel standard spaces $f : X \to Y$ is said to be Borel if $f^{-1}(B)$ is Borel in X, for every Borel subset $B \subseteq Y$.

Given a Polish space X, let $\mathcal{F}(X)$ be the collection of all closed subsets of X. The σ-algebra on $\mathcal{F}(X)$ generated by

$$A_U = \{F \in \mathcal{F}(X) \,:\, F \cap U \neq \emptyset\},$$

where U is an open subset of X, is called the *Effros Borel structure* on the closed subsets of X. It is not hard to see that $\mathcal{F}(X)$ equipped with this Borel structure is a Borel standard space. $\mathcal{SB}(X)$ denotes the collection of infinite dimensional linear subspaces $Y \in \mathcal{F}(X)$ equipped with the relative Effros Borel structure. This framework allows us to identify every class of subspaces of a Banach space X with a subset of $\mathcal{SB}(X)$ in which its complexity can be measured. For instance, since $C(2^{\mathbb{N}})$ is isometrically universal for all separable Banach spaces, we can consider the set $\mathcal{SB}(C(2^{\mathbb{N}}))$ as the standard Borel space of all separable Banach spaces. With this

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identification, properties of separable Banach spaces become sets in $\mathcal{SB}(C(2^{\mathbb{N}}))$. In [5] it was proved that the relation of isomorphism between separable Banach spaces is an analytic and not borelian subset of $\mathcal{SB}(C(2^{\mathbb{N}}))^2$.

The central notion to study the complexity of analytic and Borel equivalence relations on Borel standard spaces is the concept of *Borel reducibility*, which originated from the works of H. Friedman and L. Stanley [20] and independently from the works of L. A. Harrington, A. S. Kechris, and A. Louveau [22].

Definition 1.1. Let R and S be two Borel equivalence relations on Borel standard spaces X and Y, respectively. One says that R is Borel reducible to S (denoted by $R \leq_B S$) if there exists a Borel function $\phi : X \to Y$ such that

$$xRy \iff \phi(x)S\phi(y),$$

for all $x, y \in X$. The relation R is *Borel bireducible* to S (denoted by $R \sim_B S$) whenever both $R \leq_B S$ and $S \leq_B R$ hold.

This can be interpreted as meaning that the equivalence relation R is classified by a Borel assignment of invariants provided by equivalence classes for S. Observe that a Borel reduction induces an embedding from the quotient space X/R to Y/S, so X/R has less than or equal cardinality to that of Y/S.

Ferenczi, Louveau, and Rosendal [16] proved that the relation of isomorphism between separable Banach spaces is a complete analytic equivalence relation, i.e., that any analytic equivalence relation Borel reduces to it.

For X a Polish space, let $\mathrm{id}(X)$ be the identity relation on the space X. Since any two standard Borel spaces with the same cardinality are Borel isomorphic, it follows that for any uncountable X,

$$\mathrm{id}(X) \sim_B \mathrm{id}(\mathbb{R}).$$

Among the uncountable Borel equivalence relations, the simplest is $\mathrm{id}(\mathbb{R})$. In fact, it was proved by Silver [39] that given a Borel equivalence relation (X, R), either it has countable many classes of equivalence or $\mathrm{id}(\mathbb{R})$ is Borel reducible to (X, R). An equivalence relation admitting the reals as a complete invariant is called *smooth*, that is, when it is reducible to $\mathrm{id}(\mathbb{R})$.

The simplest example of a non-smooth equivalence relation is the relation of eventual agreement E_0 on $2^{\mathbb{N}}$; i.e., for $x, y \in 2^{\mathbb{N}}$,

$$xE_0y \iff (\exists N \in \mathbb{N})(x(n) = y(n),\, n \geq N).$$

Harrington, Kechris, and Louveau [22] proved that E_0 is minimal among non-smooth Borel equivalence relations with respect to $\leq_B$.

The following notion measures the complexity of the relation of isomorphism between subspaces of a separable Banach space and was introduced by Ferenczi and Rosendal [17].

Definition 1.2. A separable Banach space X is ergodic if

$$(2^{\mathbb{N}}, E_0) \leq_B (\mathcal{SB}(X), \simeq).$$

It follows that an ergodic Banach space has at least $2^{\mathbb{N}}$ non-isomorphic subspaces and the equivalence relation of isomorphism between its subspaces is non-smooth.

Rosendal [38] notices that every hereditarily indecomposable (H.I) Banach space (i.e., a space in which no closed infinite dimensional subspace can be written as the direct sum of two closed infinite dimensional subspaces) is ergodic. By Gowers dichotomy [21], every Banach space contains an H.I subspace or an unconditional basic sequence. Since every Banach space containing an ergodic subspace must be ergodic, one can approach the study of ergodicity by first restricting to spaces with unconditional basis.

Ferenczi and Rosendal [17] proved that a non-ergodic Banach space X with unconditional basis satisfies some regularity properties such as being isomorphic to its square and to its hyperplanes, and more generally must be isomorphic to $X \oplus Y$ for any subspace Y of X generated by a subsequence of the basis. It was conjectured in [17] that every separable Banach space not isomorphic to ℓ_2 must be ergodic.

Dilworth, Ferenczi, Kutzarova, and Odell [13] proved that every Banach space X with a strongly asymptotic ℓ_p basis ($1 \leq p \leq \infty$) not equivalent to the unit vector basis of ℓ_p (or c_0 if $p = \infty$) is ergodic. This result was generalized by R. Anisca [2], who constructed explicit Borel reductions to prove that every separable asymptotically Hilbertian space (and therefore every weak Hilbert space) not isomorphic to ℓ_2 is ergodic.

Recall that a Banach space X is called (*complementably*) *minimal* (notions due to Pełczyński and Rosenthal, respectively) if every infinite dimensional closed subspace Y of X contains a (complemented) subspace Z isomorphic to X. Clearly, every (complemented) subspace of a (complementably) minimal space is also a (complementably) minimal space. Ferenczi [14] proved that a separable Banach space without minimal subspaces must be ergodic. Hence, the conjecture in [17] is related to the following problem: Is every minimal Banach space not isomorphic to ℓ_2 ergodic?

It is well known that c_0 and ℓ_p ($1 \leq p < \infty$) are complementably minimal spaces, while the dual of the Tsirelson space T^* is an example of a minimal but not complementably minimal space [8]. The first example of a complementably minimal space other than c_0 and the ℓ_p's is the Schlumprecht space and its dual [41]. The list of minimal spaces known so far is completed with the family of Schlumprecht type spaces and their duals constructed by complex interpolation methods in [9] and every infinite dimensional closed subspace of each of the above. For classical spaces, it was proved in [15] that c_0 and ℓ_p for $1 \leq p < 2$ are ergodic. Rosendal [37] proved that the dual of the Tsirelson space is ergodic. In this work we prove ergodicity for a general family of Banach spaces including all the other minimal spaces not isomorphic to ℓ_2 listed above. More specifically, given a Banach space X, let

$$p(X) = \sup\{p : X \text{ has type } p\},$$
$$q(X) = \inf\{q : X \text{ has cotype } q\}.$$

Recall that a Banach space X is said to be *near Hilbert* when $p(X) = q(X) = 2$. We give a criterion for ergodicity which together with the Johnson and Szankowski construction of subspaces without the approximation property allows us to prove that a non-ergodic Banach space must be near Hilbert. In particular, we solve the question of [17] about the ergodicity of the ℓ_p spaces, for $p > 2$. We also prove that the family of Schlumprecht type spaces and its dual are not near Hilbert, and therefore they are ergodic spaces.

Finally, as an application of the criterion for ergodicity, we prove that for every non-near Hilbert space X there does not exist a separable Banach space which is complementably universal for the class of all subspaces of X. In particular, this is true for $X = \ell_p$, $p \neq 2$. This solves a problem left open by Johnson and Szankowski in their 1976 paper [26] and mentioned again in [25]. (Johnson and Szankowski verified the case $2 < p < \infty$ in [26].)

2. Criterion for ergodicity

A Banach space X has the approximation property (AP) if the identity operator on X can be approximated uniformly on compact subsets of X by linear operators of finite rank. The Banach space X is said to have the bounded approximation property (BAP) if there exists $\lambda > 0$ such that the finite rank operator T in the definition of AP can be taken with norm $\|T\| \leq \lambda$. In 1973, Enflo [12] presented the first example of Banach space without the AP and therefore without a Schauder basis. Enflo's construction was simplified by Davie [11], who used probabilistic methods to construct such examples inside ℓ_p-spaces ($2 < p \leq \infty$). Later, in 1978, Szankowski [40] proved that the other range of ℓ_p-spaces ($1 \leq p < 2$) also has subspaces failing AP. The criterion we introduce to study ergodicity in Banach spaces is based on a criterion introduced by Enflo and used in the works of Davie and Szankowski to prove that a space fails the AP.

We first introduce some notation used throughout the paper. For every $n \in \mathbb{N}$, denote $I_n = \{2^n, 2^n+1, \dots, 2^{n+1}-1\}$. Given a Banach space X and sequences of vectors $(z_{n,\epsilon})_{n\in\mathbb{N}}$ in X, $(z^*_{n,\epsilon})_{n\in\mathbb{N}}$ in X^*, ($\epsilon = 0, 1$), we denote $Z = \overline{span}\{z_{j,\epsilon} : j \in \mathbb{N}, \epsilon = 0, 1\}$ and we shall consider for every $t \in 2^{\mathbb{N}}$ the closed subspace

$$X_t = \overline{\text{span}}\left\{z_{j,t(n)} : j \in I_n,\, n = 1, 2, 3, \dots\right\}.$$

If $T : X_t \to Z$ is a bounded and linear operator we define the n-trace of T as

$$\beta^n_t(T) = 2^{-n} \sum_{j\in I_n} z^*_{j,t(n)} T(z_{j,t(n)}).$$

Definition 2.1. A Banach space X satisfies the Cantorized-Enflo criterion if there exist bounded sequences of vectors $(z_{n,\epsilon})_{n\in\mathbb{N}}$ in X, $(z^*_{n,\epsilon})_{n\in\mathbb{N}}$ in X^* ($\epsilon = 0, 1$), and a sequence of real scalars $(\alpha_n)_n$ such that

(1) $z^*_{i,\epsilon}(z_{j,\tau}) = \delta_{ij}\delta_{\epsilon\tau}$ for all $i, j \in \mathbb{N}$ and $\epsilon, \tau = 0, 1$.

(2) For every $t, s \in 2^{\mathbb{N}}$ and every operator $T : X_t \to X_s$,
$$\left|\beta^n_t(T) - \beta^{n-1}_t(T)\right| \leq \alpha_n \|T\|.$$

(3) $\sum_n \alpha_n < \infty$.

Recall that a subset of a topological space is said to be meagre if it is the countable union of nowhere dense subsets (sets whose closure has empty interior). An equivalence relation on a standard Borel space X is said to be meagre if it is a meagre subset of X^2.

Let $t \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$. We denote by $t/n = \{k \leq n : t(k) = 1\}$. The E'_0 equivalence relation on $2^{\mathbb{N}}$ is defined as

$$xE'_0y \iff \exists n(|t/n| = |s/n|) \wedge (t(k) = s(k),\, k \geq n).$$

E'_0 is a refinement of E_0, that is, $E'_0 \subseteq E_0$. In connection with Borel reducibility ordering we shall use the following result from Rosendal [38].

Proposition 2.2 ([38, Proposition 15])**.** *Let E be a meagre equivalence relation on $2^{\mathbb{N}}$ containing E'_0. Then $E_0 \leq_B E$.*

Recall that a Banach space X is said to be *complementably universal* for a family $\mathcal{A}$ of Banach spaces if every space in $\mathcal{A}$ is isomorphic to a complemented subspace of X. In [26] Johnson and Szankowski proved that there is no separable Banach space which is complementably universal for the class $\mathcal{A}_p$ of all subspaces of ℓ_p, $2 < p < \infty$. As observed by Johnson [24], it follows that a complementably universal Banach space for the class $\mathcal{A}_p$ ($2 < p < \infty$) must have density character at least the continuum, where the density character of a topological space X is the least cardinality of a dense subset of X. In particular, this shows that the family of non-isomorphic subspaces of ℓ_p, for ($2 < p < \infty$), has the cardinality of the continuum. We use the ideas of the proof in [26] (see also [27]) to establish a criterion for ergodic Banach spaces.

Lemma 2.3. *Let X be a Banach space satisfying the Cantorized-Enflo criterion, and let Γ be an uncountable subset of $2^{\mathbb{N}}$. Then every Banach space which is complementably universal for the family $\{X_t\}_{t\in\Gamma}$ has density character at least cardinality of Γ.*

Proof. Let X be a Banach space satisfying the Cantorized-Enflo criterion and consider sequences $(z_{n,\epsilon})_{n\in\mathbb{N}}$ on X, $(z^*_{n,\epsilon})_{n\in\mathbb{N}}$ on X^*, ($\epsilon = 0, 1$), and real scalars $(\alpha_n)_n$ as in Definition 2.1. Suppose that there is an uncountable set $\Gamma \subseteq 2^{\mathbb{N}}$ and a Banach space W with density character less than the cardinality of Γ such that for every $t \in \Gamma$, X_t is isomorphic to a complemented subspace of W. For each $t \in \Gamma$ we fix an embedding $T_t : X_t \to W$ and a projection onto $P_t : W \to T_tX_t$. We claim that there exist $\lambda > 0$ and a set $\Gamma' \subseteq \Gamma$ with the same cardinality of Γ such that $\|T_t\|\|T_t^{-1}\| \le \lambda$ and $\|P_t\| \le \lambda$ for every $t \in \Gamma'$. This follows, since $\Gamma = \bigcup_{n\in\mathbb{N}}\{t \in \Gamma : \|T_t\|\|T_t^{-1}\| + \|P_t\| \le n\}$ and from the fact that Γ is uncountable. Now replacing T_t by $\|T_t^{-1}\|T_t$, we may assume that for every $t \in \Gamma'$,

$$\|x\| \le \|T_tx\| \le \lambda\|x\| \quad \text{for every} \;\; x \in X_t.$$

Take $\delta > 0$. It follows by conditions (2) and (3) in Definition 2.1 that there exists $k = k(\delta)$ such that for every $m > k$,

$$\left|\beta_t^m(T) - \beta_t^k(T)\right| \le \delta\|T\|,$$

for every $t, s \in 2^{\mathbb{N}}$ and any operator $T : X_t \to X_s$. We observe that there is a subset $\Gamma'_k \subseteq \Gamma'$ with the same cardinality of Γ' such that for every $t, s \in \Gamma'_k$, $t(i) = s(i)$ ($i = 1, 2, \ldots, k$). Since the density character of W is less than the cardinality of Γ'_k, there exists a pair $t \neq s \in \Gamma'_k$ such that

$$\|T_t(z_{j,t(k)}) - T_s(z_{j,s(k)})\| \le 1/\lambda 2^k, \qquad j \in I_k.$$

Now define $T : X_t \to X_s$ by $T = T_s^{-1}P_sT_t$, where $T_s^{-1} : T_sX_s \to X_s$. We have $T_s^{-1}P_s(T_t(z_{j,t(k)}) - T_s(z_{j,s(k)})) = T(z_{j,t(k)}) - z_{j,s(k)}$ and therefore

$$\sum_{j\in I_k} \|Tz_{j,t(k)} - z_{j,s(k)}\| \le \sum_{j\in I_k} \|T_s^{-1}P_s\|\|T_t(z_{j,t(k)}) - T_s(z_{j,s(k)})\| \le 1.$$

From this we deduce, by using $t(k) = s(k)$, that

$$\left|\beta_t^k(T)\right| \ge 1 - 2^{-k} \sum_{j\in I_k} \left\|z^*_{j,s(k)}\left(z_{j,s(k)} - Tz_{j,t(k)}\right)\right\| \ge 1 - 2^{-k}.$$

Now since $t(m) \neq s(m)$ for some $m > k$ and $(z^*_{j,\epsilon}, z_{j,\epsilon})$ is a biorthogonal system, we have

$$\beta^m_t(T) = 0.$$

Therefore,

$$\|T\| \geq \delta^{-1} \left|\beta^m_t(T) - \beta^k_t(T)\right| \geq (1/2)\delta^{-1}.$$

On the other hand,

$$\|T\| \leq \|T_s^{-1}\|\|P_s\|\|T_t\| \leq \lambda^2.$$

Since δ was arbitrary, we get a contradiction. □

Theorem 2.4. *Every separable Banach space satisfying the Cantorized-Enflo criterion is ergodic.*

Proof. Let X be a separable Banach space satisfying the Cantorized-Enflo criterion. Define an equivalence relation E on $2^{\mathbb{N}}$ by setting sEt if and only if X_s is isomorphic to X_t. We observe that E is E_0'-invariant. Indeed, if $tE_0's$, then X_t and X_s are generated by the same sequence of vectors except for finite sets of the same cardinality and therefore are isomorphic spaces. By Lemma 2.3 each equivalence class of E is countable and then a meagre subset of $2^{\mathbb{N}}$. It is a general fact that an equivalence relation is meagre whenever each of its equivalence class is meagre [30]. Hence E is a meagre equivalence relation on $2^{\mathbb{N}}$, and we have from Proposition 2.2 that $E_0 \leq_B E$. It is clear that the function $\phi : 2^{\mathbb{N}} \to \mathcal{SB}(X)$ given by $\phi(t) = X_t$ is Borel. In consequence, X is ergodic. □

Remark 2.5. A Banach space satisfying the Cantorized-Enflo criterion has a continuum of non-isomorphic subspaces failing the bounded approximation property.

Proof. We observe that the spaces X_t used in the reduction fails the BAP for every $t \in 2^{\mathbb{N}}$. Assume without loss of generality that the vectors in the Cantorized-Enflo criterion satisfy $\|z_{i,\epsilon}\| \leq 1$ and $\|z^*_{i,\delta}\| \leq 1$, for every $\epsilon, \delta = 0, 1$ and every $i \in \mathbb{N}$. Given $\lambda > 0$, let $n \in \mathbb{N}$ be such that $\lambda \sum_{k>n} \alpha_k \leq 1/2$. Let $T : X_t \to X_t$ be an operator with $\|T\| \leq \lambda$. Since $|\beta^n_t(U)| \leq \|U_{|Z_n}\|$ for every $U : X_t \to Z$, where $Z_n = \{z_{i,t(n)},\, i \in I_n\}$ is a compact set, we have

$$\begin{aligned}\|(\mathrm{Id} - T)_{|Z_n}\| &\geq |\beta^n_t(\mathrm{Id} - T)| \geq 1 - |\beta^n_t(T)| \geq 1 - \sum_{k>n} |\beta^k_t(T) - \beta^{k-1}_t(T)| \\ &\geq 1 - \|T\| \sum_{k>n} \alpha_k > 1/2.\end{aligned}$$

□

Remark 2.6. Actually, if in Definition 2.1, we have for every $n \in \mathbb{N}$,

$$\left|\beta^n_t(T) - \beta^{n-1}_t(T)\right| \leq \sup\{\|Tz\|,\, z \in F_n\}$$

for a finite set F_n of vectors in X, such that $\sum_n \sup\{\|z\|,\, z \in F_n\} < \infty$, then every X_t fails the AP ([40, Proposition 1]).

Remark 2.7. We also proved that E_0 is Borel reducible to the relation of complemented biembeddability between the subspaces of a separable Banach space satisfying the Cantorized-Enflo criterion.

Let X and Y be two Banach spaces and let there be a constant $K > 0$. Recall that X is said to be K-*crudely finitely representable* in Y if for every finite dimensional subspace F of X there exist a linear isomorphism $T : F \to T(F) \subseteq Y$ so that $\|T\|\|T^{-1}\| \leq K$. X is said to be *finitely representable* in Y if X is $(1+\epsilon)$-crudely finitely representable in Y for every $\epsilon > 0$. A classical result of Maurey and Pisier [35] states that $l_{p(X)}$ and $\ell_{q(X)}$ are finitely representable in X, for any Banach space X. The following remark is stated in the classical book [34].

Remark 2.8. It follows from the proof of [33, Theorem 1.a.5] that if ℓ_p is K-crudely finitely representable in Y, for some $1 \leq p \leq \infty$, then Y has a subspace X which has a Schauder decomposition into $\{X_n\}_{n=1}^{\infty}$ with $d(X_n, \ell_p^n) \leq K+1$ for every $n \in \mathbb{N}$.

Proposition 2.9. *If ℓ_p is crudely finitely representable in a Banach space X for some $p > 2$, then X satisfies the Cantorized-Enflo criterion.*

Proof. The proof of Johnson and Szankowski [26, Section IV] that there does not exist a separable Banach space which is complementably universal for the class of subspaces of ℓ_p $(2 < p < \infty)$ is by modifying Davie's construction of a subspace of ℓ_p $(2 < p < \infty)$ failing AP. We observe that the Johnson and Szankowski construction yields that ℓ_p $(2 < p < \infty)$ satisfies the Cantorized-Enflo criterion.

Indeed, fix $p > 2$. For every $n \in \mathbb{N}$, we denote by $(f_j^n)_{j=1}^{3.2^n}$ the unit vector basis of $\ell_p^{3.2^n}$. Using the notation of [11, 26], let for $j \in I_n$ and $\epsilon = 0, 1$,

$$z_{j,\epsilon} = e_{j+\epsilon 2^n}^{n+1},$$

$$z_{j,\epsilon}^* = \alpha_{j+\epsilon 2^n}^{n+1},$$

where the vectors e_j^k defined in [26] have the form

$$e_j^k = \sum_{l=1}^{3.2^{k-1}} \lambda_j^k(l) f_l^k + \sum_{l=1}^{3.2^k} \delta_j^k(l) f_l^k.$$

Also the functionals α_j^k are linear combinations of the biorthogonal functionals $(f_l^{k^*})_{l=1}^{3.2^k}$ of $\ell_q^{3.2^k}$, satisfying $\alpha_l^k(e_j^i) = \delta_{ki}\delta_{lj}$. For any operator $T : X_t \to \ell_p$,

$$\left|\beta_t^n(T) - \beta_t^{n-1}(T)\right| \leq \sup\{\|T\Phi_l^{k,t}\|,\, l \in F_n\},$$

for some vectors $\Phi_l^{k,t}$ and a finite set F_n, where $\|\Phi_l^{k,t}\| \leq A(n+1)^{1/2} 2^{-n(p-2)/2p}$, uniformly on t and l. Therefore ℓ_p satisfies the Cantorized-Enflo criterion.

Actually, we notice that the previous construction only uses that ℓ_p has a natural Schauder decomposition into $\{\ell_p^{3.2^n}\}_{n=2}^{\infty}$. Therefore, if ℓ_p $(p > 2)$ is crudely finitely representable in X, then using Remark 2.8, there exist a constant $K > 0$ and a subspace Y of X admitting a Schauder decomposition into $\{X_n\}_{n=1}^{\infty}$, such that $d(X_n, \ell_p^{3.2^n}) \leq K$. Hence, the analogous construction of vectors e_j^k and α_j^k can be done as vectors supported in X_{k-1} and X_k. □

Corollary 2.10. *If ℓ_p $(p > 2)$ is crudely finitely representable in X, then X is ergodic.*

We observe that the construction of Johnson and Szankowski [26, Section IV] satisfies the Cantorized-Enflo criterion in the form of Remark 2.6, so each of the X_t constructed fails the AP.

3. Case $p(X) < 2$

In this section we prove ergodicity for separable Banach spaces such that $p(X) < 2$. The particular case for the ℓ_p spaces ($1 \leq p < 2$) was proved by Ferenczi and Galego [15], where they actually reduce the relation E_{K_σ} and use only subspaces with unconditional bases. Their approach relies on certain lower estimates on successive vectors which have no reason to hold in the case when ℓ_p is only crudely finitely representable on X.

Our approach is to obtain the 'Cantorized version' of the subspaces of ℓ_p ($1 \leq p < 2$) without AP constructed by Szankowski [40]. The advantage of this method is that the nature of that construction allows us to pass the Cantorized-Enflo criterion from ℓ_p to a Banach space X for which ℓ_p is crudely finitely representable in X.

Before the proof, we need to define the functions $f_k : \mathbb{N} \to \mathbb{N}$, $k \leq 8$, $g_k : \mathbb{N} \to \mathbb{N}$, $k \leq 15$, $h_k : \mathbb{N} \to \mathbb{N}$, $k \leq 32$ to encode the support of some vectors used in that construction. The main difference from [40] is that our construction uses vectors with support of length twelve instead of six of the original one:

$$\begin{aligned} &f_k(16i+l) = 8i+k-1,\ i = 2,3,4,\dots \quad 0 \leq l \leq 15, \quad 1 \leq k \leq 8,\\ &g_k(16i+l) = 16i+(l+k) \mod 16, \quad i = 2,3,4,\dots \quad 0 \leq l \leq 15, \quad 1 \leq k \leq 15,\\ &h_k(16i+l) = 32i+k-1,\ i = 2,3,4,\dots \quad 0 \leq l \leq 3, \quad 1 \leq k \leq 32. \end{aligned}$$

We denote $I_n^j = \{k \in I_n : k \cong j \pmod{16}\}$, $j = 0,1,2\dots,15$. The following is a modified version of the key Szankowski combinatorial argument [40] (see also [34, Proposition 1.g.5]) adapted to our set of functions $\{f_k, g_k, h_k\}$.

Lemma 3.1. *There exist partitions Δ_n and ∇_n of I_n into disjoint sets and a sequence of integers $(m_n)_n$ with $m_n \geq 2^{n/32-1}$, $n = 2,3,\dots$ such that:*

1. *For every $A \in \nabla_n$, $m_n \leq |A| \leq 2m_n$ and it is contained in some I_n^j.*
2. *For every $A \in \nabla_n$ and every $B \in \Delta_n$, $|A \cap B| \leq 1$.*
3. *For every $A \in \nabla_n$ and every function ξ in $\{f_k, g_k, h_k\}$, the set $\xi(A)$ is contained entirely in an element of Δ_{n-1}, Δ_n, or Δ_{n+1}.*

Proof. Consider the functions $\varphi_n^j : I_n^0 \to I_n^j$ given by $\varphi_n^j(k) = k+j$ ($j = 0,1,\dots,15$). For $n \geq 4$ and $r = 0,1$ we let $\psi_n^r : I_n^0 \to I_{n+1}^0$ be the map defined by $\psi_n^r(k) = 2k+16r$. The above functions are 1-1 and have disjoint ranks with $I_{n+1}^0 = \psi_n^0(I_n^0) \cup \psi_n^1(I_n^0)$.

Inductively, for $n \geq 4$ we can represent I_n^0 as the cartesian product $C_n \times D_n$, where $|D_{n+1}| = |C_n|$, $|C_{n+1}| = 2|D_n|$ and such that:

1. For every $c \in C_{n+1}$ there exist $d \in D_n$ and $r = 0,1$ such that $\psi_n^r(C_n \times \{d\}) = \{c\} \times D_{n+1}$.
2. For every $d \in D_{n+1}$ there exists $c \in C_n$ such that $\psi_n^0 \cup \psi_n^1(\{c\} \times D_n) = C_{n+1} \times \{d\}$.

This means that the functions ψ_n^r send columns of $C_n \times D_n$ onto rows of $C_{n+1} \times D_{n+1}$ in a way that every column of $C_{n+1} \times D_{n+1}$ is the image of a row of $C_n \times D_n$ by $\psi_n^0 \cup \psi_n^1$. Notice that $|C_n|, |D_n| \geq 2^{n/2-2}$.

Now we split each D_n as a cartesian product of sixteen factors $D_n = \prod_{l=0}^{15} D_n^l$ such that

$$|D_n^0| \leq |D_n^1| \leq \cdots \leq |D_n^{15}| \leq 2|D_n^0|.$$

The partitions are then defined as

$$\nabla_n = \left\{ \varphi_n^l \left(\{c\} \times D_n^l \right) \,:\, c \in C_n \times \prod_{i \neq l} D_n^i,\, 0 \leq l \leq 15 \right\},$$

$$\Delta_n = \left\{ \varphi_n^l \left(C_n \times \prod_{i \neq l} D_n^i \times \{d\} \right) \,:\, d \in D_n^l,\, 0 \leq l \leq 15 \right\}.$$

The conditions (1), (2), and (3) are satisfied in the same way as [40]. □

Theorem 3.2. *If ℓ_p is crudely finitely representable in a Banach space X, for some $1 \leq p < 2$, then X satisfies the Cantorized-Enflo criterion.*

Proof. Let X be a Banach space such that ℓ_p is crudely finitely representable, for some $1 \leq p < 2$. For every $n \in \mathbb{N}$, we fix Δ_n and ∇_n partitions of I_n obtained by Lemma 3.1. It follows by Remark 2.8 that there exist a constant $K > 0$ and a subspace Y of X admitting a Schauder decomposition into $\{X_n\}_{n=1}^{\infty}$ such that $d(X_n, \ell_p^{2^n}) \leq K$, for every $n \in \mathbb{N}$. Let $(x_j)_{j=1}^{\infty}$ be a bounded sequence of vectors in Y with $x_j \in X_n$ when $j \in I_n$ such that for every n,

(3.1)

$$K^{-1} \left(\sum_{B \in \Delta_n} \left(\sum_{j \in B} |a_j|^2 \right)^{p/2} \right)^{1/p} \leq \left\| \sum_{j \in I_n} a_j x_j \right\| \leq K \left(\sum_{B \in \Delta_n} \left(\sum_{j \in B} |a_j|^2 \right)^{p/2} \right)^{1/p},$$

for any sequence of scalars $(a_j)_{j=1}^{\infty}$. Let $(x_j^*)_{j=1}^{\infty}$ be a sequence of functionals in Y^* such that $x_j^*(x_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}$ and

(3.2)

$$K^{-1} \left(\sum_{B \in \Delta_n} \left(\sum_{j \in B} |b_j|^2 \right)^{q/2} \right)^{1/q} \leq \left\| \sum_{j \in I_n} b_j x_j^* \right\|_{Y^*} \leq K \left(\sum_{B \in \Delta_n} \left(\sum_{j \in B} |b_j|^2 \right)^{q/2} \right)^{1/q}$$

for any sequence of scalars $(b_j)_{j=1}^{\infty}$ and every n, where $1/p + 1/q = 1$.

We now define the sequence of vectors $(z_{i,\epsilon})_i$, $\epsilon = 0, 1$, in Y by setting:

$$\begin{aligned} z_{i,0} &= (x_{8i} - x_{8i+1}) + (x_{8i+2} - x_{8i+3}) + x_{16i} + x_{16i+1} + x_{16i+4} + x_{16i+5} + x_{16i+8} \\ &\quad + x_{16i+9} + x_{16i+12} + x_{16i+13}, \\ z_{i,1} &= (x_{8i+4} - x_{8i+5}) + (x_{8i+6} - x_{8i+7}) + x_{16i+2} + x_{16i+3} + x_{16i+6} + x_{16i+7} \\ &\quad + x_{16i+10} + x_{16i+11} + x_{16i+14} + x_{16i+15}. \end{aligned}$$

Recall that $Z = \overline{span}\{z_{j,\epsilon} \,:\, j \in \mathbb{N},\, \epsilon = 0, 1\}$. Notice that for every $i \in \mathbb{N}$,

$$\begin{aligned} (x_{8i}^* - x_{8i+1}^*)_{|Z} = (x_{8i+2}^* - x_{8i+3}^*)_{|Z} &= 1/2(x_{16i}^* + x_{16i+1}^* + x_{16i+8}^* + x_{16i+9}^*)_{|Z} \\ &= 1/2(x_{16i+4}^* + x_{16i+5}^* + x_{16i+12}^* + x_{16i+13}^*)_{|Z}. \end{aligned}$$

Indeed, all four formulas give 2 when evaluated on $z_{i,0}$ and give 0 when evaluated on $z_{j,\epsilon} \neq z_{i,0}$. Analogously, for every $i \in \mathbb{N}$,

$$\begin{aligned}(x^*_{8i+4} - x^*_{8i+5})_{|Z} = (x^*_{8i+6} - x^*_{8i+7})_{|Z} &= 1/2(x^*_{16i+2} + x^*_{16i+3} + x^*_{16i+10} + x^*_{16i+11})_{|Z} \\ &= 1/2(x^*_{16i+6} + x^*_{16i+7} + x^*_{16i+14} + x^*_{16i+15})_{|Z}.\end{aligned}$$

All four formulas above give 2 when evaluated on $z_{i,1}$ and 0 when evaluated on $z_{j,\epsilon} \neq z_{i,1}$. We define the sequence of functionals $(z^*_{n,\epsilon})_{n\in\mathbb{N}}$, $\epsilon = 0, 1$, on Z^* by setting

$$z^*_{i,\epsilon} = 1/2(x^*_{8i+4\epsilon} - x^*_{8i+4\epsilon+1})_{|Z}.$$

Hence,

$$\begin{aligned}z^*_{i,0} = 1/2(x^*_{8i+2} - x^*_{8i+3})_{|Z} &= 1/4(x^*_{16i} + x^*_{16i+1} + x^*_{16i+8} + x^*_{16i+9})_{|Z} \\ &= 1/4(x^*_{16i+4} + x^*_{16i+5} + x^*_{16i+12} + x^*_{16i+13})_{|Z},\end{aligned}$$

$$\begin{aligned}z^*_{i,1} = 1/2(x^*_{8i+6} - x^*_{8i+7})_{|Z} &= 1/4(x^*_{16i+2} + x^*_{16i+3} + x^*_{16i+10} + x^*_{16i+11})_{|Z} \\ &= 1/4(x^*_{16i+6} + x^*_{16i+7} + x^*_{16i+14} + x^*_{16i+15})_{|Z}.\end{aligned}$$

For $t \in 2^{\mathbb{N}}$, recall that $X_t = \overline{span}\{z_{j,t(n)} : j \in I_n,\, n \in \mathbb{N}\}$. If $T : X_t \to Z$ is a linear and bounded operator, the n-trace of T has been defined as

$$\beta^n_t(T) = 2^{-n} \sum_{j\in I_n} z^*_{j,t(n)} T(z_{j,t(n)}).$$

We need to verify that the $\beta'_n s$ satisfy the conditions of the Cantorized-Enflo criterion (Definition 2.1).

Case 1. $t(n) = t(n-1) = 0$.

$$\begin{aligned}\beta^n_t(T) - \beta^{n-1}_t(T) &= 2^{-n} \sum_{i\in I_n} z^*_{i,0} T(z_{i,0}) - 2^{-n+1} \sum_{i\in I_{n-1}} z^*_{i,0} T(z_{i,0}) \\ &= 2^{-n} \sum_{i\in I_n} 2^{-1} \left(x^*_{8i} - x^*_{8i+1}\right) T\left(z_{i,0}\right) \\ &\quad - 2^{-n+1} \sum_{i\in I_{n-1}} 2^{-2}(x^*_{16i} + x^*_{16i+1} + x^*_{16i+8} + x^*_{16i+9}) T\left(z_{i,0}\right) \\ &= 2^{-n-1} \sum_{i\in I_{n-1}} \{x^*_{16i} T(z_{2i,0} - z_{i,0}) + x^*_{16i+1} T\left(-z_{2i,0} - z_{i,0}\right) \\ &\qquad + x^*_{16i+8} T(z_{2i+1,0} - z_{i,0}) + x^*_{16i+9} T(-z_{2i+1,0} - z_{i,0})\}.\end{aligned}$$

The elements in parentheses above will be called $y_{16i}, y_{16i+1}, y_{16i+8}, y_{16i+9}$, respectively; thus

$$\beta^n_t(T) - \beta^{n-1}_t(T) = 2^{-n-1} \sum_{j\in I_{n+3}(0,0)} x^*_j T(y_j),$$

where $I_n(0, 0) = I^0_n \cup I^1_n \cup I^8_n \cup I^9_n$.

Case 2. $t(n) = 0$, $t(n-1) = 1$.

$$\begin{aligned}\beta_t^n(T) - \beta_t^{n-1}(T) &= 2^{-n} \sum_{i \in I_n} z_{i,0}^* T(z_{i,0}) - 2^{-n+1} \sum_{i \in I_{n-1}} z_{i,1}^* T(z_{i,1}) \\ &= 2^{-n} \sum_{i \in I_n} 2^{-1} \left(x_{8i+2}^* - x_{8i+3}^*\right) T\left(z_{i,0}\right) \\ &\quad - 2^{-n+1} \sum_{i \in I_{n-1}} 2^{-2} (x_{16i+2}^* + x_{16i+3}^* + x_{16i+10}^* + x_{16i+11}^*) T\left(z_{i,1}\right) \\ &= 2^{-n-1} \sum_{i \in I_{n-1}} \{x_{16i+2}^* T(z_{2i,0} - z_{i,1}) + x_{16i+3}^* T\left(-z_{2i,0} - z_{i,1}\right) \\ &\qquad + x_{16i+10}^* T(z_{2i+1,0} - z_{i,1}) + x_{16i+11}^* T(-z_{2i+1,0} - z_{i,1})\}.\end{aligned}$$

The elements in parentheses above will be called $y_{16i+2}, y_{16i+3}, y_{16i+10}, y_{16i+11}$, respectively; thus

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(0,1)} x_j^* T(y_j),$$

where $I_n(0,1) = I_n^2 \cup I_n^3 \cup I_n^{10} \cup I_n^{11}$.

Case 3. $t(n) = 1$, $t(n) = 0$.

$$\begin{aligned}\beta_t^n(T) - \beta_t^{n-1}(T) &= 2^{-n} \sum_{i \in I_n} z_{i,1}^* T(z_{i,1}) - 2^{-n+1} \sum_{i \in I_{n-1}} z_{i,0}^* T(z_{i,0}) \\ &= 2^{-n} \sum_{i \in I_n} 2^{-1} \left(x_{8i+4}^* - x_{8i+5}^*\right) T\left(z_{i,1}\right) \\ &\quad - 2^{-n+1} \sum_{i \in I_{n-1}} 2^{-2} (x_{16i+4}^* + x_{16i+5}^* + x_{16i+12}^* + x_{16i+13}^*) T\left(z_{i,0}\right) \\ &= 2^{-n-1} \sum_{i \in I_{n-1}} \{x_{16i+4}^* T(z_{2i,1} - z_{i,0}) + x_{16i+5}^* T\left(-z_{2i,1} - z_{i,0}\right) \\ &\qquad + x_{16i+12}^* T(z_{2i+1,1} - z_{i,0}) + x_{16i+13}^* T(-z_{2i+1,1} - z_{i,0})\}.\end{aligned}$$

The elements in parentheses above will be called $y_{16i+4}, y_{16i+5}, y_{16i+12}, y_{16i+13}$, respectively; thus

$$\beta_t^n(T) - \beta_t^{n-1}(T) = 2^{-n-1} \sum_{j \in I_{n+3}(1,0)} x_j^* T(y_j),$$

where $I_n(1,0) = I_n^4 \cup I_n^5 \cup I_n^{12} \cup I_n^{13}$.

Case 4. $t(n)=1$, $t(n-1)=1$.

$$\begin{aligned}\beta_t^n(T)-\beta_t^{n-1}(T) &= 2^{-n}\sum_{i\in I_n} z_{i,1}^*T(z_{i,1})-2^{-n+1}\sum_{i\in I_{n-1}} z_{i,1}^*T(z_{i,1})\\
&= 2^{-n}\sum_{i\in I_n}2^{-1}\left(x_{8i+6}^*-x_{8i+7}^*\right)T\left(z_{i,1}\right)\\
&\quad -2^{-n+1}\sum_{i\in I_{n-1}}2^{-2}(x_{16i+6}^*+x_{16i+7}^*+x_{16i+14}^*+x_{16i+15}^*)T\left(z_{i,1}\right)\\
&= 2^{-n-1}\sum_{i\in I_{n-1}}\{x_{16i+6}^*T(z_{2i,1}-z_{i,1})+x_{16i+7}^*T\left(-z_{2i,1}-z_{i,1}\right)\\
&\qquad +x_{16i+14}^*T(z_{2i+1,1}-z_{i,1})+x_{16i+15}^*T(-z_{2i+1,1}-z_{i,1})\}.\end{aligned}$$

The elements in parentheses above will be called $y_{16i+6}, y_{16i+7}, y_{16i+14}, y_{16i+15}$, respectively; thus

$$\beta_t^n(T)-\beta_t^{n-1}(T)=2^{-n-1}\sum_{j\in I_{n+3}(1,1)}x_j^*T(y_j),$$

where $I_n(1,1)=I_n^6\cup I_n^7\cup I_n^{14}\cup I_n^{15}$.

Hence,

$$\beta_t^n(T)-\beta_t^{n-1}(T)=2^{-n-1}\sum_{j\in I_{n+3}(t(n),t(n-1))}x_j^*T(y_j).$$

We use the functions $\{f_k,\, g_k,\, h_k\}$ to describe the support of the vectors y_j. For each j, we shall need four functions of the $\{f_k,\, k\le 8\}$, nine functions of the $\{g_k,\, k\le 15\}$, and eight functions of the $\{h_k,\, k\le 32\}$. In fact, notice that

$$y_j=\sum_{k=1}^4\alpha_{j_k}x_{f_{j_k}(j)}+\sum_{t=1}^9\beta_{j_t}x_{g_{j_t}(j)}+\sum_{s=1}^8\gamma_{j_s}x_{h_{j_s}(j)},$$

where $|\alpha_{j,k}|=|\gamma_{j_s}|=|\beta_{j_t}|=1$ for all the indexes in the formula above, except for one j_{t_0} which satisfies $|\beta_{j_{t_0}}|=2$.

Given $\epsilon,\delta=0,1$, we write $\nabla_n(\epsilon,\delta)=\{A\in\nabla_n\,:\,A\subseteq I_n(\epsilon,\delta)\}$. Observe that these sets are well defined because of Lemma 3.1(1). Notice also that

$$\begin{aligned}2^{-n-1}\sum_{j\in I_{n+3}(\epsilon,\delta)}x_j^*T(y_j) &= 2^{-n-1}\sum_{A\in\nabla_{n+3}(\epsilon,\delta)}\sum_{j\in A}x_j^*T(y_j)\\
&= 2^{-n-1}\sum_{A\in\nabla_{n+3}(\epsilon,\delta)}2^{-|A|}\sum_\theta\left(\sum_{j\in A}\theta_jx_j^*\right)\left(\sum_{j\in A}\theta_jTy_j\right),\end{aligned}$$

where the sum is taken over all the choices of signs $\{\theta_j\}_{j\in A}$. Observe that by Lemma 3.1(2) and equation (3.2) above (about norm of the functionals x_j^*) we have, for every $A\in\nabla_{n+3}(\epsilon,\delta)$ and $\{\theta_j\}_{j\in A}$,

$$\begin{aligned}\left\|\sum_{j\in A}\theta_jx_j^*\right\|_{Y^*} &\le K\left(\sum_{B\in\Delta_n}\left(\sum_{j\in B\cap A}|\theta_j|^2\right)^{q/2}\right)^{1/q}\\
&= K|A|^{1/q}\le K(2m_{n+3})^{1/q},\end{aligned}$$

where $1/p+1/q=1$. By Lemma 3.1(3) we have, for every $A \in \nabla_{n+3}(\epsilon,\delta)$, $\{\theta_j\}_{j\in A}$, and any function ξ in $\{f_k,\, g_k,\, h_k\}$,

$$\left\|\sum_{j\in A}\theta_j x_{\xi(j)}\right\| \le K|A|^{1/2} \le K(2m_{n+3})^{1/2}.$$

It follows that

$$\left\|\sum_{j\in A} y_j\right\| = \left\|\sum_{j\in A}\sum_{k=1}^{21}\lambda_{j_k} x_{\xi_{l_k}(j)}\right\| \le 42K|A|^{1/2} \le 42K(2m_{n+3})^{1/2}.$$

Notice that by construction $|\nabla_n(\epsilon,\delta)| = 2^{-2}|\nabla_n| \le 2^{n-2}m_n^{-1}$. Hence,

$$\begin{aligned}\left|\beta_t^n(T)-\beta_t^{n-1}(T)\right| &= \left|2^{-n-1}\sum_{j\in I_{n+3}(\epsilon,\delta)} x_j^* T(y)\right|\\ &\le 2^{-n-1}(2^{n+1}m_{n+3}^{-1})K(2m_{n+3})^{1/q}42K\sqrt{2}m_{n+3}^{1/2}\|T\|\\ &\le 84K^2(m_{n+3})^{1/q+1/2-1}\|T\|.\end{aligned}$$

Since $\alpha = 1/2+1/q-1 = 1/2-1/p<0$, the series $\sum_n m_n^\alpha \le \sum_n 2^{\alpha(n/32-1)} < \infty$. Therefore, X satisfies the Cantorized-Enflo criterion. □

Remark 3.3. The proof of Theorem 3.2 is based on the idea from [40] where subspaces of ℓ_p ($1\le p<2$) without AP were constructed. It was pointed out by Szankowski [40] (see also [34, Remark 2, p. 111]) that the mentioned idea can be easily adapted to obtain subspaces of ℓ_p ($2<p<\infty$) without the AP. This implies that the method used in the proof of Theorem 3.2 is also valid for Banach spaces X in which ℓ_p ($2<p<\infty$) is crudely finitely representable. Indeed, the same definition of vectors $z_{i,\epsilon}$ and functionals $z_{i,\epsilon}^*$ works; it is only necessary to modify the construction of the partitions Δ_n and ∇_n in Lemma 3.1. This gives us an independent proof of Proposition 2.9.

We observe that the construction of Theorem 3.2 satisfies the Cantorized-Enflo criterion in the form of Remark 2.6. Therefore every X_t constructed above fails the AP. We can conclude the following.

Theorem 3.4. *Every separable Banach space not near Hilbert satisfies the Cantorized-Enflo criterion and therefore is ergodic. Furthermore, the reduction uses subspaces without the AP.*

The following remark is due to Anisca.

Remark 3.5. There do exist near Hilbert spaces satisfying the Cantorized-Enflo criterion. Indeed, Casazza, García and Johnson [7] constructed an asymptotically Hilbertian space which fails the AP. Their approach follows closely the Davie construction [11] of a subspace of $\ell_p = (\sum_n \ell_p^{3.2^n})_p$ ($2<p<\infty$) failing the AP. The space in [7] is instead a subspace of $Z = (\sum_n \ell_{p_n}^{3.2^n})_2$ where $p_n \downarrow 2$ appropriately. One can combine the arguments of Proposition 2.9 and those in [26, Section IV] to construct a version of the Casazza, García, and Johnson space satisfying the Cantorized-Enflo criterion. Also, the arguments of Theorem 3.4 can be used in the context of construction by Anisca and Chlebovec [3] to obtain that spaces of the form $\ell_2(X)$, with X of cotype 2 and having the sequence of Euclidean distances of order at least $(\log n)^\beta$ $(\beta > 1)$, satisfy the Cantorized-Enflo criterion.

As a direct consequence of Lemma 2.3 and Theorem 3.2, we can now extend the result of Johnson and Szankowski [26] about complementably universal spaces for the family of subspaces of ℓ_p $(2 < p < \infty)$.

Theorem 3.6. *There is no separable Banach space which is complementably universal for the class of all subspaces of X when X is not near Hilbert.*

Corollary 3.7. *There is no separable Banach space which is complementably universal for the family of subspaces of ℓ_p $(1 \leq p < 2)$.*

In the limit case, Johnson and Szankowski [28] constructed a separable space, non-isomorphic to the Hilbert, such that all subspaces have the BAP and it is complementably universal for all its subspaces. Also, if every subspace of X has the BAP (for example if X is weak Hilbert), then Pełczyński's universal space (see [33, Theorem 2.d.10(a)]) is complementably universal for the family of all subspaces of X.

4. The Schlumprecht type space $S_{p,r}$ is not near Hilbert

Schlumprecht [1,41] constructed the first example of a complementably minimal Banach space S different from the classical spaces c_0 and ℓ_p $(1 \leq p < \infty)$. In [9], the Schlumprecht construction was extended to uniformly convex examples using interpolation techniques. In fact, they constructed a family of uniformly convex complementably minimal spaces by interpolating S and ℓ_q.

The approach in [9] deals with Banach spaces X defined by lattice norms $\|.\|_X$ on c_{00}. In this context, if X and Y are two such spaces and $0 < \theta < 1$, then $X^{1-\theta}Y^\theta$ is defined as the space Z with the norm $\|z\|_Z = \inf\{\|x\|_X^{1-\theta}\|y\|_Y^\theta,\ z = |x|^{1-\theta}|y|^\theta\}$. When we consider the complex scalars and either X or Y is separable, then Z coincides with the usual complex interpolation space $[X, Y]_\theta$ (see [6]).

Definition 4.1. For every $1 \leq p < r \leq \infty$, the Schlumprecht type space $S_{p,r}$ is defined as the interpolated space $\ell_t^{1-\theta}S^\theta$, where $\theta = \frac{1}{p} - \frac{1}{r}$ and $t = (1-\theta)r$.

Proposition 4.2 ([9, Proposition 3 and Theorem 8])**.** *For any $1 \leq p < r \leq \infty$, the space $S_{p,r}$ and its dual are complementably minimal. Furthermore, $S_{p,r}$ has a 1-unconditional normalized basis $(e_n)_{n=1}^\infty$ such that*

$$\left\|\sum_{i=1}^n e_i\right\|_{S_{p,r}} = n^{1/p}\log_2(n+1)^{1/r-1/p},$$

for every $n \in \mathbb{N}$.

Notice that S is simply $S_{1,\infty}$. We use the ideas from [10] and estimates of the norm of some combination of the vector basis to compute $p(S_{p,r})$ and $q(S_{p,r})$.

Proposition 4.3. *Let $1 \leq p < r \leq \infty$. For every $n \in \mathbb{N}$ and $\epsilon > 0$, there exists a sequence of vectors $v_1, \ldots, v_n$ in c_{00} such that:*

1. *The set of vectors $\{v_1, \ldots, v_n\}$ is disjointly supported.*
2. *$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|_{S_{p,r}} \leq (1+\epsilon)^\theta n^{1/r}$, for any $\epsilon_1, \ldots, \epsilon_n$ of modulus 1.*

Proof. D. Kutzarova and P. K. Lin [32] proved that there exist vectors $v_1, \dots, v_n$ in c_{00} which are disjointly supported such that $\|v_1 + \cdots + v_n\|_S \leq (1+\epsilon)$, where each v_j is of the form $\frac{m_j}{\log_2(m_j+1)} \sum_{i \in M_j} e_i$, $|M_j| = m_j$. Since the basis of S is 1-unconditional, we have $\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|_S \leq (1+\epsilon)$ for any $(\epsilon_j)_j^n$ of modulus 1.

Letting $v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$, it follows from the Lozanovskii formula that

$$\|v\|_{S_{p,r}} \leq \|v\|_t^{1-\theta} \|v\|_S^{\theta} \leq (1+\epsilon)^{\theta} n^{(1-\theta)/t} = (1+\epsilon)^{\theta} n^{1/r}.$$

□

Proposition 4.4. *Let $1 \leq p < r \leq \infty$. The family of Schlumprecht type spaces $S_{p,r}$ and their duals are not near Hilbert. In particular, they are ergodic spaces.*

Proof. Let $1 \leq p < r \leq \infty$. Assume that $S_{p,r}$ has type t. Then by Proposition 4.2, $n^{1/p} \log_2(n+1)^{1/r-1/p} \leq T_t n^{1/t}$ for some constant T_t, and then $t \leq p$. Hence $p(S_{p,r}) \leq p$. Analogously, if $S_{p,r}$ has cotype t, then by Proposition 4.3 $t \geq r$, and then $q(S_{p,r}) \geq r$. We have that $p(S_{p,r}) \leq p < r \leq q(S_{p,r})$, and it follows that $S_{p,r}$ is not near Hilbert. Also, since a Banach space X is near Hilbert if and only if X^* is near Hilbert, the dual space $S_{p,r}^*$ is not near Hilbert. □

5. Final remarks

Of course, the main question concerning ergodic spaces is whether ℓ_2 is the only non-ergodic Banach space. The conclusion of Theorem 3.4 restricts the question of ergodicity to the case of near Hilbert spaces, but our technique uses a reduction throughout subspaces without the AP. A Banach space in which all of its subspaces have AP is said to have the *hereditary approximation property* (HAP). Szankowski [40] proved that every HAP space must be near Hilbert. The first example of a HAP space not isomorphic to a Hilbert space was constructed by Johnson [23]. Later, Pisier [36] proved that every weak Hilbert space has the HAP. The space constructed by Johnson is asymptotically Hilbertian and therefore ergodic by the Anisca [2] result. In 2010 Johnson and Szankowski [28] constructed a HAP space with a symmetric basis but not isomorphic to ℓ_2 and hence not asymptotically Hilbertian. Hence, a natural question is the following:

Problem. Is the HAP non-asymptotically Hilbertian space contructed in [28] ergodic?

Or more generally:

Problem. Is every HAP not isomorphic to the Hilbert space ergodic?

Another interesting class of near Hilbert spaces are the twisted Hilbert spaces. The most important example of non-trivial twisted Hilbert space is the Kalton-Peck space Z_2 [29]. Z_2 is not asymptotically Hilbertian, and it is not known whether it has the HAP.

Problem. Does there exist an ergodic non-trivial twisted Hilbert space?

Another natural question is:

Problem. Is every minimal Banach space not isomorphic to a Hilbert space ergodic?

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