

## PERIOD INTEGRALS AND MUTATION

KETIL TVEITEN

ABSTRACT. We define what it means for a Laurent polynomial in two variables to be *mutable*. For a mutable Laurent polynomial we prove several results about  $f$  and its period  $\pi_f$  in terms of the Newton polytope of  $f$ . In particular, we give an in principle complete description of the monodromy of  $\pi_f$  around the origin. Special attention is given to the class of *maximally mutable* Laurent polynomials, which has applications to the conjectured classification of Fano manifolds via mirror symmetry.

### 1. INTRODUCTION

Let  $N \simeq \mathbb{Z}^2$  be a lattice, and let  $\mathbb{C}[N]$  be the ring of Laurent polynomials in two variables. Let  $P \subset N$  be a lattice polytope, and let  $f(a, x) = \sum_{m \in P} a_m x^m$  be a generic Laurent polynomial with Newton polytope  $\text{Newt}(f) = P$ . For such an  $f$ , and  $C$  a 2-cycle in  $H_2(\{x \in (\mathbb{C}^*)^2 \mid f(x) \neq 0\}, \mathbb{C})$ , consider the integral

$$\phi_f(a) = \int_C \frac{1}{f(a, x)} \frac{dx_1}{x_1} \frac{dx_2}{x_2}.$$

As a function of the coefficients  $a_m$  of  $f$ , this integral satisfies a system of differential equations of the GKZ type, as follows: Let  $P'$  be the image of  $P$  under the embedding  $N \hookrightarrow \mathbb{Z} \times N$  “at height 1” given by  $m \mapsto (1, m)$ . Then  $\phi_f$  is a solution to the GKZ system  $H_\gamma(P')$ , where  $\gamma = (0, 0, -1)$  (see [SST00, 5.4.2]).

An interesting specialization of this is the *classical period integral* of  $f$ ,

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^2 \int_{|x_1|=|x_2|=1} \frac{1}{1 - tf(a, x)} \frac{dx_1}{x_1} \frac{dx_2}{x_2},$$

which is a (possibly multivalued) holomorphic function of  $t$  in a punctured disk around  $t = 0$ . When the polytope  $P$  is a *Fano polytope* (see Definition 2.1), the classical period plays an important role in the conjectured classification of Fano manifolds via mirror symmetry, for a certain class of Laurent polynomials  $f$  called *maximally mutable* [CCG<sup>+</sup>13, ACC<sup>+</sup>16] (see Definition 3.14); understanding this class of Laurent polynomials and the associated  $D$ -module generated by  $\pi_f(t)$  is the primary motivation for this paper.

We will study the local behaviour of the  $D$ -module  $D \cdot \pi_f(t)$  around the origin by using tools from toric geometry and the relation between properties of this module and the combinatorial data of the Newton polytope of  $f$ , in particular through the operation of *mutation* (see Definitions 3.9 and 3.12, and also [ACGK12]). In addition, we try to extract some information about its global behaviour from the

---

Received by the editors March 3, 2015, and, in revised form October 13, 2015, April 22, 2016, and March 17, 2017.

2010 *Mathematics Subject Classification*. Primary 32S40, 14J33.

*Key words and phrases*. Monodromy, periods, mirror symmetry.

combinatorics of  $\text{Newt}(f)$ . We solve the former problem completely; for the latter the best we can do is make plausible conjectures backed up by empirical data. We also prove several results about the relationship between Laurent polynomials and their Newton polygons.

For a mutable Laurent polynomial  $f$  with Newton polygon  $P$ , the pencil  $|\mathbf{1}, f|$  defines a rational map  $Y_P \dashrightarrow \mathbb{P}^1$ , where  $Y_P$  is the toric variety defined by the normal fan of  $P$ . If  $X$  is the minimal resolution of the base locus of this map and the singularities of  $Y_P$ , there is a morphism  $X \rightarrow \mathbb{P}^1$ , and if  $X_t$  denotes the fiber of this map at  $t \in \mathbb{P}^1$ , the monodromy of  $\pi_f$  around some point is the monodromy of  $H_1(X_t, \mathbb{Z})$  around this point, for  $t$  generic (see Section 2 for the details of this construction).

When discussing mutation of two-dimensional polytopes (or equivalently, of two-dimensional toric del Pezzo surfaces), the principal invariant of mutation is the *singularity content* due to Akhtar and Kasprzyk [AK14]; see Definition 3.3. In Section 4, we show the main result of this paper, computing the monodromy at zero by combining the contribution of each element in the singularity content.

**Theorem 1.1.** *Let  $P$  be a Fano polygon with singularity content  $(k, \mathcal{B})$ , and let  $X_t$  be the general fiber of a generic maximally mutable Laurent polynomial with  $\text{Newt}(f) = P$ . Then the monodromy of  $H_1(X_t, \mathbb{Z})$  around  $t = 0$  is given by the composition  $[(\alpha, \beta) \mapsto (\alpha + k\beta, \beta)] \circ \bigcirc_{\sigma \in \mathcal{B}} \psi_\sigma$ , where the  $\psi_\sigma$  are the automorphisms defined in Lemma 4.9, and  $\alpha, \beta$  are the cycles defined at the beginning of Section 4.*

The terms and objects involved in this statement are defined in Sections 2, 3, and 4. From the main theorem we can deduce another important result.

**Theorem 1.2.** *Let  $P$  be a Fano polygon, let  $f$  be a maximally mutable Laurent polynomial with  $\text{Newt}(f) = P$ , and let  $\pi_f$  be its classical period. The monodromy at zero of  $\pi_f$  determines and is determined by the singularity content of  $P$ , thought of as a multiset.*

In particular, we state the version of the main theorem that applies in the simplest nontrivial case, considered in [ACC<sup>+</sup>16].

**Theorem 1.3.** *Let  $P$  be a Fano polygon with singularity content  $(k, \{n \times \frac{1}{3}(1, 1)\})$ , and let  $X_t$  be the generic fiber of the morphism defined by a generic maximally mutable Laurent polynomial  $f$  with  $\text{Newt}(f) = P$ . Then there is a basis  $\{\alpha, \beta, a_1^1, a_2^1, \dots, a_1^n, a_2^n\}$  of cycles in  $H_1(X_t, \mathbb{Z})$  such that in terms of this basis, the monodromy automorphism  $\phi$  of  $H_1(X_t, \mathbb{Z})$  is given by*

- $\phi(\alpha) = \alpha + (k + n - 12)\beta + \sum_{j=1}^n a_1^j + \sum_{j=1}^n a_2^j$ ,
- $\phi(\beta) = \beta$ ,
- $\phi(a_1^j) = a_2^j$  for  $1 \leq j \leq n$ , and
- $\phi(a_2^j) = \beta - a_1^j - a_2^j$  for  $1 \leq j \leq n$ .

Note that the singularity content is not a complete invariant; there are polygons with the same singularity content that are not mutation-equivalent. In Section 5 we conjecture an improvement to this: A local system  $V$  on  $\mathbb{P}^1 \setminus S$  (where  $S$  is a finite set) has monodromy  $T_s$  around each point  $s \in S$ , and we can gather up some information about the total monodromy group in a quantity called the *ramification* of  $V$ , defined by

$$\text{rf}(V) = \sum_{s \in S} \dim(V_x/V_x^{T_s}) - 2rk(V),$$

where  $x \in \mathbb{P}^1 \setminus S$  is some point (it doesn't matter which, as  $T_s$  is only defined up to conjugation, i.e., up to the choice of base point). It is a general fact that  $\text{rf}(V) \geq 0$ , in particular local systems with  $\text{rf}(V) = 0$  seem interesting in their own right (also see [CCG<sup>+</sup>13]); the local systems arising from classical periods of maximally mutable Laurent polynomials have unusually low ramification, often minimal. Based on some empirical evidence we conjecture the following.

**Conjecture 1.4.** *Let  $P$  be a Fano polygon, and let  $f$  be a generic standard maximally mutable Laurent polynomial with  $\text{Newt}(f) = P$ . Then, the singularity content of  $P$  together with the ramification of the local system  $\text{Sol}(D \cdot \pi_f)$  completely determines the mutation class of  $P$ .*

2. PRELIMINARIES

As we will discuss the operation of *mutation* of polygons and Laurent polynomials, we must restrict ourselves to the class of *Fano polygons*, where this operation is well behaved.

**Definition 2.1.** Let  $N$  be a two-dimensional lattice, and let  $P \subset N \otimes \mathbb{R}$  be a convex lattice polygon such that

- (1)  $\dim P = 2$ ;
- (2)  $0 \in \text{int}(P)$ , that is, the origin is a strict interior point of  $P$ ; and
- (3) the vertices of  $P$  are primitive lattice points.

Such a polygon is called a *Fano polygon* (see [KN12]).

In the remainder, all polygons are assumed to be Fano polygons, and all Laurent polynomials are such that  $\text{Newt}(f)$  is Fano. We consider two polygons to be equal if they differ by an element of  $\text{GL}(N)$ , and similarly consider two Laurent polynomials to be equal if they are related by an automorphism of  $\mathbb{C}[N]$  induced by an element of  $\text{GL}(N)$ . The one-dimensional faces of a polygon are called *edges* and the zero-dimensional faces are called *vertices*.

Let  $P \subset N$  be a Fano polygon, let  $M = \text{Hom}(N, \mathbb{Z})$ , and let  $Y_P$  be the toric del Pezzo surface defined by the normal fan of  $P$  in  $M$ . Recall that a variety  $Y$  is *Fano* if the anticanonical divisor  $-K_Y$  is ample (two-dimensional Fano varieties are usually called *del Pezzo surfaces* for historical reasons). The rays  $u_i$  generating the normal fan of  $P$  are the inward normals to the edges  $E_i$  of  $P$ , so for each edge  $E_i$  of  $P$ , let  $D_{E_i} = D_i$  denote the corresponding divisor on  $Y_P$ . There is a distinguished Cartier divisor on  $Y_P$ ,

$$D_P = \sum_i h_i D_i;$$

here  $h_i = -\langle u_i | E_i \rangle$  is the lattice height of the edge  $E_i$ . The associated line bundle  $\mathcal{O}(D_P)$  is very ample, and its global sections  $\Gamma(Y_P, \mathcal{O}(D_P))$  can be identified with the set of Laurent polynomials with Newton polygon  $\text{Newt}(f) = P$  (see [CLS11, 4.3.3/4.3.7]). In particular, the origin  $0 \in P$  corresponds to a distinguished element  $\mathbf{1} \in \Gamma(Y_P, \mathcal{O}(D_P))$ , with  $\text{div}(\mathbf{1}) = D_P$ .

A section  $f$  determines a rational map

$$\tau := \frac{\mathbf{1}}{f} : Y_P \dashrightarrow \mathbb{P}^1.$$

Let  $Z_P \rightarrow Y_P$  be the resolution of the indeterminacy locus  $\tau^{-1}(0) \cap \tau^{-1}(\infty)$ , and let  $\widehat{D}$  be the strict transform of  $D_P$ . The induced map  $\widehat{\tau} : Z_P \rightarrow \mathbb{P}^1$  is a morphism,

with central fiber  $\widehat{\tau}^{-1}(0) = \widehat{D}$ . The variety  $Z_P$  is usually not smooth, however, so let  $\Gamma_{\widehat{\tau}}$  be the graph of this morphism, and let  $X \rightarrow \Gamma_{\widehat{\tau}}$  be the minimal resolution. Via the embedding  $Z_P \subset \Gamma_{\widehat{\tau}}$  there is an induced morphism  $X \rightarrow \mathbb{P}^1$ , which we call  $\widetilde{\tau}$ . Let  $D'$  be the pullback of  $\widehat{D}$  to  $X$ ; the central fiber  $\widehat{\tau}^{-1}(0)$  is equal to  $D'$ .

In fact, we will work with not merely a single  $f$ , but rather a linear system of Laurent polynomials (the linear systems of mutable Laurent polynomials associated to a polygon  $P$  generally have nonzero dimension) and so really a parametrised family of  $\tau$ 's; this does not change the above description, as the base point locus of this linear system coincides with the indeterminacy locus of  $\tau$  (regardless of which section  $f$  in the linear system one uses to define  $\tau$ ). As nothing we discuss really depends on these parameters, we suppress the mention of them.

From  $D$ -module theory the module  $D \cdot \pi_f$  is the degree zero part of the  $D$ -module theoretic direct image  $\widetilde{\tau}_+ \mathcal{O}_X$  (see [SST00, 5.5.1], in whose terminology this is the *integration module* of  $\mathcal{O}_X$ ). It follows from [BGK<sup>+</sup>87, VII.9.6, VIII.13.4 and VIII.14.5.1] that  $\text{Sol}(\widetilde{\tau}_+ \mathcal{O}_X)$  is isomorphic to  $R\widetilde{\tau}_! \mathbb{C}_X$  (via the standard isomorphisms of functors  $\text{Sol} \simeq D \circ DR$ ,  $DR \circ \widetilde{\tau}_+ \simeq R\widetilde{\tau}_* \circ DR$ , and  $D \circ R\widetilde{\tau}_* \simeq R\widetilde{\tau}_! \circ D$ , where  $D$  is the Verdier duality and  $DR$  is the De Rham functor from  $D$ -modules to constructible sheaves, and such that  $DR(\mathcal{O}_X)$  is a resolution of  $\mathbb{C}_X$ ), so  $\text{Sol}(D \cdot \pi_f) \simeq R^1 \widetilde{\tau}_! \mathbb{C}_X$  as constructible sheaves. From [Dim04, 2.3.21, 2.3.26] we have  $(R^1 \widetilde{\tau}_! \mathbb{C}_X)_t \simeq H_c^1(X_t, \mathbb{C}_X) \simeq H_1(X_t, \mathbb{C})$  for  $t \in \mathbb{P}^1$ . As  $R^1 \widetilde{\tau}_! \mathbb{C}_X$  is constructible with respect to the stratification given by the critical values of  $\widetilde{\tau}$  (there are two strata: the critical values and their complement), the restriction of  $R^1 \widetilde{\tau}_! \mathbb{C}_X$  to the complement of the critical values of  $\widetilde{\tau}$  (the top stratum) is a local system with fiber  $H_1(X_t, \mathbb{C})$  for generic  $t$ . Recalling the well-known fact that the monodromy of a holomorphic function is integral (see, e.g., [Zol06, 5.4.32]), the monodromy of  $\text{Sol}(D \cdot \pi_f)$  is thus the monodromy of  $H_1(X_t, \mathbb{Z})$  around zero.

See Remark 3.16 for an alternate equivalent description of  $X$ .

### 3. MUTATION

In this section we will introduce the operations of *mutation* (of Fano polygons and Laurent polynomials, respectively), introduce the class of *maximally mutable* Laurent polynomials, and prove some results that prepare us for the monodromy computation in the next section. First, we must introduce some notation and terminology, which will remain in force for the remainder of the paper.

Let  $P \subset N$  be as before, with vertices  $p_i$  and edges  $E_i$  with inward normal vectors  $u_i \in M$ . We number these so that  $E_i$  is the edge between  $p_i$  and  $p_{i+1}$ . The *lattice height* (or simply *height*) of an edge  $E_i$  is  $-\langle u_i | E_i \rangle$ , and the *lattice width* (or simply *width*) of  $E_i$  is  $\langle u_{i-1} | p_i - p_{i+1} \rangle = \langle u_{i+1} | p_{i+1} - p_i \rangle$ ; the lattice width is equal to the number of lattice points on  $E_i$  minus one. The following definitions are due to Akhtar and Kasprzyk (see [AK14]).

**Definition 3.1.** Let  $C \subset N$  be a primitive lattice cone of lattice height  $h$  and lattice width  $w$ . If  $h = w$ , we say that  $C$  is a *primitive  $T$ -cone*. If  $w$  is a positive multiple of  $h$ , we say that  $C$  is a  *$T$ -cone*. If  $w$  is strictly less than  $h$ , we say that  $C$  is an  *$R$ -cone*.

Let  $E$  be an edge of  $P$  of height  $r$  and width  $w = mr + w_0$ , where  $0 \leq w_0 < r$ . Then we can subdivide the cone in  $P$  spanned by  $E$  into  $m$  primitive  $T$ -cones and one  $R$ -cone of width  $w_0$ ; we say these are cones *on the edge*  $E$ . There are  $m + 1$

ways to do this, depending on where you place the  $R$ -cone, but by [AK14, 2.3] the type (see Definition 3.7) of the  $R$ -cone depends only on  $E$  and not on the ordering. We therefore talk about, e.g., “the  $R$ -cone on  $E$ ” and the “set of  $R$ -cones in  $P$ ” without ambiguity.

**Definition 3.2.** Let  $P$  be a Fano polygon. A lattice point in the interior of  $P$  that is either the origin or inside an  $R$ -cone is called a *residual point* of  $P$ .

**Definition 3.3** ([AK14]). Let  $P$  be a Fano polygon, let  $k$  be the number of  $T$ -cones in  $P$ , and let  $\mathcal{B}$  be the cyclically ordered list of types of  $R$ -cones in  $P$ . The set  $\mathcal{B}$  is called the *singularity basket* of  $P$ . The *singularity content* of  $P$  is the pair  $(k, \mathcal{B})$ .

*Remark 3.4.* Note that in Theorem 1.2, the singularity basket is thought of as a *multiset* rather than a cyclically ordered list; see Remark 4.10 for a discussion of this defect.

Next we state the main results of this section, Theorem 3.5 and Proposition 3.6, from which we see that the number of residual points is important. Note that some of the terms in these statements are not yet defined; the definitions appear in the text immediately following the statements of Theorem 3.5 and Proposition 3.6.

**Theorem 3.5.** *Let  $f$  be a generic maximally mutable Laurent polynomial with  $\text{Newt}(f) = P$ . The general fiber  $X_t \subset X$ , which is the desingularization of the curve  $f = 0$  in  $Y_P$ , has genus equal to the number of residual points of  $P$ . We call this number the mutable genus of  $Y_P$  and denote it by  $g_{\text{mut}}(Y_P)$ ; it is mutation-invariant.*

**Proposition 3.6.** *The dimension of the linear system of standard maximally mutable Laurent polynomials with Newton polygon  $P$  is equal to the number of residual points of  $P$ .*

A proof of Theorem 3.5 will be given at the end of this section; a proof of Proposition 3.6 is given in [KT16] (the obvious generalization from standard to generic maximally mutable to arbitrarily mutable is proved in the same way).

**Definition 3.7.** Let  $C \subset N$  be a primitive lattice cone, with primitive spanning vectors  $u$  and  $v$ . If  $\{u, v\}$  is a lattice basis for  $N$ , we say that  $C$  is *smooth*. If  $C$  is not smooth, there is a point  $p \in N$  such that  $p = \frac{1}{r}u + \frac{a-1}{r}v$ , and  $\{u, p\}$  and  $\{v, p\}$  are lattice bases for  $N$ ; in this case we say that  $C$  is of type  $\frac{1}{r}(1, a - 1)$ .

*Remark 3.8.* The type of cones parallels the classification of cyclic quotient singularities; a cone of type  $\frac{1}{r}(1, a - 1)$  defines a toric variety isomorphic to the cyclic quotient singularity  $\mathbb{C}^2/\mu_r$ , where  $\mu_r$  acts with weight  $(1, a - 1)$ . See [AK14] for further details on  $R$ - and  $T$ -cones, and the corresponding singularities, called  $R$ - and  $T$ -singularities. It is conjectured in [ACC<sup>+</sup>16] that there is an injective correspondence from the set of Fano polygons up to *mutation* to the set of orbifold del Pezzo surfaces up to  $\mathbb{Q}$ -Gorenstein deformation. We will now describe this operation of mutation.

**Definition 3.9** ([ACGK12]). Let  $P \in N$  be a Fano polygon, and let  $E$  be an edge of  $P$ , with inward normal vector  $u$ , lattice height  $-r$ , and lattice width  $w = mr + w_0$ , where  $m > 0, r \geq w_0 \geq 0$  are integers (in other words,  $E$  supports  $m$   $T$ -cones and an  $R$ -cone of width  $w_0$ ). Let  $r'$  be the maximal lattice height (with respect to  $u$ ) of the points in  $P$ , let  $F \in u^\perp \subset N$  be a primitive lattice vector, and let  $\overline{F}$  be the

one-dimensional lattice polygon  $[0, F]$ . For any  $s \in \mathbb{Z}$  let  $P_s = \{p \in P \mid u(p) = s\}$  be the points of  $P$  at height  $s$  with respect to  $u$  (in particular,  $P_{-r} = E$ ). Notice that we can for each  $0 > s \geq -r$  write  $P_s$  as a Minkowski sum  $m \cdot \bar{F} + Q_s$ , where  $Q_s$  is some (possibly empty) polygon. The *mutation of  $P$*  with respect to the *mutation datum*  $(u, F)$  is the polygon  $P' = \text{mut}_u(P)$  defined by

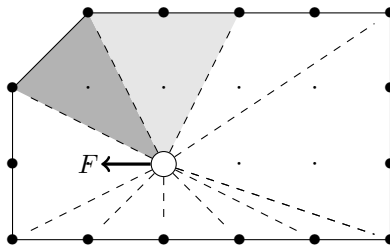
$$P'_s = \begin{cases} (m-1)s\bar{F} + Q_s & 0 \geq s \geq -r, \\ P_s + sF & r' \geq s > 0. \end{cases}$$

Intuitively, we are removing slices  $s\bar{F}$  from each negative height  $s < 0$  and adding slices  $s'\bar{F}$  at each positive height  $s' > 0$ . Equivalently, we are contracting a  $T$ -cone from  $E$  and putting in a  $T$ -cone on the opposite side of  $P$ . We call  $F$  the *factor of mutation*.

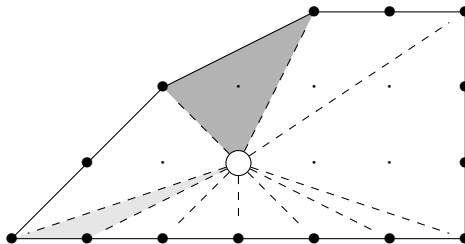
Any polygons  $P, Q$  related by a chain of mutations are said to be *mutation equivalent*.

We observe that we cannot mutate an edge with only a single  $R$ -cone, as that edge will not have sufficient width to permit the procedure. The result of this is that  $R$ -cones are essentially unchanged by mutation, more precisely the cyclically ordered set of  $R$ -cones is invariant under mutation.

**Example 3.10.** Let  $P$  be the Fano polygon with vertices  $(-2, 1), (-1, 2), (3, 2), (3, -1),$  and  $(-2, -1)$ . It has one  $R$ -cone of type  $\frac{1}{3}(1, 1)$  (shaded dark grey), and nine primitive  $T$ -cones. We will perform a mutation with factor  $F = (-1, 0)$  (indicated by an arrow) and height function  $h((x, y)) = -y$ , which will contract away the lightly shaded  $T$ -cone, and add a new  $T$ -cone on the other side of the polygon.



After the mutation, we have this picture; the lightly shaded  $T$ -cone has been contracted, a new  $T$ -cone has been added to the opposite side, and the  $R$ -cone and the  $T$ -cone beneath it have been sheared to fit.



Observe that the number of  $T$ - and  $R$ -cones are unchanged, and the type of the  $R$ -cone is preserved.

Any mutation removes one  $T$ -cone and adds another, so the total number of  $T$ -cones is unchanged. The  $R$ -cones and their relative order is unchanged by mutation, so the singularity content is an invariant under mutation (see [AK14]).

**Example 3.11.** The polygons in example Example 3.10 have singularity content  $(9, \{\frac{1}{3}(1, 1)\})$ .

**Definition 3.12.** Let  $f$  be a Laurent polynomial with Newton polygon  $P$ , and let  $P'$  be the mutation of  $P$  with mutation datum  $(u, F)$ . The map  $\mu : x^a \mapsto x^a(\gamma + \delta x^F)^{\langle u|a \rangle}$  (where  $a \in N, \gamma, \delta \in \mathbb{C}$ ) defines an automorphism of  $\mathbb{C}(N)$ , the rational functions in two variables, and is called a *cluster transformation*. We say that  $f$  is *mutable* with respect to  $u, (\gamma + \delta x^F)$  if  $f' = f \circ \mu$  is in  $\mathbb{C}[N]$  (notice that  $\text{Newt}(\gamma + \delta x^F) = [0, F]$ ), i.e., is a Laurent polynomial, and in this case that  $f'$  is a *mutation of  $f$* ; the Newton polygon of  $f'$  is  $P'$ . Any two Laurent polynomials related by a chain of mutations are said to be *mutation equivalent*.

*Remark 3.13.* Note that if  $P$  and  $Q = \text{mut}(P)$  are polygons with associated Laurent polynomials  $f$  and  $g = \text{mut}(f)$ , the cluster transformation of  $\mathbb{C}(N)$  defining the mutation corresponds to a rational map  $Y_P \dashrightarrow Y_Q$ . Suppose  $E$  is the edge of  $P$  with the  $T$ -cone that is contracted, and suppose  $E'$  is the edge of  $Q$  where the new  $T$ -cone is inserted. The map  $Y_P \dashrightarrow Y_Q$  is given in suitable coordinates by  $(x, y) \mapsto (x, \frac{y}{\gamma + \delta x})$ , with inverse  $(\xi, \eta) \mapsto (\xi, \eta(\gamma + \delta \xi))$ , and one can factor the map through first blowing up the point  $x = -\gamma/\delta$  on  $D_E \subset Y_P$  and then blowing down to the point  $\xi = -\gamma/\delta$  on  $D_{E'} \subset Y_Q$ . This enables an equivalent reformulation of the notion of mutability in more geometric terms:

Let  $f$  be a Laurent polynomial with Newton polygon  $P$ , and let  $D_P = \sum r_E D_E$  be the distinguished divisor on  $Y_P$  defined by  $P$ , so  $f$  is an element of  $\Gamma(Y_P, \mathcal{O}(D_P))$ . Let  $(\gamma : \delta)$  be a point on a divisor  $D_E$  different from  $(1 : 0)$  and  $(0 : 1)$ , let  $Z \rightarrow Y_P$  be the blow-up at this point, and let  $\widehat{D}$  be the strict transform of  $D_P$ . We say that  $f$  is mutable with respect to  $(\gamma : \delta), E$  if  $f$  pulls back to an element of  $\Gamma(Z, \mathcal{O}(\widehat{D}))$ .

This brings us to our main interest, the (*generic*) *maximally mutable* Laurent polynomials.

**Definition 3.14.** Let  $P$  be a Fano polygon, and let  $D_P = \sum r_E D_E$  be the distinguished divisor on  $Y_P$ . For every edge  $E$  of width  $w = mr + w_0$ , select  $m$  points  $p_i := (\gamma_i : \delta_i)$  on the curve  $\mathbb{P}^1 \simeq D_E \subset Y_P$ . Let  $\overline{Y} \rightarrow Y_P$  be the blowup at these points such that if any of the points  $p_i$  coincide, the blowup should be done sequentially, at each step blowing up at the corresponding point on the strict transform under the previous blowup, and let  $\overline{D}$  be the strict transform of  $D_P$ . A Laurent polynomial  $f$  with Newton polygon  $P$  is *maximally mutable* if it pulls back to a section in  $\Gamma(\overline{Y}, \mathcal{O}_{\overline{Y}}(\overline{D}))$ .<sup>1</sup> We say that  $f$  is *mutable with respect to* the points  $p_i$ . We say that  $f$  is *generic maximally mutable* if the points  $p_i$  are distinct. In the special case where on each edge the points  $p_i$  coincide at the point with canonical coordinate  $(1 : -1)$ , we say that  $f$  is *standard* maximally mutable.

We may for simplicity refer to maximally mutable Laurent polynomials as *MMLP's*.

The standard MMLP's are of particular importance for the mirror symmetry classification of Fano manifolds; see, e.g., [ACC<sup>+</sup>16, CCG<sup>+</sup>13, OP15]. Here we will

---

<sup>1</sup>In this case,  $\overline{Y}, \overline{D}$  equals  $Z_P, \widehat{D}$ .

focus on the generic case, but we will take a look at the specializations of coefficients in the final section. We should also point out that the notions of mutation, Fano polygons, and MMLP's make sense in higher dimension, but the formulations in this article are particular to the two-dimensional situation and do not directly generalize; for the case of higher dimension see [KT16].

The relevant fact for us is that mutation of  $f$  preserves the classical period integral  $\pi_f(t)$ . An explicit proof of this fact can be found in [ACGK12], though it also follows from the construction in Remark 3.16. The analysis of  $\pi_f(t)$  and  $D \cdot \pi_f$  is then independent of which  $f$  in the mutation class we use, which allows for a great deal of flexibility.

*Remark 3.15.* It is of course possible to consider Laurent polynomials that are only mutable with respect to some smaller number of points; this corresponds to only allowing mutations of  $\text{Newt}(f)$  involving a subset of the  $T$ -cones. The results analogous to Theorem 3.5 and Proposition 3.6 are obvious, replacing the number of residual points by the corresponding number of lattice points in “unmutable” cones of  $P$ . In particular, when one permits no mutations at all, the analogue of Theorem 3.5 is the well-known result that the *sectional genus* of a toric surface is the number of internal lattice points of the associated polygon, and the analogue of Proposition 3.6 is the obvious statement that the dimension of the linear system is equal to the number of basis elements.

*Remark 3.16.* There is a sense in which mutation-equivalent Laurent polynomials are in fact the *same* function, which we sketch here briefly. Recall that a mutation between polygons  $P$  and  $Q$  corresponds to a rational map  $Y_P \dashrightarrow Y_Q$  which factors through blow-ups at chosen points. If  $f$  and  $g$  are Laurent polynomials such that  $g = \text{mut}(f)$  under the same mutation, it is clear that the pullbacks to the blow-up surface are the same function.

So, if  $f$  is a mutable Laurent polynomial with  $\text{Newt}(f) = P$ , blowing up all the chosen points on  $Y_P$  that define the mutability of  $f$  gives us the surface  $Z_P$  to which all the mutation-equivalent  $g = \text{mut}(f)$  on different  $Y_{\text{mut}(P)}$  pull back to become the same function. If we remove the proper transform of the divisor  $D_P$ , i.e., the polar locus of this function, we get a universal variety  $Z'_P$  which has a cover by the inverse images of the toruses  $(\mathbb{C}^*)^2$  from each of the toric varieties  $Y_{\text{mut}(P)}$ , and the pullbacks of maximally mutable Laurent polynomials are exactly the global sections  $\Gamma(Z'_P, \mathcal{O}_{Z'_P})$ . This variety  $Z'_P$  is called the *cluster variety* associated to the mutation class of  $P$ ; it is clearly mutation-invariant, and so provides us with a new way of seeing that  $D \cdot \pi_f$  is mutation-invariant (the pencils  $\tau_f$  and  $\tau_g$  are in this perspective *the same function on the same variety*). Our surface  $X$  can be constructed from  $Z_P$  by resolving the base locus of the pencil  $\tau_f$ , and so is a natural compactification of the cluster variety.

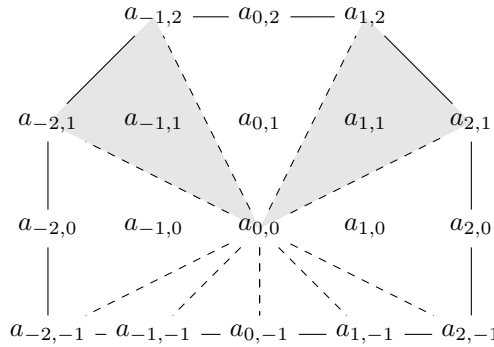
This notion of cluster variety is treated in greater detail in [GHK15] (for the case of classical cluster algebras) and [GU10] (for the analogous case which concerns us).

The process of finding the MMLP's for a given polygon  $P$  is best illustrated with an example.

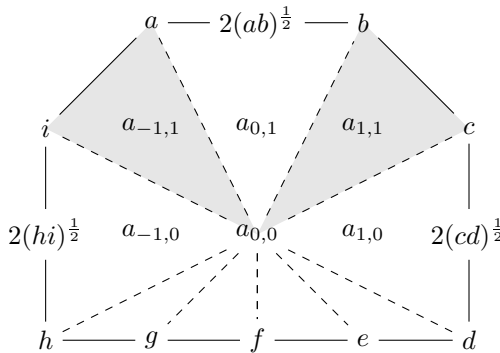
**Example 3.17.** Let  $P$  be the polygon with vertices  $(-1, 2)$ ,  $(1, 2)$ ,  $(2, 1)$ ,  $(2, -1)$ ,  $(-2, -1)$ , and  $(-2, 1)$ ; this has two  $R$ -cones of type  $\frac{1}{3}(1, 1)$  and seven  $T$ -cones. It is easiest to show the process of finding the MMLP's by labelling the vertices of  $P$  by the associated coefficients. The cones are indicated; the  $R$ -cones are shaded grey,



the  $T$ -cones are white. We begin with generic coefficients:



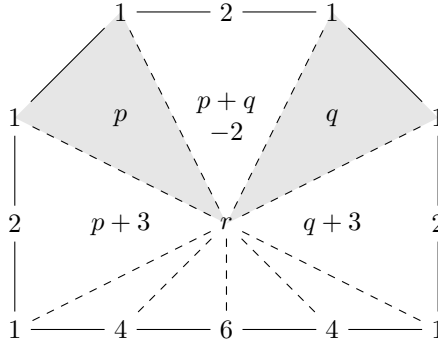
First impose the factorization conditions along the edges with a linear factor  $(\gamma + \delta x)$  for each  $T$ -cone. This will determine the “internal” coefficients on the edges with  $T$ -cones of height 2, e.g.,  $a_{-1,2} + a_{0,2}x + a_{1,2}x^2 = (\gamma + \delta x)^2$  (for some  $\gamma, \delta$ ) implies that  $a_{0,2} = 2(a_{-1,2}a_{1,2})^{\frac{1}{2}}$ . In the same way,  $a_{2,0} = 2(a_{2,1}a_{2,-1})^{\frac{1}{2}}$  and  $a_{-2,0} = 2(a_{-2,1}a_{-2,-1})^{\frac{1}{2}}$ . To reduce visual clutter, we rename the free parameters on the edges by  $a, b, c, \dots$ , etc.



We now require the polynomial  $i\frac{y}{x^2} + a_{-1,1}\frac{y}{x} + a_{0,1}y + a_{1,1}xy + cx^2y$  along the  $y = 1$  row to be divisible by  $a^{\frac{1}{2}} + b^{\frac{1}{2}}x$ , the polynomial  $a\frac{y^2}{x} + a_{-1,1}\frac{y}{x} + a_{-1,0}\frac{1}{x} + g\frac{1}{xy}$  along the  $x = -1$  line to be divisible by  $h^{\frac{1}{2}} + i^{\frac{1}{2}}x$ , and the polynomial  $bxy^2 + a_{1,1}xy + a_{1,0}x + e\frac{x}{y}$  along the  $x = 1$  line to be divisible by  $c^{\frac{1}{2}} + d^{\frac{1}{2}}x$ . Solving the equations this imposes, we get

- $a_{-1,0} = \frac{h^{\frac{1}{2}}}{i^{\frac{1}{2}}}a_{-1,1} - \frac{ah}{i} + \frac{gi^{\frac{1}{2}}}{h^{\frac{1}{2}}}$ ,
- $a_{0,1} = \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}a_{-1,1} + \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}a_{1,1} - \frac{ac}{b} - \frac{bi}{a}$ ,
- $a_{1,0} = \frac{d^{\frac{1}{2}}}{c^{\frac{1}{2}}}a_{1,1} - \frac{bd}{c} + \frac{c^{\frac{1}{2}}e}{d^{\frac{1}{2}}}$ .

If for simplicity we let  $f$  be *standard* maximally mutable, imposing binomial edge coefficients, we get the following picture (where we set  $a_{-1,1} = p, a_{1,1} = q, a_{0,0} = r$  for readability):



Observe that there are 12 free parameters  $a, b, c, d, e, f, g, h, i, a_{-1,1}, a_{1,1}, a_{0,0}$  in the coefficients; there is one for each vertex, and one for each residual point of  $P$ ; in the standard case there are only free parameters corresponding to the residual points, as predicted by Proposition 3.6.

One can detect mutability of a Laurent polynomial via geometrical properties.

**Proposition 3.18.** *Let  $P$  be a Fano polygon, and let  $f$  be a generic Laurent polynomial with  $\text{Newt}(f) = P$ . Let  $E$  be an edge of  $P$  of height  $r$ , and let  $(\gamma : \delta) \in D_E$  be a point with respect to which  $f$  is mutable (as in Remark 3.13). Then  $f$  has an ordinary multiple point of multiplicity  $r$  at  $(\gamma : \delta)$  on  $D_E$ . In particular, a generic maximally mutable Laurent polynomial has one multiple point of multiplicity  $r_i$  for each primitive  $T$ -cone of  $P$  of height  $r_i$ .*

*Proof.* Recall from Definition 3.12 the conditions for mutability: choose local coordinates  $x, y$  so the edge  $E$  is contained in the hyperplane  $y = r$ , and let  $f_s$  be the polynomial made up of terms of  $f$  corresponding to points at height  $s$  (using the same height function). Then in these coordinates, we can write

$$f_s = (\gamma + \delta x)^s y^s h_s,$$

where  $h_s = h_s(x)$  is some Laurent polynomial in  $x$ . In local coordinates at the point  $x = -\gamma/\delta$ , this becomes

$$f_s = x^{r-s} y^s h'_s,$$

where  $h'_s$  is a rational function of  $x$ . We can now rewrite  $f$  as

$$f = \sum_{i=0}^r a_{r-i,i} x^{r-i} y^i + (\text{terms of degree } > r)$$

for some numbers  $a_{i,j}$ , where at least  $a_{r,0}$  (generically) and  $a_{0,r}$  are nonzero. To see that there are  $r$  distinct branches at the point, notice that the origin of  $P$  is a residual point, and so the constant term of  $f$  is a free parameter; this parameter appears only in one of the coefficients  $a_{r-i,i}$ , namely  $a_{r,0}$ . For generic values of the parameter we fall outside the discriminant of the polynomial  $\sum_{i=0}^r a_{r-i,i} x^{r-i} y^i$ , as the leading coefficient of a polynomial is not a factor of its discriminant. This gives  $r$  distinct factors, i.e.,  $r$  distinct branches through the chosen point. □

*Remark 3.19.* If there are more than one  $T$ -cone on the same edge, and  $f$  is mutable with respect to the same chosen point for each of them, then the obvious generalization— $f$  has a multiple point with higher-order tangency—is proved in a similar way: now instead we have  $f_s = (\gamma + \delta x)^{m_s} y^s h_s$  and so in local coordinates we have  $f = \prod (b_j x^m + c_j y) +$  (higher-order terms).

*Proof of Theorem 3.5.* Recall that the genus  $g(D_P)$  of the desingularization of  $D_P$ , called the *sectional genus* of  $Y_P$ , is equal to the number of internal lattice points of  $P$  [CLS11, 10.5.8]. This is the genus of a generic curve in the complete linear system of curves linearly equivalent to  $D_P$ ; this linear system lifts to  $X$ , so we may consider it as the linear system of curves equivalent to  $D'$ . The curves defined by generic MMLP's lift to  $X$  to form a base point-free linear subspace of this linear system in the obvious way (passing to  $X$  we blow up all the base points), and to find the genus of such a curve, we need to examine how an MMLP  $f$  differs from a generic section of  $D_P$ .

Proposition 3.18 shows that the curve defined by an MMLP has an  $r$ -uple point for every point at which we blow up; these points are generic and we have imposed no other conditions, so there are no other special points that affect the genus.

The effect of an ordinary  $r$ -uple point on the genus of a curve is well known (see, e.g., [GH94, pp. 500–508]): the genus drops by  $\frac{1}{2}r(r - 1)$  for every such point. Thus, the genus of the curve defined by  $f$  is  $g(D_P) - \sum \frac{1}{2}r_i(r_i - 1)$ , where the sum runs over the  $T$ -cones of  $\text{Newt}(f)$  and the  $i$ th cone has lattice height  $r_i$ .

Now observe that  $\frac{1}{2}r(r - 1)$  is exactly the number of internal lattice points in a  $T$ -cone of height  $r$  (this follows directly from Pick's formula [CLS11, Ex. 9.4.4]), so the genus of  $f$  is equal to  $g(D_P) - \sum \frac{1}{2}r_i(r_i - 1) = |\text{int}(P) \cap N| - |\text{int}(P) \cap N \cap (T\text{-cones})|$ , that is, the number of residual points of  $P$ .

To see that this genus  $g$  is mutation-invariant, it is enough to recall that the singularity content of  $P$ , in particular the set of  $R$ -cones, is invariant under mutation (Definition 3.3; see also [AK14]), which of course implies that the number of residual points is preserved. □

*Remark 3.20.* We remark that one gets the same number for the genus even if some of the chosen mutation points on an edge  $D_E$  are allowed to coincide (the standard MMLP's are the extreme case where all the points on each edge are equal), i.e., if  $f$  is mutable with the same factor  $(\gamma + \delta x)$  multiple times (as in Remark 3.19). This is because at such a point, the curve  $\{f = 0\}$  will have an  $r$ -uple point with higher order tangency, so in local coordinates it will factor as  $\prod (b_i x^m + c_i y)$  (plus higher degree terms), similar to Proposition 3.18. Similar to how an ordinary  $r$ -uple point drops the genus by  $\frac{1}{2}r(r - 1)$ , an  $r$ -uple point with tangency of order  $m$  drops the genus by  $m$  for each branch (again, see [GH94, pp. 500–508]), and so the total defect for the MMLP is still  $m \cdot \frac{1}{2}r(r - 1)$ , the same as for  $m$  ordinary  $r$ -uple points.

#### 4. MONODROMY AT $t = 0$

In this section we will prove Theorems 1.2 and 1.3. To compute the monodromy of  $H_1(X_t, \mathbb{Z})$ , we need to find a suitable basis of cycles and a description of the monodromy automorphism. We will do this by explicitly constructing a model for  $X_t$  by means of local calculations, explicitly carrying out the resolution  $X \rightarrow Y_P$ .

Let us recap what we know so far: The general fiber  $X_t \subset X$  is a genus  $g_{mut}$  curve, the special fiber  $X_0$  is equal to the divisor  $D'$ , and there is a retraction of

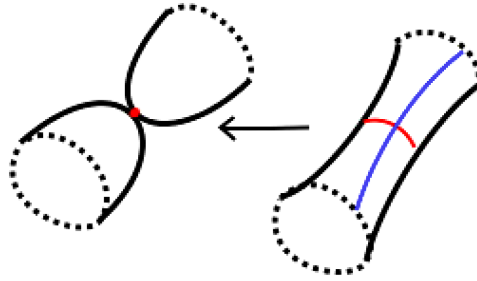


FIGURE 1. Local picture of the degeneration over an intersection between components of  $D_P$ ; the vanishing cycle is indicated in red, and the relative cycle in blue.

a neighbourhood (in  $X$ ) of the special fiber  $X_0$  on to  $X_0$  that induces a surjection  $X_t \rightarrow X_0$ . Topologically this divisor  $D'$  is a necklace of spheres with some chains of spheres branching off at certain points; this will be apparent from the resolution we perform.

Recall from Theorem 3.5 that  $g_{mut}(Y_P)$ , the genus of the general fiber  $X_t$ , is equal to the number of residual points of  $P$ , which always includes the origin as  $P$  by assumption is Fano. A necklace of spheres is a degeneration of a topological surface of genus at least one, which accounts for the contribution to the genus from the lattice point at the origin. By Theorem 3.5 the rest of the genus comes from the internal points of the  $R$ -cones of  $P$ , so the remaining genus must result from resolving the base points on the components of  $D_P$  corresponding to the edges of  $P$ .

We may thus reduce to a series of local considerations, which we will refer to as the *contributions* from the vertices and edges, and  $R$ -cones, respectively. The contribution from the vertices is this: intersection points between the components are degenerations of the form  $\{x^m y^n = t\} \rightarrow \{x^m y^n = 0\}$ , and we must describe which of these occur and what the monodromy does to them (see Figure 1). The contributions from the  $R$ -cones is this: on the components of  $D_P$  corresponding to edges with  $R$ -cones, we must identify what singular points occur and resolve them to get a positive-genus curve  $\tilde{C} \rightarrow \mathbb{P}^1$ ; then find an appropriate automorphism of  $\tilde{C}$  that fixes the inverse images of all the singular points and intersection points with the adjoining components of  $D_P$  (see Figure 2).

To compute the whole monodromy action on  $X_t$ , we will then cut the curve into pieces  $Z$  corresponding to the edges  $E$  of  $P$ , or equivalently summands  $D_E$  of  $D'$  with boundaries  $\partial Z$ , and consider each piece by itself, and then assemble the results afterwards. For each piece  $Z$  the following is true: some of the basis cycles of  $H_1(X_t)$  will exist entirely within these pieces (that is, they are homologous to homology classes in  $H_1(Z, \mathbb{Z})$ ), and these will be cycles that degenerate to a point in the special fiber and are as such called *vanishing cycles*; there is a distinguished cycle which enters and exits the local pieces through the cuts (that is, has homology class in  $H_1(Z, \partial Z, \mathbb{Z})$ ), called a *relative cycle*; and there is a distinguished vanishing cycle homologous to one of the two components<sup>2</sup> of  $\partial Z$ .

We fix some notation: the monodromy restricted to a piece  $Z$  is the *variation diffeomorphism*  $\phi_Z$  (or merely  $\phi$  if  $Z$  is understood), the relative cycle will be

<sup>2</sup>The two components are homologous, so it doesn't matter which one we pick.

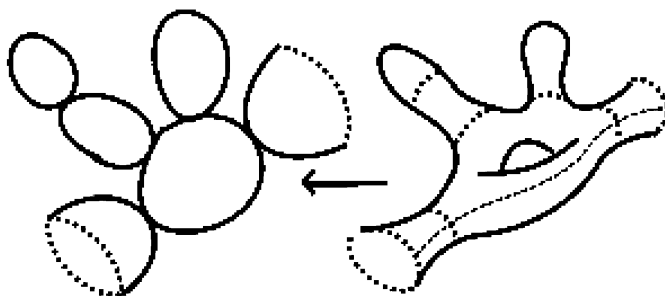


FIGURE 2. The component of  $D'$  corresponding to an  $R$  cone, showing on the left the curve  $D_E$  as the central sphere, exceptional curves of some resolved singular points (these parts make up the piece  $Z$  discussed in the text), and parts of the  $D_{E'}$  for adjoining edges (shown as half-spheres); this is a degeneration of a higher-genus surface (on the right), vanishing and relative cycles indicated. Notice how the monodromy automorphism of  $X_t$  must fix these vanishing cycles to degenerate correctly to the special fiber.

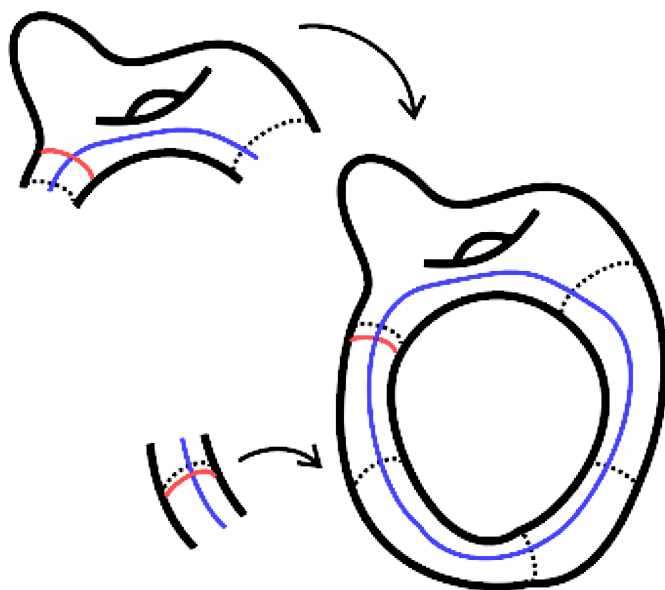


FIGURE 3. The cycle  $\alpha$  marked in blue and the vanishing cycle  $\beta$  marked in red, and their images in the local pieces.

denoted  $\alpha$ , and the distinguished vanishing cycle on the boundary is denoted  $\beta$  (see Figure 3). It is important to remark that  $\phi_Z$  does not fix  $\partial Z$  pointwise, but rotates one component a rational multiple of  $2\pi$ ; this means that the coefficients of  $\beta$  in  $\phi_Z$  may be rational (all other cycles will have integral coefficients).

Observe that when adding up the local pieces, those rational coefficients must add up to something integral, as we know the monodromy of  $H_1(X_t)$  is integral. Exploiting this fact will be crucial in some of the local calculations.

**4.1. The singularities of  $Y_P$  and the intersections between the components.** After we have resolved the singularities of  $Y_P$ , we may look at the monodromy action over the intersections between the components of  $D'$ . Locally at the intersection between two components of this divisor, of multiplicities  $m$  and  $n$ , respectively, in the local coordinates given by the toric chart corresponding to the vertex of intersection, we can write  $\text{supp}(D')$  as  $\{x^m y^n = 0\}$ . In these local coordinates, the global sections  $\mathbf{1}$  and  $f$  become  $x^m y^n$  and  $1 + (\text{higher-order terms})$ , respectively, and we can write  $1 - tf$  as  $x^m y^n - t(1 + (\text{higher-order terms}))$ , locally analytically equivalent to  $x^m y^n - t$ . The degeneration when  $t \rightarrow 0$  is now equivalent to  $\{x^m y^n = t\} \rightarrow \{x^m y^n = 0\}$ . The monodromy action is then locally the monodromy of the curve  $x^m y^n = t$  as  $t$  goes around zero.

**Lemma 4.1.** *Let  $\beta$  be the vanishing cycle of  $x^m y^n = t$  when  $t \rightarrow 0$  (with positive orientation), and let  $\alpha$  be the relative cycle. The monodromy action on  $\alpha, \beta$  in  $x^m y^n = t$  as  $t$  goes around  $t = 0$  in the positive direction is given by  $\beta \mapsto \beta, \alpha \mapsto \alpha - \frac{1}{mn}\beta$ .*

*Proof.* Consider the Riemann surface of  $y = \sqrt[n]{\frac{t}{x^m}}$ , for fixed  $t$ . This is an  $n$ -sheeted covering of the punctured complex plane with a singularity at  $x = 0$ , where as you trace along the surface around the singularity,  $y$  will alternate between approaching  $+\infty$  and  $-\infty$  as  $x$  approaches zero, alternating a total of  $m$  times (see Figure 4 for a picture of what this looks like). Notice the  $m$ -fold rotational symmetry of the surface.

Write  $t = e^{i\theta}$  and  $x = e^{i\tau}$  (we may ignore the magnitude, as only the argument is relevant to the monodromy action); we may now express the surface as

$$y = (tx^{-m})^{\frac{1}{n}} = (e^{i(\theta - m\tau)})^{\frac{1}{n}}.$$

In other words, when  $t$  moves around the origin, the resulting surface satisfies an equation  $y = (x_\theta^{-m})^{\frac{1}{n}}$ , where  $x_\theta = e^{i\tau(\theta)}$ , and the argument satisfies  $-m\tau(\theta) = \theta - m\tau$ . From this,  $\tau(\theta) = -\theta/m + \tau$ , we see that the surface will rotate in the same direction as  $t$ , with  $\frac{1}{m}$ th the speed. Thus, when  $t$  has completed a full revolution, the surface will have rotated by an angle of  $\frac{2\pi}{m}$ , or one step along the  $m$ -fold rotational symmetry.

To find the effect of this on the cycles  $\alpha$  and  $\beta$ , we give an explicit model for each. The vanishing cycle  $\beta$  is homologous to the curve  $\{(e^{i\theta}, e^{-\frac{i\theta}{n}}) | 0 \leq \theta \leq 2n\pi\}$  that winds around the singularity  $n$  times, following the sheets until it meets itself. This curve is preserved under the rotational symmetry of the surface, so the monodromy action on  $\beta$  is the identity. The relative cycle  $\alpha$  can be modelled by a curve going along the topmost sheet of the surface from  $(\varepsilon, \varepsilon^{-\frac{m}{n}})$  to  $(K, K^{-\frac{m}{n}})$ , where  $\varepsilon \ll 1$  and  $K \gg 1$  are real numbers (note the orientation). The monodromy action can be modelled by pinning the initial point  $(\varepsilon, \varepsilon^{-\frac{m}{n}})$  in place (i.e., letting it rotate along with the surface) while holding the other fixed over  $x = K$ . After the monodromy action, the initial point has been moved to  $(\varepsilon \cdot e^{2\pi i/m}, \varepsilon^{-\frac{m}{n}} e^{2\pi i/m})$ , while the final point, fixed to lie over  $x = K$ , will be on the sheet immediately below the topmost one. The resulting curve is homologous to  $\alpha - \frac{1}{mn}\beta$ , as the  $m$ -fold rotational symmetry moves a point  $\frac{1}{mn}$ th of the length of  $\beta$ .  $\square$

We now find the pullback of  $\widehat{D}$  when resolving the singularities of  $Z_P$ ; to get  $D'$ , this will give us all the points on  $X_0$  that locally are of the form  $x^m y^n = 0$ , and

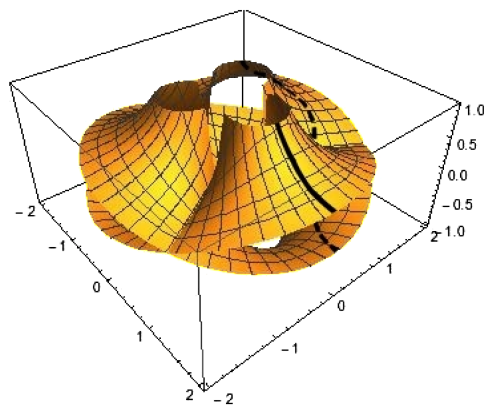


FIGURE 4. The Riemann surface of  $y = \operatorname{Re}(\sqrt{\frac{1}{x^3}})$ ; the solid curve is the relative cycle  $\alpha$ , and the vanishing cycle  $\beta$  can be identified with the outer boundary of the displayed surface. The dashed curve is the cycle  $\alpha - \frac{1}{6}\beta$ .

now section 4.1 tells us what the local monodromy action is. Notice that we can combine the local actions without a problem, as the vanishing cycles  $\beta$  appearing in all of them are homologous, so the local actions commute. Also note that as  $Z_P \rightarrow Y_P$  is a blow-up of smooth points of  $Y_P$ , and we are now looking at the singular points of  $Z_P/Y_P$ , for this purpose  $\hat{D}$  (on  $Z_P$ ) and  $D_P$  (on  $Y_P$ ) can be thought of as interchangeable.

Recall that  $D_P = \sum h_i D_i$ , where  $h_i$  are the lattice heights of the edges  $E_i$  of  $P$  corresponding to the divisors  $D_i$ . The resolved divisor can be written as  $D' = D_P + \sum m_j F_j$ , where  $F_j$  are some exceptional curves and  $m_j$  are their multiplicities. An interesting fact is that the numbers  $m_j$  are such that it makes sense to think of the  $F_j$  as corresponding to “edges” of  $P$  of width zero and height  $m_j$ ; we will however not need this.

Suppose now that  $v$  is a vertex of  $P$ , corresponding to a cone in the normal fan where  $Y_P$  has a singularity of type  $\frac{1}{r}(1, a - 1)$ , and that  $v$  is joining edges  $E$  and  $E'$ , of heights  $h$  and  $h'$ . The singularity is resolved according to [CLS11, Chapter 10]; recall in particular the notion of Hirzebruch–Jung continued fractions, denoted as follows:

$$[b_1, b_2, \dots, b_k] = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_k}}}.$$

We introduce some notation: suppose the Hirzebruch–Jung continued fraction expansion of  $r/(a - 1)$  is  $[b_1, \dots, b_k]$ ; let  $s_1 = t_k = 1$ , and define positive integers  $s_i, t_i$  by

$$\begin{aligned} s_i/s_{i-1} &:= [b_{i-1}, \dots, b_1], & 2 \leq i \leq k, \\ t_i/t_{i+1} &:= [b_{i+1}, \dots, b_k], & 1 \leq i \leq k - 1. \end{aligned}$$

Note that we may extend this to letting  $s_0 = t_{k+1} = 0$  and  $s_{k+1} = t_0 = r$ .

When resolving the singularity at  $v$ , we get  $k$  exceptional curves  $F_1, \dots, F_k$ , with self-intersections  $F_i^2 = -b_i$ . Let  $m_i$  denote the multiplicity of  $E_i$  in  $D$ ; these multiplicities are determined by the criterion that  $E_i \cdot D = 0$ .

**Lemma 4.2.**

- (1)  $s_{i+1} + s_{i-1} = b_i s_i$  and  $t_{i+1} + t_{i-1} = b_i t_i$ .
- (2)  $m_i = \frac{1}{r}(t_i m_0 + s_i m_{k+1})$ .
- (3)  $m_0 = s_{i+1} m_i - s_i m_{i+1}$ .

*Proof.*

(1) By definition,  $s_{i+1}/s_i = [b_i, \dots, b_1] = b_i - 1/[b_{i-1}, \dots, b_1] = b_i - s_{i-1}/s_i$ , and it follows that  $s_{i+1} + s_{i-1} = b_i s_i$ . A similar rearrangement shows the other identity.

(2) Recall that the  $m_i$  are defined by the system of equations  $E_i \cdot D = 0$ . As the only components of  $D$  that are involved are  $D_F, D_{F'}$ , and the  $E_i$ 's, and the intersection numbers are 1 for adjacent components and 0 for nonadjacent components, we get equations  $m_{i-1} - b_i m_i + m_{i+1} = 0$  (for  $1 \leq i \leq k$ ). Successive elimination, applying item (1) at each step, now yields the desired conclusion.

(3) We show this by induction. The base case is the equation  $m_{i-1} - b_i m_i + m_{i+1} = 0$  for  $i = 1$ , using that  $s_1 = 1$  and  $s_2 = b_1$ . The induction step is to show that  $s_{i+1} m_i - s_i m_{i+1} = s_{i+2} m_{i+1} - s_{i+1} m_{i+2}$ ; rearranging we have  $s_{i+1} m_i + s_{i+1} m_{i+2} = s_{i+2} m_{i+1} + s_i m_{i+1}$ , and applying the identity  $s_{i+2} + s_i = b_{i+1} s_{i+1}$  on the right-hand side and the equation  $m_i + m_{i+2} = b_{i+1} m_{i+1}$  on the left-hand side, we see that both sides equal  $b_{i+1} s_{i+1} m_{i+1}$ . □

**Lemma 4.3.**  $\sum_{i=0}^k \frac{1}{m_i m_{i+1}} = \frac{r}{m_0 m_{k+1}}$ .

*Proof.* Observe first that  $\frac{1}{m_0 m_1} + \frac{1}{m_1 m_2} = \frac{1}{m_1} \frac{m_2 + m_0}{m_0 m_2}$ , and by Lemma 4.2(3)  $m_2 + m_0 = s_1 m_2 + m_0 = s_2 m_1$ , so we get  $\frac{1}{m_0 m_1} + \frac{1}{m_1 m_2} = \frac{s_2}{m_0 m_2}$ . In a similar fashion we see that  $\frac{s_i}{m_0 m_i} + \frac{1}{m_i m_{i+1}} = \frac{1}{m_i} \frac{s_i m_{i+1} + m_0}{m_0 m_{i+1}} = \frac{s_{i+1}}{m_0 m_{i+1}}$ , so by induction we have  $\sum_{i=0}^k \frac{1}{m_i m_{i+1}} = \frac{s_{k+1}}{m_0 m_{k+1}} = \frac{r}{m_0 m_{k+1}}$ . □

**Proposition 4.4.** *The contribution to the global monodromy of the cycles  $\alpha, \beta \in H_1(X_t)$  from the vertices of  $P$  is  $\alpha \mapsto \alpha - (K_{\Pi}^2)\beta$  and  $\beta \mapsto \beta$ , where  $\Pi$  is the toric variety defined by the spanning fan of  $P$  and  $K_{\Pi}$  is its canonical divisor.*

*Proof.* Combining Lemmas 4.1, 4.2, and 4.3 tells us that the contribution from a vertex of  $P$  is  $\alpha \mapsto \alpha - \frac{r}{mn}\beta$ , where  $m, n$  are the lattice heights of the adjoining edges and the singularity of  $Y_P$  in the corresponding chart is of type  $\frac{1}{r}(1, a - 1)$  (or if  $Y_P$  is smooth here, take  $r = 1$ ).

It is well known that  $K_{\Pi}^2$  is equal to the lattice volume of the dual polytope  $P^\circ \subset M_{\mathbb{R}}$  of  $P$  (see [CLS11, 13.4.1]). To show the claim, it is enough to show that the volume of the cone  $C_v$  in  $P^\circ$  corresponding to the vertex  $v$  of  $P$  is equal to  $\frac{r}{mn}$ . Let  $u, w$  be primitive lattice generators of  $C_v$ . By Definition 3.7,  $C_v$  is of type  $\frac{1}{r}(1, a - 1)$  when  $\{\frac{1}{r}u + \frac{a-1}{r}w, w\}$  is a lattice basis for  $M$ . As  $\{\frac{1}{r}u + \frac{a-1}{r}w, w\}$  is a lattice basis, we have  $\det(\frac{1}{r}u + \frac{a-1}{r}w, w) = 1$ , and it follows that  $\det(u, v) = r$ . Observing now that  $C_v$  is spanned by  $\frac{1}{m}u$  and  $\frac{1}{n}w$ , we are done as  $\det(\frac{1}{m}u, \frac{1}{n}w) = \frac{r}{mn}$ . □

**4.2. Monodromy over an  $R$ -cone.** It remains to see what happens over the  $R$ -cones, so assume we have an edge  $E$  of  $P$  supporting a single cone, of height  $r$  and width  $w$ , and denote by  $X_E$  the inverse image under the map  $X_t \rightarrow X_0$  of the pullback of  $D_E$  under the resolution  $X \rightarrow Z_P$ . The strict transform of  $D_E$  is a  $\mathbb{P}^1$ , and it intersects the adjacent components of the pullback of  $\widehat{D}$ , as well as the exceptional curves coming from resolving the indeterminacy points of  $f$  on  $D_E$ . The inverse image  $X_E$  is then the part of  $X_t$  bounded by the vanishing cycles over



these points of intersection. From this and Theorem 3.5 we see that topologically,  $X_E$  is a surface with genus equal to  $\frac{1}{2}(r - 1)w$ , the number of internal points in the  $R$ -cone (this follows directly from Pick’s formula), with two punctures, one for each of the intersection points.

As before, we may in suitable coordinates write  $f = \sum f_s$ , where  $f_s$  are the terms “at height  $s$ ”, corresponding to lattice points in  $\text{Newt}(f)$  at height  $s$  relative to the height function given by the normal vector of the edge  $E$ . In particular,  $f_r = y^r \prod_{i=1}^w (x - \eta_i)$  (up to some monomial in  $x$  which is not important). In the local coordinate chart of  $Y_P$  corresponding to a vertex of  $E$ , we can write  $f$  as

$$f = \prod (x - \eta_i) + y h_1(x) + y^2 h_2(x) + \dots,$$

where  $h_i(x)$  are some polynomials in  $x$ . In the same coordinate chart,  $\mathbf{1}$  becomes  $x^e y^r$  (where  $e$  is some positive integer; it is not important which). At each of the points  $(\eta_i, 0)$  on  $D_E$ , it is easy to see (do a coordinate change  $x \mapsto x + \eta_i$ ) that  $f$  becomes  $x + y +$  (terms of higher degree), so locally at these points we have  $f \sim x + y$  analytically; and that  $\mathbf{1}$  becomes  $y^r (x + \eta_i)^e \sim y^r$ . We see that the graph  $\Gamma_\tau$ , which is given by  $\mathbf{1} - tf$ , is at the corresponding points analytically equivalent to  $y^r - t(x + y)$ . In other words, on  $D_E \subset Y_P \subset \Gamma_\tau$  there are  $w$  special points where the graph  $\Gamma_\tau$  is locally equivalent to  $y^r - t(x + y)$ , a singularity of type  $A_{r-1}$ .

Thus, we have on  $D_E$  the  $w$  singular points of  $f$ , each of type  $A_{r-1}$ , and the two points of intersection with the adjacent components of  $D_P$ . As  $D_E$  has multiplicity  $r$ , we can now model our  $X_E$  as a ramified degree  $r$  cover of  $\mathbb{P}^1$ , with two ramification points of ramification index  $r$  (corresponding to the intersection points with the adjacent divisors), and  $w$  ramification points corresponding to the singular points of  $f$ . More precisely,  $X_E$  is homotopic to such a surface, punctured at the two ramification points over the intersections with the adjacent components. The ramification index  $e_p$  of the remaining  $w$  points is found by the Riemann–Hurwitz formula: setting  $g = \frac{1}{2}(r - 1)w$  in

$$2g - 2 = -2r + 2(r - 1) + w(e_p - 1)$$

gives  $e_p = r$ , so we have a degree  $r$  map, ramified at  $w + 2$  points of ramification index  $r$ .

The local monodromy action on  $H_1(X_E)$  must then be induced by an automorphism of  $X_E$  with  $w + 2$  fixed points, near which the automorphism has order  $r$  (to be compatible with the ramification index); this implies the automorphism has order  $r$  everywhere (a priori it has order a factor of  $r$ ). We have shown the following.

**Lemma 4.5.** *Let  $E$  be a edge of  $P$  with an  $R$ -cone of height  $r$  and width  $w$ . Then the local monodromy action on  $H^1(X_E)$ , where  $X_E$  is as above, is given by an order  $r$  automorphism of a genus  $\frac{1}{2}(r - 1)w$  surface with  $w + 2$  fixpoints.*

From the above discussion and fixing some value for  $t$  in a local expression for  $\mathbf{1} - tf$ , it is clear which surface to use as a model for  $X_E$ : the Riemann surface of the function  $\sqrt[r]{(x - x_1) \cdots (x - x_{w+2})}$ , for some values  $x_1 \cdots x_{w+2}$ . We will now construct a topological model of this surface that suits our purposes, and with it an automorphism with the desired properties.

Denote the  $w + 2$  fixpoints by  $p_1, p_2, q_1, \dots, q_w$ . The points labelled  $p_1, p_2$  correspond to the punctures described above, while the  $q_i$ ’s are the remaining fixpoints. Now take  $r$  sheets<sup>3</sup>  $S_1, \dots, S_r$  marked with  $w + 2$  points with the appropriate labels,

<sup>3</sup>That is, copies of  $\mathbb{P}^1$ , as usual.

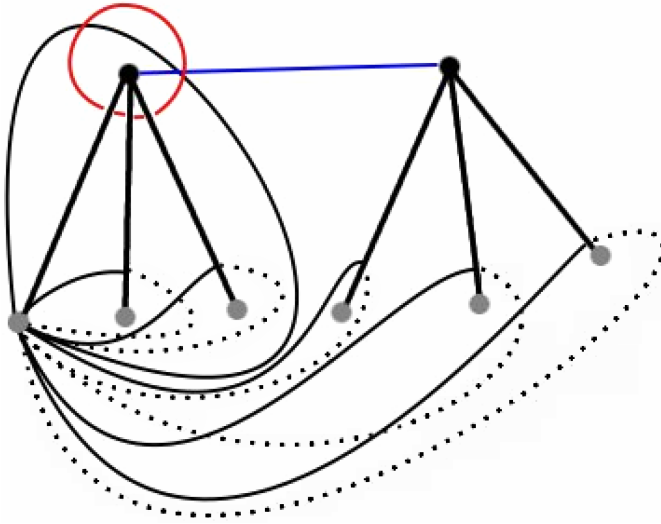


FIGURE 5. The first sheet in the described arrangement of cuts and cycles in the case  $w = 6$  with  $r$  coprime to 3. The black points are the points  $p_1, p_2$  and the grey points are  $q_1, \dots, q_6$ ; cuts are indicated by black lines. The blue cycle is  $\alpha$ ; the red cycle is the parts of  $\beta$  that are on this sheet; the remaining cycles  $a_i^j$  for  $i = 1, \dots, 6$  are indicated, with solid curves being the part on  $S_1$  and dashed curves the part on  $S_2$ .

and make cuts from each  $q_i$  to one of the  $p_j$ 's in such a way that the number of  $q_i$ 's joined to each  $p_j$  is coprime to  $h$  (this is of course always possible). Next glue the sheets along these cuts such that when viewed from a  $p_j$ , going in the positive direction across any cut takes you one step "down" in the stack of sheets, from  $S_i$  to  $S_{i+1}$  (and from  $S_r$  to  $S_1$ ). The readers can easily verify for themselves that the result is a surface of genus  $g = \frac{1}{2}(r-1)w$ .

Now we choose a basis of homology cycles; we need  $2g$  cycles, plus the relative and boundary cycles  $\alpha$  and  $\beta$ . For  $\alpha$ , choose a path on  $S_1$  from  $p_1$  to  $p_2$ . For  $\beta$ , choose a path passing around  $p_1$  until it returns to the origin; the parts of this cycle on each sheet will form a circle around the point  $p_1$ . We will permit homology deformations of cycles to cross the points marked  $q_i$  (where this makes sense), but not the points  $p_1, p_2$  (as these represent the boundary of  $X_E$ ); with this in mind the reader can easily verify that the similar path around  $p_2$  with the opposite orientation is homologous to the one we have chosen. For the remaining  $2g$  cycles, choose paths as follows: on each sheet  $S_i$  denote by  $a_i^1$  the path from  $q_1$  around the assembly of cuts connected to  $q_1$ ; denote by  $a_i^j$  the path from  $q_1$  crossing the cut connected to  $q_j$  and returning to  $q_1$  on the next sheet  $S_{i+1}$ . This totals  $rw$  cycles (of which we need only  $(r-1)w$  to form a basis); however, as the readers may verify for themselves, the cycle  $\sum_{i=1}^r a_i^1$  is homologous to  $\beta$ , while the cycles  $\sum_{i=1}^r a_i^j$  (for  $j \neq 1$ ) are homologous to the trivial cycle. Thus, we can choose the cycles  $a_i^j$  with  $i = 1, \dots, r-1$  as a basis. We refer to Figure 5 for a visual aid.

With the model of the surface in hand, it is easy to describe an automorphism with the desired order and number of fixpoints: let  $\phi_{X_E}$  take any point of the surface

to the point immediately below it on the next sheet. Clearly this has order  $r$ , as there are  $r$  sheets, and the fixpoints are exactly the  $w + 2$  marked points.

What remains is to write this action in terms of the basis cycles. Clearly we have  $\beta \mapsto \beta$  and  $a_i^j \mapsto a_{i+1}^j$ , and as

$$\sum_{i=1}^r a_i^j = \begin{cases} \beta & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have in terms of our chosen basis that  $a_{r-1}^1 \mapsto \beta - \sum_{i=1}^{r-1} a_i^1$  and for  $j \neq 1$  that  $a_{r-1}^j \mapsto -\sum_{i=1}^{r-1} a_i^j$ . The interesting thing is what happens to  $\alpha$ .

The relative cycle  $\alpha$ , which goes from  $p_1$  to  $p_2$  on the first sheet  $S_1$ , is sent to the similar path on  $S_2$ . We can join these paths by attaching a curve homologous to  $\frac{1}{r}\beta$  at each end (see Figure 6), and we get a closed cycle  $\phi(\alpha) - \alpha + \frac{2}{r}\beta$ . As one can easily verify (see Figure 7), this cycle is homologous to  $a_1^1 + a_2^1 - a_1^j$  (for some  $j$ , as indicated), so rearranging we have

$$\phi(\alpha) = \alpha - \frac{2}{r}\beta + a_1^1 + a_2^1 - a_1^j.$$

We may of course relabel the points  $q_i$  such that  $j = 2$  in this formula. The special case of  $w = 1$  is slightly different; as one can see in Figure 8, here we instead have  $\phi(\alpha) = \alpha - \frac{2}{r}\beta + a_1^1 + a_2^1$  (which makes sense as there is no cycle  $a_1^2$  when  $w = 1$ ).

We have shown the following.

**Proposition 4.6.** *On a Riemann surface of genus  $g = \frac{1}{2}(r-1)w$  with two punctures and  $w$  other marked points, there exists an order  $r$  automorphism  $\phi$  fixing these points and punctures and a basis of homology cycles  $\{\alpha, \beta, a_1^1, \dots, a_{r-1}^w\}$  such that*

- $\phi(\alpha) = \begin{cases} \alpha - \frac{2}{r}\beta + a_1^1 + a_2^1 - a_1^2 & \text{if } w \neq 1 \\ \alpha - \frac{2}{r}\beta + a_1^1 + a_2^1 & \text{if } w = 1, \end{cases}$
- $\phi(\beta) = \beta,$
- $\phi(a_i^j) = a_{i+1}^j$  for  $i < r - 1$ , and
- $\phi(a_{r-1}^j) = \begin{cases} \beta - \sum_{i=1}^{r-1} a_i^1 & \text{if } j = 1, \\ -\sum_{i=1}^{r-1} a_i^1 & \text{if } j \neq 1. \end{cases}$

Any power of  $\phi$  also has the same order and fixpoints, so it remains to find which power is the right one; the reader may be pleasantly surprised by the answer.

Recall from Proposition 4.4 that the total monodromy from the vertices of  $P$  is equal to the degree  $K_{\Pi}^2$  of the toric variety defined by the spanning fan of  $P$ . We may reformulate this to a count of contributions from the edges by using a result of Akhtar and Kasprzyk ([AK14, Prop. 3.3]). Recall also from section 4.1, for a singularity  $\sigma$  of type  $\frac{1}{r}(1, a - 1)$ , the numbers  $b_1, \dots, b_{k_\sigma}$  making up the Hirzebruch–Jung continued fraction expansion of  $r/(a - 1)$ , and the numbers  $s_i, t_i$  ( $1 \leq i \leq k_\sigma$ ) defined in terms of the  $b_i$ . Also let  $d_i = (s_i + t_i)/r - 1$ , and let  $A(\sigma) = k_\sigma + 1 - \sum_{i=1}^{k_\sigma} d_i^2 b_i + 2 \sum_{i=1}^{k_\sigma-1} d_i d_{i+1}$ . Note that if  $\sigma$  is a primitive  $T$ -cone,  $A(\sigma) = 1$ .

**Proposition 4.7** (Akhtar-Kasprzyk, [AK14]). *Let  $\Pi$  be a complete toric surface with singularity content  $(n, \mathcal{B})$ . Then*

$$K_{\Pi}^2 = 12 - n - \sum_{\sigma \in \mathcal{B}} A(\sigma).$$

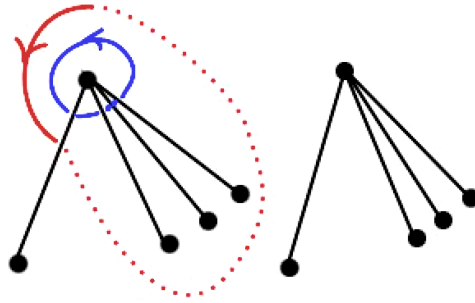


FIGURE 6. The cycles made by joining together one copy of the depicted curves for each of the  $r$  sheets of  $X_E$  (as before, a dotted line indicates “on the sheet below”) are both homologous to the boundary cycle  $\beta$ . The depicted curves can then be said to represent  $(1/r)\beta$ .

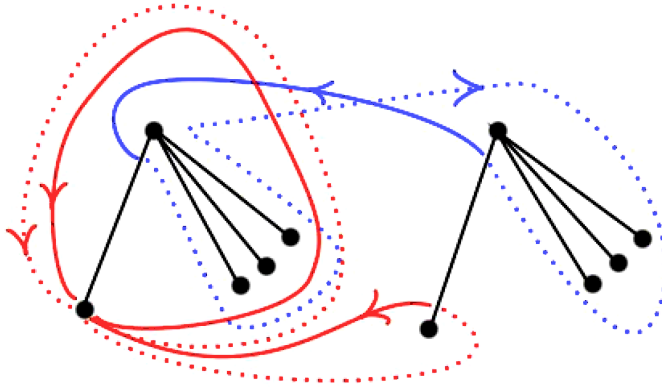


FIGURE 7. The blue cycle is  $\phi(\alpha) - \alpha + \frac{2}{r}\beta$ ; the red cycle is  $a_1^1 + a_2^1 - a_1^j$  (here  $j = 5$ ). The blue and red cycles are homologous.

This and Proposition 4.4 together give that for each  $R$ -cone of type  $\sigma$ , the contribution to the total monodromy of the relative cycle  $\alpha$  from the vertices is  $\alpha \mapsto \alpha + (n - 12 + A(\sigma))\beta$ . The number  $A(\sigma)$  is in general not an integer, and so the action on  $\alpha$  in the local monodromy of the corresponding  $H_1(X_E)$  must be of the form  $\alpha \mapsto (\text{something}) + B(\sigma)\beta$ , where  $A(\sigma) + B(\sigma)$  is an integer.

**Lemma 4.8.** *Let  $\sigma$  be of type  $\frac{1}{r}(1, a - 1)$ , and let the numbers  $b_i, s_i, t_i$ , and  $d_i$  be as above. Then  $A(\sigma) = k_\sigma + 1 - \sum_{i=1}^{k_\sigma} (b_i - 2) - \frac{2a}{r}$ .*

*Proof.* From the recurrence relations  $b_i s_i = s_{i-1} + s_{i+1}$  and  $b_i t_i = t_{i-1} + t_{i+1}$  (from subsection 4.2) follows the relation  $d_{i+1} + d_{i-1} - b_i d_i = b_i - 2$ . This enables us to rewrite the definition for  $A(\sigma)$  as

$$A(\sigma) = k_\sigma + 1 + \sum_{i=1}^{k_\sigma} d_i (b_i - 2)$$

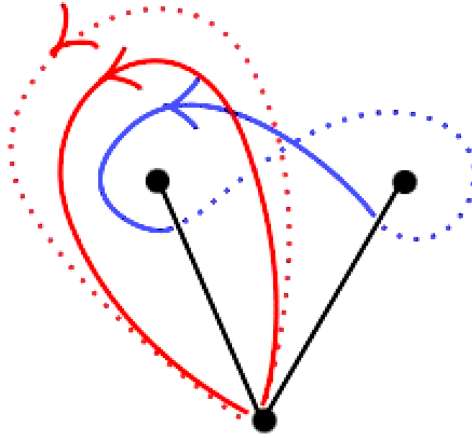


FIGURE 8. The monodromy of the relative cycle  $\alpha$  over an  $R$ -cone of width 1. The blue cycle is  $\phi(\alpha) - \alpha + \frac{2}{r}\beta$ ; the red cycle is  $a_1^1 + a_2^1$ . The blue and red cycles are homologous.

as the sums involving  $d_i$ 's telescope using the recurrence relation. Now  $d_i(b_i - 2) = \frac{1}{r}(b_i(s_i + t_i) - 2(s_i + t_i) - (b_i - 2)r)$ , and summing over  $i$  now cancels all the terms where  $s_i, t_i$  occur (using the recurrences for  $s_i, t_i$ ), except for the first and last, where there remains  $-s_1 - t_1 - s_k - t_k$ . We have  $s_1 = t_k = 1$  by definition, and from  $t_0 = s_{k+1} = r$  it follows from the recurrences that  $s_k = t_1 = a - 1$ .  $\square$

By Proposition 4.6 the number  $B(\sigma)$  is some multiple of  $-\frac{2}{r}$ , and the fractional part of  $A(\sigma)$  is  $-\frac{2a}{r}$ , so to determine the correct power  $p$  of  $\phi$  one solves the congruence  $-2p - 2a \equiv_r 0$ ; as  $\phi$  has order  $r$  this uniquely determines the correct monodromy automorphism. The result below follows immediately; we use the fact that a cone of type  $\frac{1}{r}(1, a - 1)$  has height  $r/\gcd(r, a)$  and width  $\gcd(r, a)$  ([AK14][2.2]).

**Lemma 4.9.** *Let  $\sigma$  be an  $R$ -cone of type  $\frac{1}{r}(1, a - 1)$ , and let  $\phi_{h,w}$  be the automorphism defined in Proposition 4.6 for an  $R$ -cone of width  $w = \gcd(r, a)$  and height  $h = r/\gcd(r, a)$ . Then the contribution of  $\sigma$  to the total monodromy is  $\psi_{\frac{1}{r}(1,a-1)} := \phi_{h,w}^{-a}$ .*

We are now in a position to prove the main theorems.

*Proof of Theorem 1.1.* This follows from Propositions 4.4 and 4.6 and Lemma 4.9; all that remains is to see that the local automorphisms commute. The only cycles appearing in more than one of these maps are  $\alpha$  and  $\beta$ , so it is enough to check that the restrictions to these cycles commute. This is immediate, however, as they are all shear maps of the form  $(\alpha, \beta) \mapsto (\alpha + q\beta, \beta)$  (where  $q$  is some rational number).  $\square$

*Proof of Theorem 1.3.* From Proposition 4.6 we have that for  $R$ -cones of type  $\frac{1}{3}(1, 1)$  (which is a cone of height 3 and width 1) the variation diffeomorphism on the piece  $X_E$  is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

or its inverse. From Lemma 4.9 we have that the correct power is congruent to  $-2$  modulo 3 (here  $a = 2$ ), so it is 1; in other words the above matrix is the right one. Putting this together with the vertex contributions that each  $\frac{1}{3}(1, 1)$ -cone gives  $\alpha \mapsto \alpha + A(\sigma)\beta$ , and that  $A(\sigma) = \frac{5}{3}$ , we have that each  $R$ -cone of type  $\frac{1}{3}(1, 1)$  contributes one to the coefficient of  $\beta$  in the image of  $\alpha$ . Combining this with the rest of Propositions 4.4 and 4.7, we get the desired result.  $\square$

*Proof of Theorem 1.2.* It is clear by Propositions 4.4, 4.7 and 4.6 that the singularity content determines the monodromy.

Suppose now that the singularity content of  $P$  is  $(k, \mathcal{B})$  and that we are given the monodromy matrix in the bases we have described. By Theorem 1.1 this matrix is of the form

$$\begin{pmatrix} 1 & 0 & & \mathbf{0} \\ B & 1 & & \mathbf{r} \\ & & \mathbf{M}_1 & \\ \mathbf{c} & \mathbf{0} & & \mathbf{M}_2 \\ & & & \ddots \\ & & & & \mathbf{M}_n \end{pmatrix}.$$

Here  $\mathbf{r}$  and  $\mathbf{c}$  are some vectors and  $\mathbf{M}_i$  are block matrices of size  $2g_i \times 2g_i$ , where  $g_i$  is the genus of the local piece  $X_{E_i}$ , and  $B = k - 12 + \sum_{\sigma \in \mathcal{B}} m(\sigma)$ , where  $m(\sigma) = A(\sigma) - \frac{2a}{r}$  (for  $\sigma = \frac{1}{r}(1, a - 1)$ ).

Each block  $\mathbf{M}_i$  is associated to an  $R$ -cone of type  $\frac{1}{r_i}(1, a_i - 1)$ . The sizes  $2g_i$  of the blocks  $\mathbf{M}_i$  give us the  $r$  in  $\frac{1}{r}(1, a - 1)$ , as  $2g_i = w_i(h_i - 1)$  ( $h_i$  and  $w_i$  are the height and width of the  $R$ -cone, respectively), and necessarily the matrix  $\mathbf{M}_i$  has order  $h_i$ , so we can solve for  $w_i$  and get  $r_i = h_i w_i$ ; then  $a_i$  can be deduced from the local automorphism  $\phi_{E_i}$  of Proposition 4.6 corresponding to that height and width, by taking powers until you get  $\mathbf{M}_i$  as a block in the lower right corner;  $a_i$  is then minus that power (as in Lemma 4.9).

Now having identified the elements of the singularity basket  $\mathcal{B}$ , we deduce the number of  $T$ -cones as  $k = 12 + B - \sum_{\sigma \in \mathcal{B}} m(\sigma)$ .  $\square$

*Remark 4.10.* We remarked after the definition of singularity content (Definition 3.3) that the singularity basket should be viewed as a cyclically ordered list, but we are only able to see it in the monodromy as a multiset. That we cannot recover the cyclical order of the singularity basket from the monodromy follows from the easily verified fact that the local monodromy automorphisms over the  $R$ -cones commute; also we may reorder the blocks  $\mathbf{M}_i$  as desired by reordering the basis.

**Corollary 4.11.** *With the assumptions of Theorem 1.2, suppose the singularity basket contains  $n_i$   $R$ -cones of height  $r_i$ . Then the monodromy of  $D \cdot \pi_f$  at zero has eigenvalues 1 with multiplicity 2, and each  $r_i$ th root of unity (other than 1) with multiplicity  $n_i$ .*

### 5. RAMIFICATION AND THE PICARD–FUCHS OPERATOR

The singularity content does not completely classify Fano polygons up to mutation, as there are nonequivalent polygons with the same singularity content. In this section we note that the *ramification* of the MMLPs associated to a polygon is a mutation invariant quantity, and present some evidence that these two invariants together might give a complete classification.

**Example 5.1.** These Fano polygons both have singularity content  $(5, \{1 \times \frac{1}{3}(1, 1)\})$ .



However, they are not mutation-equivalent, as one can detect by the period sequences of their standard MMLPs: the first has period

$$\pi_{f_1}(t) = 1 + 8t^2 + 6at^3 + 168t^4 + 240at^5 + \dots,$$

while the second has

$$\pi_{f_2}(t) = 1 + 2(1 + a)t^2 + 18t^3 + 6(7 + 4a + a^2)t^4 + 20(10 + 9a)t^5 + \dots.$$

Notice that the number of vertices is 3 and 4; there is no way to mutate the second polygon into a triangle.

Let  $Ann_D(\pi_f)$  be the annihilator ideal of  $D \cdot \pi_f$  in  $D = \mathbb{C}\langle t, \nabla \rangle$  (where  $\nabla = t \frac{d}{dt}$ ), that is, the ideal such that  $D \cdot \pi_f \simeq D/Ann_D(\pi_f)$ . Elements  $\theta$  of  $D$  can be written in a standard normal form  $\theta = \sum_i^n p_i(t)\nabla^i$ ; we say that the *order* of  $\theta$  is  $n$ , the highest occurring power of  $\nabla$  in this expression, and the *degree* of  $\theta$  is the highest degree among the polynomials  $p_i(t)$ . We now define the *Picard–Fuchs operator* of  $\pi_f$  to be the minimal generator of  $Ann_D(\pi_f)$ , that is, the generator with lowest degree of  $p_n(t)$  among those with lowest order. We denote the Picard–Fuchs operator by  $L_f$ . In nice cases (and there is numerical data to suggest this is the generic behaviour),  $L_f$  captures the monodromy of  $D \cdot \pi_f$ .

**Proposition 5.2.** *Let  $f$  be a Laurent polynomial with  $Newt(f) = P$ , and let  $t \in \mathbb{P}^1$  be a noncritical value of  $\tau$ , which is outside the singular set of  $L_f$ . Then if the order of the Picard–Fuchs operator  $L_f$  is 2 times the genus of  $X_t$ ,  $Sol(D \cdot \pi_f)$  is locally isomorphic to  $Sol(D/L_f)$  away from the critical locus of  $L_f$  (which contains the singular points of  $\tau$ ).*

*Proof.* As  $L_f \in Ann(\pi_f)$  there is a surjection  $D/\langle L_f \rangle \twoheadrightarrow D \cdot \pi_f$ , and correspondingly an injection  $Sol(D/L_f) \hookrightarrow Sol(D \cdot \pi_f)$ .

It follows from the Cauchy–Kovalevski theorem ([Hör90, 9.4.5]) that the order of  $L_f$  is equal to the rank of its solution space; and it is a well-known fact that  $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$  if  $X$  is a compact Riemann surface of genus  $g$ , so  $Sol(D \cdot \pi_f)$  has rank  $2g(X_t)$  by Theorem 3.5. Thus, if  $L_f$  has order  $2g$ ,  $Sol(D/L_f)$  has rank  $2g$ , and so the cokernel of  $Sol(D/L_f) \hookrightarrow Sol(D \cdot \pi_f)$  has finite support. In particular, in a small punctured disk around any singular point of  $D \cdot \pi_f$  the two local systems will be equal.  $\square$

It is in general very expensive to compute  $L_f$  and  $\pi_f$ , even with the latest technology (see [Lai14]). However, in the cases where we have computed the Picard–Fuchs operators of (generic) MMLPs (e.g., for all the 10 smoothable Fano polygons and the 26 classes with singularity content  $(k, \{n \times \frac{1}{3}(1, 1)\})$ ), one indeed gets the order  $2 \cdot g_{mut}(Y_P)$  and  $L_f$  captures the monodromy at zero. It is not an unreasonable conjecture that this is true in general.

Recall from the Introduction the *ramification* of a local system. The ramification of  $Sol(D \cdot \pi_f)$  is invariant under mutation (as  $f$  is). This quantity appears linked

to the degree of  $L_f$  and the shape of  $P = \text{Newt}(f)$  in the following way. Suppose  $P$  has singularity content  $(k, \{n \times \frac{1}{3}(1, 1)\})$  ( $n$  may be zero, which is the smooth case), let  $f$  be a standard MMLP with  $\text{Newt}(f) = P$ , and let  $L_f$  be the associated Picard–Fuchs operator. There are 10 smooth such  $P$ 's and 26 with  $\frac{1}{3}(1, 1)$  cones, and all 36 of these satisfy the following:

- (1) the degree of  $L_f$  is equal to  $n^2 + 5n + 3 + 2(n + 1) \text{rf}(L_f)$ ,
- (2) the ramification index  $\text{rf}(L_f)$  is equal to  $n + k_{\text{eff}} - 3$ , where  $k_{\text{eff}}$  is the number of multiple points on the curve  $f = 0$ , and
- (3) when  $f$  is a *standard* MMLP, the number  $k_{\text{eff}}$  is equal to the smallest number of edges among the polygons mutation-equivalent to  $P$ .

The number  $k_{\text{eff}}$  here is equal to  $k$  for generic MMLP's, and it drops by one whenever two  $T$ -cones (that can be mutated to be) on the same edge of  $P$  have the same associated factor  $(a + bx)$  in  $f$ . Thus the minimal case is the standard MMLP case where this is just the number of edges. We should also point out that we do not know how to generalise the formula for the degree to more complicated singularity baskets; the only method available to us at present—compute many examples and make educated guesses—is not feasible when considering “large” singularity baskets, as the relevant computations become too expensive (even for  $\mathcal{B} = \{2 \times \frac{1}{5}(1, 1)\}$  the computation is infeasible). Presumably there exists a combinatorially derived formula for the degree, but this has so far eluded us.

**Example 5.3.** The computations are expensive, as noted above, and the output is very large and not particularly enlightening, so we'll show only the simplest few examples here. The simplest smooth Fano polygon is the one with vertices  $(0, 1)$ ,  $(1, 0)$ , and  $(-1, -1)$ , with singularity content  $(3, \emptyset)$ : the standard MMLP is  $x + y + \frac{1}{xy}$  and  $L_f$  is  $\nabla^2 - 27t^3(\nabla + 1)(\nabla + 2)$  (as before,  $\nabla = t\partial_t$ ); this has ramification index zero, degree 3, and order 2. The second simplest smooth Fano polygon is the one with vertices  $(0, 1)$ ,  $(1, 0)$ ,  $(-1, -1)$ , and  $(1, 1)$ , with singularity content  $(4, \emptyset)$ ; here the standard MMLP is  $x + y + \frac{1}{xy} + xy$  and  $L_f$  is  $8\nabla^2 + t\nabla(17\nabla - 1) - t^2(5\nabla + 8)(11\nabla + 8) - 12t^3(30\nabla^2 + 78\nabla + 47) - 4t^4(\nabla + 1)(103\nabla + 147) - 99t^5(\nabla + 1)(\nabla + 2)$  (this has ramification index 1, order 2, and degree 5); there is no way to mutate this polygon into one with three vertices.

The nonequivalent polygons from Example 5.1 had 3 and 4 vertices, respectively; their Picard–Fuchs operators have, respectively, order 4, degree 9, and ramification zero; and order 4, degree 13, and ramification one.

We are now in the apparent situation that we can distinguish nonequivalent Fano polygons by either the minimal number of edges in their mutation class (which we can currently bound from above, but not prove minimality of) or the ramification of the Picard–Fuchs operators (which is easy to compute given  $L_f$ ). This motivates Conjecture 1.4 from the Introduction.

#### ACKNOWLEDGMENTS

This paper came out of the project to classify Fano manifolds via mirror symmetry, to which I was introduced during the PRAGMATIC 2013 Research school in Algebraic Geometry and Commutative Algebra “Topics in Higher Dimensional Algebraic Geometry” held in Catania, Italy, in September 2013. I am very grateful to Alfio Ragusa, Francesco Russo, and Giuseppe Zappalá, the organizers of the PRAGMATIC school, for making all that happen. I am also grateful to Alessio Corti



and Al Kasprzyk for introducing me to this topic and for the fruitful collaboration that has followed, and to Tom Coates and Rikard Bøgvad for helpful comments. I also wish to thank the anonymous reviewer for several helpful suggestions.

## REFERENCES

- [ACC<sup>+</sup>16] Mohammad Akhtar, Tom Coates, Alessio Corti, Liana Heuberger, Alexander Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, and Ketil Tveiten, *Mirror symmetry and the classification of orbifold del Pezzo surfaces*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 513–527. MR3430830
- [ACGK12] Mohammad Akhtar, Tom Coates, Sergey Galkin, and Alexander M. Kasprzyk, *Minkowski polynomials and mutations*, SIGMA Symmetry Integrability Geom. Methods Appl. **8** (2012), Paper 094, 17. MR3007265
- [AK14] Mohammad Akhtar and Alexander Kasprzyk, *Singularity content*, arXiv:1401.5458v1 (2014).
- [BGK<sup>+</sup>87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics, vol. 2, Academic Press, Inc., Boston, MA, 1987. MR882000
- [CCG<sup>+</sup>13] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander Kasprzyk, *Mirror symmetry and Fano manifolds*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2013, pp. 285–300. MR3469127
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322
- [Dim04] Alexandru Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004. MR2050072
- [GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Reprint of the 1978 original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. MR1288523
- [GHK15] Mark Gross, Paul Hacking, and Sean Keel, *Birational geometry of cluster algebras*, Algebr. Geom. **2** (2015), no. 2, 137–175. MR3350154
- [GU10] Sergey Galkin and Alexandr Usnich, *Mutations of potentials*, preprint IPMU 10-0100 (2010).
- [Hör90] Lars Hörmander, *The analysis of linear partial differential operators. I*, Distribution theory and Fourier analysis, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1990. MR1065993
- [KN12] Alexander M. Kasprzyk and Benjamin Nill, *Fano polytopes*, Strings, Gauge Fields, and the Geometry Behind – the Legacy of Maximilian Kreuzer (Anton Rebhan, Ludmil Katzarkov, Johanna Knapp, Radoslav Rashkov, and Emanuel Scheidegger, eds.), World Scientific, 2012, pp. 349–364.
- [KT16] Alexander M. Kasprzyk and Ketil Tveiten, *Maximally mutable laurent polynomials*, in preparation (2016).
- [Lai14] Pierre Lairez, *Computing periods of rational integrals*, Math. Comp. **85** (2016), no. 300, 1719–1752. MR3471105
- [OP15] Alessandro Oneto and Andrea Petracci, *On the quantum periods of del pezzo surfaces with  $\frac{1}{3}(1, 1)$  singularities*, in preparation (2015).
- [SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000. MR1734566
- [Żoł06] Henryk Żołądek, *The monodromy group*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], vol. 67, Birkhäuser Verlag, Basel, 2006. MR2216496