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A LOCAL RAMSEY THEORY FOR BLOCK SEQUENCES

IIAN B. SMYTHE

Abstract. We develop local forms of Ramsey-theoretic dichotomies for block sequences in infinite-dimensional vector spaces, analogous to Mathias's selective coideal form of Silver's theorem for analytic partitions of $[\mathbb{N}]^\infty$. Under large cardinals, these results are extended to partitions in $\mathbf{L}(\mathbb{R})$, and $\mathbf{L}(\mathbb{R})$-generic filters of block sequences are characterized. Variants of these results are also established for block sequences in Banach spaces and for projections in the Calkin algebra.

1. Introduction

Ramsey-theoretic techniques have a long history of use in Banach space theory; see, e.g., [4]. Most relevant for the present work is Gowers's dichotomy for infinite block sequences in Banach spaces:

Theorem (Gowers [18], [19]). *Let B be an infinite-dimensional Banach space with a Schauder basis. If $\mathbb{A}$ is an analytic set of normalized block sequences, then for any $\Delta > 0$, there is a block sequence Y such that either*

(i) *every normalized block subsequence of Y is in $\mathbb{A}^c$, or*
(ii) *II has a strategy in the Gowers game $G^*[Y]$ for playing into $\mathbb{A}_\Delta$.*

Loosely speaking, this result says that for $\mathbb{A}$ as described, there is a block sequence Y such that either all of Y's normalized block subsequences are disjoint from $\mathbb{A}$ or there is a wealth of block subsequences of Y which are within a small perturbation of $\mathbb{A}$. This was used, together with work of Komorowski and Tomczak-Jaegerman [23], to solve (affirmatively) the homogeneous space problem.

In the setting of a discrete countably infinite-dimensional vector space E over a countable field, Rosendal isolated an "exact" version of Gowers's dichotomy which yields a much simplified proof of the original result:

Theorem (Rosendal [35]). *If $\mathbb{A}$ is an analytic set of block sequences in E, then there is a block sequence Y such that either*

(i) *I has a strategy in the infinite asymptotic game $F[Y]$ for playing into $\mathbb{A}^c$, or*
(ii) *II has a strategy in the Gowers game $G[Y]$ for playing into $\mathbb{A}$.*

These dichotomies are analogues, in the Banach space and vector space settings, respectively, of the following result for partitions of $[\mathbb{N}]^\infty$, the set of infinite subsets of the natural numbers:

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Theorem (Silver [38])**.** *If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic, then there is a $y \in [\mathbb{N}]^\infty$ with either all of its further infinite subsets disjoint from, or contained in, $\mathbb{A}$.*

While the theory of topological Ramsey spaces, in the sense of [40], encompasses many variations on this result, the dichotomies of Gowers and Rosendal highlighted above do not fall into this framework.

An important generalization of Silver's theorem is the following "local" Ramsey theorem showing that the witness y in the conclusion can always be found in a given selective coideal (or "happy family"):

Theorem (Mathias [30])**.** *Let $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ be a selective coideal. If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic, then there is a $y \in \mathcal{H}$ with either all of its further infinite subsets disjoint from, or contained in, $\mathbb{A}$.*

By passing to a forcing extension resulting from the Lévy collapse of a Mahlo cardinal, Mathias extended these results to all partitions $\mathbb{A}$ which are "reasonably definable", that is, in the definable closure of the reals $\mathbf{L}(\mathbb{R})$. Later work of Farah and Todorcevic [14] generalized this to semiselective coideals and showed that under stronger large cardinal hypotheses the passage to a forcing extension is not necessary. The extension of Silver's theorem to all partitions in $\mathbf{L}(\mathbb{R})$ is due to Shelah and Woodin [37]. Similar results have been developed recently for topological Ramsey spaces [31], [12].

The upshot of obtaining these local results is twofold: We clearly isolate the combinatorial properties which enable the original dichotomies, and we obtain greater control over the witnesses to said dichotomies.

This latter point was used by Todorcevic [14] to characterize, under large cardinal hypotheses, selective ultrafilters as being exactly those which are generic for $([\mathbb{N}]^\infty, \subseteq^*)$ over $\mathbf{L}(\mathbb{R})$. Such ultrafilters are said to possess "complete combinatorics", following Blass and Laflamme [25] who used this phrase to describe ultrafilters which are generic over $\mathbf{L}(\mathbb{R})$ after collapsing a Mahlo cardinal. We instead ask for genericity over $\mathbf{L}(\mathbb{R})$ of the ground model, at the expense of stronger large cardinal hypotheses.

Using [35] as a starting point, we develop local versions of Gowers's and Rosendal's dichotomies. When E is a countably infinite-dimensional space with basis (e_n) over some countable field F, we isolate in §2 (p^+)-*families* of block sequences, collections of block sequences closed under certain diagonalizations and witnessing a weak pigeonhole principle, and in §3 establish our local form of Rosendal's dichotomy:

Theorem 1.1. *Let $\mathcal{H}$ be a (p^+)-family of block sequences in E. If $\mathbb{A}$ is an analytic set of block sequences and $X \in \mathcal{H}$, then there is a $Y \in \mathcal{H} \restriction X$ such that either*

(i) *I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or*
(ii) *II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.*

Stronger properties of families are discussed in §4, notably *strategic* families. The existence of filters with these properties is considered in §5 and §6, where their existence is proved to be independent of the usual Zermelo–Fraenkel axioms of set theory with the Axiom of Choice (ZFC).

In §7 we show that, under large cardinal hypothesis, strategic (p^+)-filters have complete combinatorics for infinite block sequences with the block subsequence ordering, and generalize Theorem 1.1 to partitions in $\mathbf{L}(\mathbb{R})$ (the corresponding extension of Gowers's original result is due to López-Abad [28]; see also [6]). This requires an analysis of a Mathias-like notion of forcing used to build generic block sequences.

Theorem 1.2. *Assume that there is a supercompact cardinal. A filter $\mathcal{G}$ of block sequences in E is $\mathbf{L}(\mathbb{R})$-generic for the partial ordering of block sequences if and only if it is a strategic (p^+)-filter.*

Theorem 1.3. *Assume that there is a supercompact cardinal. Let $\mathcal{H}$ be a strategic (p^+)-family of block sequences in E. If $\mathbb{A}$ is a set of block sequences in $\mathbf{L}(\mathbb{R})$ and $X \in \mathcal{H}$, then there is a $Y \in \mathcal{H} \restriction X$ such that either*

(i) *I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or*
(ii) *II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.*

In §8 we consider normed vector spaces and Banach spaces. For an infinite-dimensional separable Banach space B with a Schauder basis, we develop the notion of *spread (p^*)-families*, similar to the (p^+)-families in §2, and establish the following local form of Gowers's dichotomy and its extension to $\mathbf{L}(\mathbb{R})$:

Theorem 1.4. *Let $\mathcal{H}$ be a spread (p^*)-family of normalized block sequences in B which is invariant under small perturbations. If $\mathbb{A}$ is an analytic set of normalized block sequences and $X \in \mathcal{H}$, then for any $\Delta > 0$, there is a $Y \in \mathcal{H} \restriction X$ such that either*

(i) *every normalized block subsequence of Y is in $\mathbb{A}^c$, or*
(ii) *II has a strategy in $G^*[Y]$ for playing into $\mathbb{A}_\Delta$.*

Theorem 1.5. *Assume that there is a supercompact cardinal. Let $\mathcal{H}$ be a strategic (p^*)-family of normalized block sequences in B which is invariant under small perturbations. If $\mathbb{A}$ is a set of normalized block sequences in $\mathbf{L}(\mathbb{R})$ and $X \in \mathcal{H}$, then for any $\Delta > 0$, there is a $Y \in \mathcal{H} \restriction X$ such that either*

(i) *every normalized block subsequence of Y is in $\mathbb{A}^c$, or*
(ii) *II has a strategy in $G^*[Y]$ for playing into $\mathbb{A}_\Delta$.*

It is our hope that Theorem 1.4 will afford new applications of the techniques introduced by Gowers in [19] to obtain block sequences in Banach spaces with simultaneous properties, some captured by the target set $\mathbb{A}$, while others by the family $\mathcal{H}$.

In §9 we apply these results to the study of the projections in the Calkin algebra, the quotient of the bounded operators $\mathcal{B}(H)$ on a Hilbert space H by the compact operators. The natural ordering on projections in the Calkin algebra induces an ordering $\leq_{\mathrm{ess}}$ on $\mathcal{P}_\infty(H)$, the infinite-rank projections in $\mathcal{B}(H)$. We give a version of Theorem 1.2 for filters in this ordering:

Theorem 1.6. *Assume that there is a supercompact cardinal. A filter $\mathcal{G}$ in $(\mathcal{P}_\infty(H), \leq_{\mathrm{ess}})$ is $\mathbf{L}(\mathbb{R})$-generic if and only if projections onto block subspaces are $\leq_{\mathrm{ess}}$-dense in $\mathcal{G}$ and the associated family of block sequences in H is a strategic (p^*)-family.*

Generic filters for $(\mathcal{P}_\infty(H), \leq_{\mathrm{ess}})$ induce *pure states* on $\mathcal{B}(H)$, via the theory of *quantum filters* introduced by Farah and Weaver [15]. It is known that these generic pure states are not pure on any atomic maximal abelian self-adjoint subalgebra (essentially due to Farah and Weaver [15]), and are, thus, counterexamples to a conjecture of Anderson [3]. We show that any family satisfying the hypotheses of Theorem 1.4 and generating a pure state on $\mathcal{B}(H)$ produces such a counterexample. We caution that our counterexamples remain beyond ZFC.

Theorem 1.7. *A spread* (p^*)*-family* $\mathcal{H}$ *of block sequences in* H *which is* $\leq_{\mathrm{ess}}$*-centered induces a singular pure state* ρ *on* $\mathcal{B}(H)$ *which is not pure on any atomic maximal abelian self-adjoint subalgebra.*

§10 concludes the paper with questions for future investigation.

An effort has been made to keep the set-theoretic prerequisites for understanding this work to a minimum with the hope that the material, particularly in §3 and §8, may be used for further applications in Banach space and operator theory. We assume a familiarity with the basic properties of Polish spaces, Borel sets, and analytic sets (as covered in [22]) throughout. We only make explicit use of the method of forcing and large cardinal hypotheses in §5 and §7, with occasional reference back to that material in §8 and §9. The Banach space prerequisites amount to little more than a familiarity with basic sequences (as covered in the first sections of [2]).

2. Families of block sequences

Fix a countable field F, a countably infinite-dimensional F-vector space E, and a Hamel F-basis (e_n) for E. Typically, we will think of F as a subfield of $\mathbb{C}$, but this is not necessary; F may even be finite. Given $v \in E$, say with $v = \sum_{n=0}^{N} a_n e_n$, let $\mathrm{supp}(v) = \{n \in \mathbb{N} : a_n \neq 0\}$, the *support* of v. We write $n < v$ if $n < \min(\mathrm{supp}(v))$ and $v < w$ if $\max(\mathrm{supp}(v)) < \min(\mathrm{supp}(w))$.

We say that a (finite or infinite) sequence (x_n) of nonzero vectors in E is a *block sequence* (with respect to (e_n)) if for all n, $x_n < x_{n+1}$. If $\vec{x} = (x_0, \ldots, x_n)$ is a finite block sequence, let $\mathrm{supp}(\vec{x}) = \bigcup_{i=0}^{n} \mathrm{supp}(x_i)$, and for X any block sequence, let $\langle X \rangle = \mathrm{span}(X) \setminus \{0\}$. We will abuse notation and write E for $E \setminus \{0\}$, and use "vector" to mean nonzero vector.

Let $\mathrm{bb}^\infty(E)$ be the collection of all infinite block sequences in E, which we consider as a subspace of $E^{\mathbb{N}}$, where E has the discrete topology. It is easy to check that $\mathrm{bb}^\infty(E)$ is a G_δ subset of $E^{\mathbb{N}}$, and, thus, a Polish space. Let $\mathrm{bb}^{<\infty}(E)$ be the collection of all finite block sequences in E.

For $X = (x_n)$ and $Y = (y_n)$ in $\mathrm{bb}^\infty(E)$, we write $X \preceq Y$ if (x_n) is a block sequence with respect to (y_n), sometimes called a *block subsequence* of Y, or equivalently (for block sequences), $\langle X \rangle \subseteq \langle Y \rangle$. We write $X \preceq^* Y$ if for some m, $X/m \preceq Y$, where X/m is the tail of X with supports above m. For $\vec{x} \in \mathrm{bb}^{<\infty}(E)$, write $X/\vec{x}$ for $X/\max(\mathrm{supp}(\vec{x}))$. Note that the orderings $\preceq$ and $\preceq^*$ fail to be antisymmetric, but are reflexive and transitive.

We will make repeated use of the following order-theoretic notions: A subset D of a preorder $(P, \leq)$ (that is, $\leq$ is reflexive and transitive) is *dense* if for all $p \in P$, there is a $q \in D$ with $q \leq p$. It is, moreover, *dense open*, if whenever $q \leq p \in D$, then $q \in D$. Elements p and q in P are *compatible* if they have a common lower bound in P, and are *incompatible* otherwise.

Compatibility in $(\mathrm{bb}^\infty(E), \preceq)$ is equivalent to that in $(\mathrm{bb}^\infty(E), \preceq^*)$ and we write $X \perp Y$ when X and Y are incompatible. The following observation shows that

$(\mathrm{bb}^\infty(E), \preceq)$ can be identified with a dense suborder of the lattice of all infinite-dimensional subspaces of E. In particular, X and Y are compatible if and only if $\langle X\rangle \cap \langle Y\rangle$ is infinite-dimensional.

Lemma 2.1. *If X is an infinite-dimensional subspace of E, then X contains an infinite block sequence.*

Proof. By taking appropriate linear combinations, one can show that for any $N \in \mathbb{N}$, X contains an infinite-dimensional subspace whose supports are above N. From this, it is easy to inductively construct a block sequence in X. □

Throughout, when we speak of a *family* $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$, we mean a nonempty subset which is closed upwards with respect to $\preceq^*$. For $X \in \mathcal{H}$, we denote by $\mathcal{H} \upharpoonright X = \{Y \in \mathcal{H} : Y \preceq X\}$. A *filter* $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ is a family such that for every $X, Y \in \mathcal{F}$, there is a $Z \in \mathcal{F}$ with $Z \preceq X$ and $Z \preceq Y$.

Definition 2.2.

(a) Given a descending sequence $X_0 \succeq X_1 \succeq \cdots$ in $\mathrm{bb}^\infty(E)$, we call $Y \in \mathrm{bb}^\infty(E)$ a *diagonalization* of (X_n) if for all n, $Y \preceq^* X_n$.
(b) Given a sequence $(\mathcal{D}_n)$ of subsets of $\mathrm{bb}^\infty(E)$, we call Y a *diagonalization* of $(\mathcal{D}_n)$ if for each n, there is an $X_n \in \mathcal{D}_n$ such that $Y \preceq^* X_n$.

For $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$, a set $\mathcal{D}$ is $\preceq$-*dense* (*open*) in $\mathcal{H}$ if $\mathcal{D} \cap \mathcal{H}$ is.

Definition 2.3. A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a (p)-*family*, or has the (p)-*property*, if whenever $X_0 \succeq X_1 \succeq \cdots$ is a decreasing sequence with each $X_n \in \mathcal{H}$, there is a diagonalization $Y \in \mathcal{H}$ of (X_n).

It is easy to see that $\mathrm{bb}^\infty(E)$ itself is a (p)-family. We note that every (p)-family $\mathcal{H}$ contains a diagonalization of any given sequence $(\mathcal{D}_n)$ of $\preceq$-dense open subsets in $\mathcal{H}$: build a decreasing sequence (X_n) in $\mathcal{H}$ with each $X_n \in \mathcal{D}_n$, then any diagonalization $Y \in \mathcal{H}$ of (X_n) will be a diagonalization of $(\mathcal{D}_n)$. This can be done below any given $X \in \mathcal{H}$, so the set of such diagonalizations is $\preceq$-dense in $\mathcal{H}$. This latter property, which could be called the "weak (p)-property", will be sufficient for all of the results in §3, and in particular, for Theorem 1.1.

Recall that $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ is a *coideal* if it contains all cofinite sets, is closed upwards with respect to $\subseteq$, and whenever $Y_0 \cup Y_1 \in \mathcal{H}$, then one of Y_0 or Y_1 is also in $\mathcal{H}$. This last property asserts that $\mathcal{H}$ witnesses the pigeonhole principle. In our setting, provided $|F| > 2$,[1] the "obvious" formulation of the pigeonhole principle is simply false, as the following example shows:

Example 2.4.[2] Consider the case when $F \subseteq \mathbb{R}$. Similar examples can be constructed whenever $|F| > 2$; cf. Theorem 7 in [26]. For a vector $x \in E$ define the *oscillation* $\mathrm{osc}(x)$ as the number of times the sign of the nonzero coefficients of x alternate in its expansion with respect to (e_n). So, $\mathrm{osc}(e_0 - e_1 + e_2) = 2$, $\mathrm{osc}(e_2 + e_4 - e_5 + e_7 - e_{10}) = 3$, etc.

Define $A_0 \subseteq E$ (respectively, $A_1 \subseteq E$) to be the set of all $x \in E$ such that $\mathrm{osc}(x)$ is even (respectively, odd), and let $\mathbb{A}_i = \{(x_n) : x_0 \in A_i\}$ for $i = 0, 1$. The $\mathbb{A}_i$'s are clopen sets which partition $\mathrm{bb}^\infty(E)$. Moreover, the pair $\mathbb{A}_0$, $\mathbb{A}_1$ is *asymptotic*;

[1]When $|F| = 2$, such a pigeonhole principle for block subspaces does hold; this is essentially Hindman's theorem [20].

[2]The author would like to thank Jordi López-Abad for pointing out this example which has the advantage of being well-defined at the level of the spanned subspaces.

that is, for any $X \in \mathrm{bb}^\infty(E)$ and $i = 0, 1$, there is $Y_i \preceq X$ such that $Y_i \in \mathbb{A}_i$. To see this, suppose that $X = (x_n)$ such that $X \in \mathbb{A}_0$, so $\mathrm{osc}(x_0)$ is even. If $\mathrm{osc}(x_1)$ is odd, then $(x_n)_{n\geq 1} \preceq X$ and in $\mathbb{A}_1$. If $\mathrm{osc}(x_1)$ is even, then let $x = x_0 - x_1$ if the signs of the last nonzero coefficient in x_0 and the first in x_1 agree, and $x = x_0 + x_1$ otherwise. In either case, $\mathrm{osc}(x) = \mathrm{osc}(x_0) + \mathrm{osc}(x_1) + 1$, so $(x, x_2, x_3, \ldots)$ is in $\mathbb{A}_1$.

The following is a weak analogue of the pigeonhole property of coideals.

Definition 2.5. Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ be a family.

(a) A subset $\mathcal{D} \subseteq \mathrm{bb}^\infty(E)$ is *$\mathcal{H}$-dense* below some $X \in \mathcal{H}$ if for every $Y \in \mathcal{H} \restriction X$, there is a $Z \preceq Y$ with $Z \in \mathcal{D}$. A set $D \subseteq E$ is *$\mathcal{H}$-dense* below X if $\{Z : \langle Z\rangle \subseteq D\}$ is.

(b) $\mathcal{H}$ is *full* if whenever $D \subseteq E$ (not necessarily a subspace) and $X \in \mathcal{H}$ are such that D is $\mathcal{H}$-dense below X, there is a $Z \in \mathcal{H} \restriction X$ with $\langle Z\rangle \subseteq D$.

Fullness allows one to upgrade $\{Z : \langle Z\rangle \subseteq D\}$ being $\mathcal{H}$-dense below X to being $\preceq$-dense (open) below X in $\mathcal{H}$. Obviously $\mathrm{bb}^\infty(E)$ itself is a full family. If the family in question is a filter $\mathcal{F}$, we may simplify the definition of fullness by replacing X with (e_n) (or any element of $\mathcal{F}$). We note that any full filter is maximal; this can be seen by applying the definition of fullness when D is a block subspace. It is shown in Proposition 3.6 that fullness is necessary for Theorem 1.1.

Definition 2.6. A family in $\mathrm{bb}^\infty(E)$ which is full and has the (p)-property will be called a *(p^+)-family.* Likewise for *(p^+)-filter.*

Lemma 2.7. (a) *For $X_0 \succeq X_1 \succeq \cdots$ in $\mathrm{bb}^\infty(E)$, the set*

$$\mathcal{D}_{(X_n)} = \{Y : Y \textit{ is a diagonalization of } (X_n) \textit{ or } \exists n (Y \perp X_n)\}$$

is $\preceq$-dense open.

(b) *For $D \subseteq E$ and $X \in \mathrm{bb}^\infty(E)$, the set*

$$\mathcal{D}_{D,X} = \{Z : \langle Z\rangle \subseteq D \textit{ or } \forall V \preceq X(\langle V\rangle \subseteq D \rightarrow V \perp Z)\}$$

is $\preceq$-dense open below X.

Proof. (a) Take $Y \in \mathrm{bb}^\infty(E)$ which is compatible with all of the X_n's. We can build a diagonalization $X = (x_n) \preceq Y$ by picking vectors $x_n \in \langle X_n\rangle \cap \langle Y\rangle$ with $x_n < x_{n+1}$.

(b) Take $Y \preceq X$. If there is no $Z \preceq Y$ such that $\langle Z\rangle \subseteq D$, then for any $V \preceq X$ with $\langle V\rangle \subseteq D$, it must be that $V \perp Y$, as otherwise any Z witnessing the compatibility of V and Y would satisfy $\langle Z\rangle \subseteq D$. □

Lemma 2.7 will be used to construct (p^+)-filters in §5. We will see in Corollary 6.5 that the existence of full filters is independent of ZFC.

3. Games with vectors and a local Rosendal dichotomy

The *Gowers game* played below $X \in \mathrm{bb}^\infty(E)$, denoted $G[X]$, is defined as follows: two players, I and II, alternate with I going first and playing block sequences $X_k \preceq X$, and II responding with vectors $y_k \in \langle X_k\rangle$ subject to the constraint $y_k < y_{k+1}$. The block sequence (y_k) is the *outcome* of a play of the game. Given $\vec{x} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathrm{bb}^\infty(E)$, the game $G[\vec{x}, X]$ is defined exactly as $G[X]$ except that II is restricted to playing vectors above $\vec{x}$ and the outcome is $\vec{x}^\frown(y_k)$. This is a discrete version of the game defined by Gowers in [18], [19].

A *strategy* for II in $G[\vec{x}, X]$ is a function α taking sequences $(X_0, \dots, X_k)$ of possible prior moves by I to vectors $y \in \langle X_k \rangle$, with $\vec{x} < \alpha(X_0, \dots, X_{k-1}) < y$ for all k. Given a set $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$, we say that α is a strategy in $G[\vec{x}, X]$ for *playing into* $\mathbb{A}$ if whenever II follows α (that is, at each turn, given as input I's prior moves, they play the output of α), the resulting outcome lies in $\mathbb{A}$. These notions are defined likewise for I.

The *infinite asymptotic game* [34], [35] played below X, denoted $F[X]$, is defined in a similar fashion: Two players, I and II, alternate with I going first and playing natural numbers n_k, and II responding with vectors $y_k \in \langle X/n_k \rangle$ subject to the constraint $y_k < y_{k+1}$. Again, (y_k) is the *outcome* of a play of the game. The game $F[\vec{x}, X]$ is defined as above, as are *strategies* for I and II, and the notion of having a strategy for *playing into* a set.

It is important to note that plays of $F[\vec{x}, X]$ can be considered as plays of $G[\vec{x}, X]$ where I is restricted to playing tail block subsequences of X. Consequently, if II has a strategy in $G[\vec{x}, X]$ for playing into a set $\mathbb{A}$, then II has such a strategy in $F[\vec{x}, X]$ as well. Similarly, if I has a strategy in $F[\vec{x}, X]$ for playing into $\mathbb{A}$, then they have such a strategy in $G[\vec{x}, X]$.

The following generalizes the notion of *strategically Ramsey* given in [35], where $\mathcal{H}$ was taken to be all of $\mathrm{bb}^\infty(E)$.

Definition 3.1. For $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ a family, we say that a subset $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is *$\mathcal{H}$-strategically Ramsey* if for all $\vec{y} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \restriction X$ such that either

(i) I has a strategy in $F[\vec{y}, Y]$ for playing into $\mathbb{A}^c$, or
(ii) II has a strategy in $G[\vec{y}, Y]$ for playing into $\mathbb{A}$.

Note that consequences (i) and (ii) are mutually exclusive by our comments above. The key fact about $\mathcal{H}$-strategically Ramsey sets is that the witness, Y in the above definition, can be found in $\mathcal{H}$.

Our goal for the remainder of this section is to outline the proof that, for any (p^+)-family $\mathcal{H}$, analytic sets are $\mathcal{H}$-strategically Ramsey, thereby establishing Theorem 1.1. Much of what follows closely hews to [35] and is a variation of the combinatorial forcing technique used in [40].

Definition 3.2. Let $\mathcal{H}$ be a family and let $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ be given. For $\vec{y} \in \mathrm{bb}^{<\infty}(E)$ and $Y \in \mathcal{H}$, we say that

(1) $(\vec{y}, Y)$ is *good* (*for* $\mathbb{A}$) if II has a strategy in $G[\vec{y}, Y]$ for playing into $\mathbb{A}$;
(2) $(\vec{y}, Y)$ is *bad* (*for* $\mathbb{A}$) if for all $Z \in \mathcal{H} \restriction Y$, $(\vec{y}, Z)$ is not good;
(3) $(\vec{y}, Y)$ is *worse* (*for* $\mathbb{A}$) if it is bad and there is an n such that for every $v \in \langle Y/n \rangle$, $(\vec{y}^\frown v, Y)$ is bad.

Reference to $\mathbb{A}$ and $\mathcal{H}$ will be suppressed where understood.

Lemma 3.3. *If $\mathcal{H}$ is a (p^+)-family and $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$, then for every $\vec{x} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \restriction X$ such that either*

(i) *$(\vec{x}, Y)$ is good, or*
(ii) *I has a strategy in $F[\vec{x}, Y]$ for playing into*

$$\{(z_n) : \forall n (\vec{x}^\frown (z_0, \dots, z_n), Y) \text{ is worse}\}.$$

Proof. Observe that if $(\vec{y}, Y)$ is good/bad/worse and $Z \preceq^* Y$ is in $\mathcal{H}$, then $(\vec{y}, Z)$ is also good/bad/worse. It is immediate that for each $\vec{y}$, the set

$$\mathcal{D}_{\vec{y}} = \{Y \in \mathcal{H} : (\vec{y}, Y) \text{ is either good or bad}\}$$

is $\preceq$-dense open in $\mathcal{H}$.

Claim. If $(\vec{y}, Y)$ is bad, then for all $Z \in \mathcal{H} \upharpoonright Y$, there is a $V \preceq Z$ such that for all $x \in \langle V/\vec{y}\rangle$, $(\vec{y}^\frown x, Y)$ is not good.

Proof of claim. Let $(\vec{y}, Y)$ be bad. Towards a contradiction, suppose that there is some $Z \in \mathcal{H} \upharpoonright Y$ such that for all $V \preceq Z$, there is an $x \in \langle V/\vec{y}\rangle$ such that $(\vec{y}^\frown x, Y)$ is good. We claim that $(\vec{y}, Z)$ is good. If I plays $V \preceq Z$, then by supposition there is some $x \in \langle V/\vec{y}\rangle$ such that $(\vec{y}^\frown x, Z)$ is good. Let II play that x and from then on follow the strategy given from $(\vec{y}^\frown x, Z)$ being good. This is contrary to $(\vec{y}, Y)$ being bad. $\square_{\text{(claim.)}}$

Claim. For each $\vec{y}$, the set

$$\mathcal{E}_{\vec{y}} = \{Z \in \mathcal{H} : (\vec{y}, Z) \text{ is either good or worse}\}$$

is $\preceq$-dense open in $\mathcal{H}$.

Proof of claim. Fix $\vec{y}$ and let $Y \in \mathcal{H}$. Since the sets $\mathcal{D}_{\vec{x}}$ are dense in $\mathcal{H}$ and there are only countably many $\vec{x}$, the (p)-property allows us to diagonalize all of them within $\mathcal{H}$ and assume that for all $\vec{x}$, $(\vec{x}, Y)$ is either good or bad. Suppose that $(\vec{y}, Y)$ is bad. Let $D = \{x : (\vec{y}^\frown x, Y) \text{ is not good}\}$. By the previous claim, D is $\mathcal{H}$-dense below Y. Since $\mathcal{H}$ is full, there is a $Z \in \mathcal{H} \upharpoonright Y$ such that $\langle Z\rangle \subseteq D$. If $z \in \langle Z\rangle$, then $(\vec{y}^\frown z, Z)$ is not good, hence, bad, by our choice of Y. Thus, $(\vec{y}, Z)$ is worse. $\square_{\text{(claim.)}}$

We can now prove the lemma. By the previous claim, we have a $Y \in \mathcal{H} \upharpoonright X$ so that for all $\vec{y}$, $(\vec{x}^\frown \vec{y}, Y)$ is either good or worse. If $(\vec{x}, Y)$ is good, we are done, so suppose that $(\vec{x}, Y)$ is worse. We will describe a strategy for I in $F[\vec{x}, Y]$: Suppose that at some point in the game $(z_0, \ldots, z_k)$ has been played by II so that $(\vec{x}^\frown(z_0, \ldots, z_k), Y)$ is worse. Then, there is some n such that for all $z \in \langle Y\rangle$, if $n < z$, then $(\vec{x}^\frown(z_0, \ldots, z_k)^\frown z, Y)$ is bad, hence, worse. Let I play n. $\square$

Lemma 3.4 (cf. Lemma 2 in [35])**.** *Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ a (p^+)-family. Then, open sets are $\mathcal{H}$-strategically Ramsey.*

Proof. Let $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ be open. Given $\vec{x} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, by Lemma 3.3, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either $(\vec{x}, Y)$ is good, in which case we are done, or I has a strategy in $F[\vec{x}, Y]$ to play (z_n) such that for all n, $(\vec{x}^\frown(z_0, \ldots, z_n), Y)$ is worse. In the latter case, if I follows this strategy, as II builds (z_n), for no m can II have a strategy in $G[\vec{x}^\frown(z_0, \ldots, z_m), Y]$ to play in $\mathbb{A}$. Since $\mathbb{A}$ is open, this means that $\vec{x}^\frown(z_0, z_1, \ldots) \notin \mathbb{A}$ and I has a strategy for playing into $\mathbb{A}^c$. $\square$

Proof (*sketch*) *of Theorem* 1.1. The proof closely follows that of Theorem 5 in [35], where $\mathcal{H} = \mathrm{bb}^\infty(E)$. The idea of the proof is that, given a Souslin scheme $\{\mathbb{A}_s\}_{s \in \mathbb{N}^{<\infty}}$ for an analytic set $\mathbb{A}$, we can use Lemma 3.4 and diagonalization to find a $Y \in \mathrm{bb}^\infty(E)$ such that if I does not have a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, then in $G[Y]$, II can build a sequence (z_k) such that I continues to have no strategy in $F[(z_0, \ldots, z_k), Y]$ for playing into $\mathbb{A}_s^c$, where s is an initial segment of some branch y in $\mathbb{N}^\mathbb{N}$. y will witness that II's strategy has produced an outcome in $\mathbb{A}$.

We omit the details, except to say that the arguments in [35] can be modified for our result simply by ensuring that the block sequences used are taken in $\mathcal{H}$. This can be done, in each instance, as a block sequence is obtained either by applying the result for open sets or by diagonalization. $\square$

Theorem 1.1 is consistently sharp and necessarily asymmetric, as there is a coanalytic counterexample (for $\mathcal{H} = \mathrm{bb}^\infty(E)$) in $\mathbf{L}$ [28].[3] In particular, the collection of $\mathcal{H}$-strategically Ramsey sets may fail to be a σ-algebra. It is, however, closed under countable unions. Again, the proof is nearly identical to that of the corresponding result in [35] and is omitted.

Theorem 3.5 (cf. Theorem 9 in [35])**.** *Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ be a (p^+)-family. Then, the collection of $\mathcal{H}$-strategically Ramsey sets is closed under countable unions.* $\square$

We note that fullness is a necessary assumption for our results:

Proposition 3.6. *If $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a family for which clopen sets are $\mathcal{H}$-strategically Ramsey, then $\mathcal{H}$ is full.*

Proof. Given $D \subseteq E$, $\mathcal{H}$-dense below some $X \in \mathcal{H}$, let $\mathbb{D} = \{(z_n) : z_0 \in D\}$, a clopen subset of $\mathrm{bb}^\infty(E)$. For no $Y \in \mathcal{H} \restriction X$ can II have a strategy into $\mathbb{D}^c$: Consider the round of $G[Y]$ where I starts by playing some $Z \preceq Y$ with $\langle Z \rangle \subseteq D$. Since $\mathbb{D}^c$ is $\mathcal{H}$-strategically Ramsey, there is a $Y \in \mathcal{H} \restriction X$ such that I has a strategy σ in $F[Y]$ for playing into $\mathbb{D}$. Let $Z = Y/\sigma(\emptyset) \in \mathcal{H}$. Since σ is a strategy for playing into $\mathbb{D}$, $\langle Z \rangle \subseteq D$. $\square$

4. Stronger properties of families

If an element Y in a family $\mathcal{H}$ witnesses Theorem 1.1, then either $\mathbb{A}^c$ or $\mathbb{A}$ is $\mathcal{H}$-dense below Y, depending on which half of the dichotomy holds. However, it would be desirable to ensure that $\mathcal{H}$ itself meets whichever one of $\mathbb{A}^c$ or $\mathbb{A}$ the conclusion of the dichotomy provides. To this end, we consider stronger properties of families, the first of which is based on the original definition of selectivity (or being "happy") given in [30].

Definition 4.1.

(a) For $(X_{\vec{x}})_{\vec{x} \in \mathrm{bb}^{<\infty}(E)}$ generating a filter in $\mathrm{bb}^\infty(E)$, we say that $X \in \mathrm{bb}^\infty(E)$ *strongly diagonalizes* $(X_{\vec{x}})$ if $X/\vec{x} \preceq X_{\vec{x}}$ whenever $\vec{x} \sqsubseteq X$.

(b) A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a *strong (p)-family*, or has the *strong (p)-property*, if whenever $(X_{\vec{x}})_{\vec{x} \in \mathrm{bb}^{<\infty}(E)}$ generates a filter in $\mathcal{H}$, there is a $Y \in \mathcal{H}$ which strongly diagonalizes $(X_{\vec{x}})$.

The strong (p)-property implies the (p)-property: Take $X_0 \succeq X_1 \succeq \cdots$ in $\mathcal{H}$, and define $X_{\vec{x}} = X_{|\vec{x}|}$ for $\vec{x} \in \mathrm{bb}^{<\infty}(E)$. Any X strongly diagonalizing $(X_{\vec{x}})$ will diagonalize (X_n).

[3] This counterexample is to Gowers's theorem, but the discussion in §5 of [35] shows that this also yields a counterexample to Rosendal's dichotomy.

As in Lemma 2.7, it is useful for constructing families with the strong (p)-property to know that it corresponds to certain $\preceq$-dense sets.

Lemma 4.2. *For $(X_{\vec{x}})_{\vec{x}\in \mathrm{bb}^{<\infty}(E)}$ generating a filter in $\mathrm{bb}^{\infty}(E)$, the set*

$$\{Y : Y \text{ is a strong diagonalization of } (X_{\vec{x}})_{\vec{x}\in \mathrm{bb}^{<\infty}(E)}, \text{ or } \\ \{Y\} \cup (X_{\vec{x}})_{\vec{x}\in \mathrm{bb}^{<\infty}(E)} \text{ does not generate a filter}\}$$

is $\preceq$-dense.

Proof. Fix $X \in \mathrm{bb}^{\infty}(E)$, and suppose that $\{X\} \cup (X_{\vec{x}})$ generates a filter. We build a $Y \preceq X$ which strongly diagonalizes $(X_{\vec{x}})$: Pick any $y_0 \in \langle X\rangle \cap \langle X_\emptyset\rangle$. Since X, $X_\emptyset$, and $X_{(y_0)}$ generate a filter, there is a $y_1 \in \langle X\rangle \cap \langle X_\emptyset\rangle \cap \langle X_{(y_0)}\rangle$ with $y_0 < y_1$. Continue in this fashion. □

The following result connects the strong (p)-property to the infinite asymptotic game and is based on a characterization of selective ultrafilters (Theorem 4.5.3 in [7]).

Theorem 4.3. *If $\mathcal{H} \subseteq \mathrm{bb}^{\infty}(E)$ is a strong (p)-family, then for no $X \in \mathcal{H}$ does I have a strategy in $F[X]$ for playing into $\mathcal{H}^c$.*[4]

Proof. Let σ be a strategy for I in $F[X]$ for playing into $\mathcal{H}^c$, where $X \in \mathcal{H}$. Towards a contradiction, suppose that $\mathcal{H}$ has the strong (p)-property. Define sets $\mathcal{A}_{\vec{x}} \subseteq \mathcal{H}$ as follows: $\mathcal{A}_\emptyset = \{X/\sigma(\emptyset)\}$ and inductively, for $\vec{x} = (x_0, \dots, x_{n-1})$, $\mathcal{A}_{\vec{x}}$ is the set of all X/m where m is played by I following σ in the first n rounds of $F[X]$, as II plays $x_0, x_1, \dots, x_{n-1}$. In the case that elements of a given $\vec{x}$ fail to be valid moves for II against σ, let $\mathcal{A}_{\vec{x}} = \mathcal{A}_{\vec{x}'}$ where $\vec{x}'$ is the maximal initial segment of $\vec{x}$ consisting of valid moves. Then, for all $\vec{x}$, $\mathcal{A}_{\vec{x}}$ is finite and $\mathcal{A}_{\vec{x}} \subseteq \mathcal{A}_{\vec{y}}$ whenever $\vec{x} \sqsubseteq \vec{y}$.

For each $\vec{x}$, let $M_{\vec{x}} = \max\{m : X/m \in \mathcal{A}_{\vec{x}}\}$ and $Y_{\vec{x}} = X/M_{\vec{x}}$. Clearly, $(Y_{\vec{x}})$ generates a filter in $\mathcal{H}$. By the strong (p)-property, there is a $Y = (y_n) \in \mathcal{H} \upharpoonright X$ such that $Y/\vec{y} \preceq Y_{\vec{y}}$ for all $\vec{y} \sqsubseteq Y$.

Consider the play of $F[X]$ wherein I follows σ and II plays y_0, y_1, and so on. We claim that this is a valid sequence of moves for II. Note that $y_0 \in \langle Y/\emptyset\rangle \subseteq \langle Y_\emptyset\rangle \subseteq \langle X/\sigma(\emptyset)\rangle$, so y_0 is a valid move. Inductively, suppose that $(y_0, \dots, y_k)$ is a valid sequence of moves. We have $y_{k+1} \in \langle Y/(y_0, \dots, y_k)\rangle \subseteq \langle Y_{(y_0,\dots,y_k)}\rangle \subseteq \langle X/\sigma(y_0, \dots, y_k)\rangle$, where the last containment uses our induction hypothesis. Thus, y_{k+1} is a valid move. Since the resulting outcome in this play is in $\mathcal{H}$, we have a contradiction. □

Equivalently, Theorem 4.3 says that if $\mathcal{H}$ is a strong (p)-family and σ is a strategy for I in $F[X]$, for $X \in \mathcal{H}$, then there is an outcome of σ in $\mathcal{H}$.

Lemma 4.4. *If $\mathbb{D} \subseteq \mathrm{bb}^{\infty}(E)$ is $\preceq$-dense open below $X \in \mathrm{bb}^{\infty}(E)$, then*

(a) *II has a strategy in $F[X]$ for playing into $\mathbb{D}$, and*
(b) *I has a strategy in $G[X]$ for playing into $\mathbb{D}$.*

Proof. For $F[X]$, take $Y \preceq X$ in $\mathbb{D}$, and let II always play vectors in Y. For $G[X]$, take $Y \preceq X$ in $\mathbb{D}$, and let I simply play Y repeatedly. □

[4] When $\mathcal{F}$ is a strong (p)-filter, one can improve the conclusion to the following: for no $X \in \mathcal{F}$ does I have a strategy in $G_{\mathcal{F}}[X]$ for playing into $\mathcal{F}^c$. Here, $G_{\mathcal{F}}[X]$ is the variant of the Gowers game below X where I is restricted to playing elements of $\mathcal{F}$. See §3.11 of [39].

It follows from Lemma 4.4, and Theorems 1.1 and 4.3, that whenever $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a strong (p^+)-family and $\mathbb{D}$ is a coanalytic $\preceq$-dense open set, then $\mathcal{H} \cap \mathbb{D} \neq \emptyset$. In particular, strong (p^+)-families meet all $\preceq$-dense open Borel sets. This is a special case of Theorem 1.2. The following definition is a counterpart to Theorem 4.3 for II in $G[X]$.

Definition 4.5. A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is *strategic* if whenever $X \in \mathcal{H}$ and α is a strategy for II in $G[X]$, there is an outcome of α which is in $\mathcal{H}$.

As above, if $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a strategic (p^+)-family and $\mathbb{D} \subseteq \mathrm{bb}^\infty(E)$ is an analytic $\preceq$-dense open set, then $\mathbb{D} \cap \mathcal{H} \neq \emptyset$. As a consequence for (p^+)-filters, being strategic subsumes the strong (p)-property.

Proposition 4.6. *If $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ is a strategic (p^+)-filter, then $\mathcal{F}$ is also a strong (p)-filter.*

Proof. Suppose that $\mathcal{F}$ is as described and $(X_{\vec{x}})_{\vec{x} \in \mathrm{bb}^{<\infty}(E)}$ is contained $\mathcal{F}$. Let $\mathcal{D}$ be the set given in Lemma 4.2, so that the $\preceq$-downwards closure of $\mathcal{D}$ is a $\preceq$-dense open set. Moreover, $\mathcal{D}$ is easily seen to be Borel and its $\preceq$-downwards closure analytic. By the comments above, it follows that $\mathcal{F} \cap \mathcal{D} \neq \emptyset$, and any $Y \in \mathcal{F} \cap \mathcal{D}$ must be a strong diagonalization of $(X_{\vec{x}})$. □

In §5 we will construct (under set-theoretic hypotheses) strategic (p^+)-filters. To this end, we again need to know that certain sets are $\preceq$-dense, but also that there are not "too many" of them. If α is a strategy for II in $G[X]$, then the set of outcomes which result from α, denoted by $[\alpha, X]$, is $\preceq$-dense below X. However, as strategies are functions from finite sequences in $\mathrm{bb}^\infty(E)$ to vectors, there are $2^{2^{\aleph_0}}$ many of them.

One way to resolve this is to "finitize" the Gowers game as in [5]. Given $X \in \mathrm{bb}^\infty(E)$, the *finite-dimensional Gowers game below* X, denoted by $G_f[X]$, is defined as follows: Two players, I and II, alternate with I going first and playing a nonzero vector $x_0^{(0)} \in \langle X \rangle$. II responds with either a nonzero $y_0 \in \langle x_0^{(0)} \rangle$ or 0. If II plays y_0, then the game "restarts" with I playing a nonzero vector $x_0^{(1)} \in \langle X \rangle$. If II plays 0, then I must play a nonzero vector $x_1^{(0)} \in \langle X/x_0^{(0)} \rangle$, to which II again responds with either a nonzero vector $y_0 \in \langle x_0^{(0)}, x_1^{(0)} \rangle$ or 0, and so on. The nonzero plays of II are required to satisfy $y_n < y_{n+1}$ and the outcome is the sequence (y_n). The notion of strategy for II in $G_f[X]$ is defined in the obvious way (with the added requirement that the outcome must be infinite) and we denote by $[\alpha, X]_f$ the corresponding set of outcomes.

Lemma 4.7. *If α is a strategy for II in $G[X]$, then there is a strategy α' for II in $G_f[X]$ such that $[\alpha', X]_f \subseteq [\alpha, X]$. Moreover, $[\alpha', X]_f$ is still $\preceq$-dense below X.*

Proof. The proof is identical to the $(\Leftarrow)$ direction of Theorem 1.2 in [5]. □

It is easy to see that strategies α for II in $G_f[X]$ are coded by reals and $[\alpha, X]_f$ is an analytic set. This will suffice for our constructions in §5.

5. Constructions of filters in $\mathrm{bb}^\infty(E)$

In this section we show how to construct filters $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ having all of the properties discussed in §2 and §4. These constructions use either assumptions about certain "cardinal invariants" (cf. [10]) which hold consistently with ZFC, or the method of forcing. We will see in Corollary 6.5 that we cannot hope for a construction in ZFC alone.

Definition 5.1.

(a) A *tower* (of *length* κ) in $\mathrm{bb}^\infty(E)$ is a sequence $(X_\alpha)_{\alpha<\kappa}$ such that $\alpha<\beta<\kappa$ implies $X_\beta \preceq^* X_\alpha$ and there is no $X \in \mathrm{bb}^\infty(E)$ with $X \preceq^* X_\alpha$ for all $\alpha<\kappa$.
(b) $\mathfrak{t}^*$ is the minimum length of a tower in $\mathrm{bb}^\infty(E)$.

$\mathfrak{t}^*$ is a regular cardinal and, moreover, uncountable as $\mathrm{bb}^\infty(E)$ has the (p)-property. Thus, the continuum hypothesis (CH) implies that $\mathfrak{t}^* = 2^{\aleph_0}$.

We use the following notational conventions for versions of Martin's axiom: for $\kappa < 2^{\aleph_0}$, $\mathsf{MA}(\kappa)$ is the forcing axiom for meeting κ-many dense subsets of posets having the countable chain condition (ccc), MA is $\forall\kappa < 2^{\aleph_0}(\mathsf{MA}(\kappa))$, and $\mathsf{MA}(\sigma$-centered) is MA restricted to σ-centered posets.

Lemma 5.2 (Lemma 5 in [17])**.** ($\mathsf{MA}(\sigma$-centered)) *If $\mathcal{L} \subseteq \mathrm{bb}^\infty(E)$ is linearly ordered with respect to $\preceq^*$ and $|\mathcal{L}| < 2^{\aleph_0}$, then there is a Y such that $Y \preceq^* X$ for all $X \in \mathcal{L}$. In particular, $\mathfrak{t}^* = 2^{\aleph_0}$.*

Consequently, the following theorem holds under CH or $\mathsf{MA}(\sigma$-centered):

Theorem 5.3. ($\mathfrak{t}^* = 2^{\aleph_0}$) *There exists a strategic (p^+)-filter in* $\mathrm{bb}^\infty(E)$.

Proof. Fix enumerations

(i) $\{X_\xi : \xi < 2^{\aleph_0}\} = \mathrm{bb}^\infty(E)$,
(ii) $\{(X_n^\xi) : \xi < 2^{\aleph_0}\}$ of all $\preceq^*$-decreasing sequences (X_n^ξ) in $\mathrm{bb}^\infty(E)$,
(iii) $\{D_\xi : \xi < 2^{\aleph_0}\}$ of all subsets D_ξ of E, and
(iv) $\{[\alpha_\xi, X_\xi]_f : \xi < 2^{\aleph_0}\}$ of all sets $[\alpha, X]_f$ of outcomes of α, where α is a strategy for II in $G_f[X]$.

This can be done in (i) and (ii) since $|\mathrm{bb}^\infty(E)| = 2^{\aleph_0}$, in (iii) since E is countable, and in (iv) since the strategies α are coded by reals.

Define sets for $\xi, \gamma < 2^{\aleph_0}$, with $\langle\cdot,\cdot\rangle$ a bijection $2^{\aleph_0} \times 2^{\aleph_0} \to 2^{\aleph_0}$,

$$\begin{aligned}\mathcal{D}_\xi &= \{Y : Y \text{ is a diagonalization of } (X_n^\xi) \text{ or } \exists n (Y \perp X_n^\xi)\},\\ \mathcal{F}_{\langle\xi,\gamma\rangle} &= \{Y : \langle Y\rangle \subseteq D_\xi \text{ or } \forall V \preceq X_\gamma(\langle V\rangle \subseteq D_\xi \Rightarrow V \perp Y)\},\\ \mathcal{S}_\xi &= \{Y : Y \in [\alpha_\xi, X_\xi]_f \text{ or } Y \perp X_\xi\}.\end{aligned}$$

Note that the first two sets above are $\preceq$-dense in $\mathrm{bb}^\infty(E)$ by Lemma 2.7, and the third is $\preceq$-dense by Lemma 4.7.

We construct a $\preceq^*$-descending chain (Y_η) of length $2^{\aleph_0}$ in $\mathrm{bb}^\infty(E)$ by transfinite induction on η. For $\eta = 0$, pick Y_0 below conditions in each of $\mathcal{D}_0$, $\mathcal{F}_0$, and $\mathcal{S}_0$. If we have already defined Y_β for all $\beta < \eta$, pick Y_η below each Y_β for $\beta < \eta$ and below conditions in each of $\mathcal{D}_\eta$, $\mathcal{F}_\eta$, and $\mathcal{S}_\eta$. This is possible since $t^* > \eta$.

Let $\mathcal{F}$ be the filter generated by $\{Y_\eta : \eta < 2^{\aleph_0}\}$ in $\mathrm{bb}^\infty(E)$. To see that $\mathcal{F}$ is a (p)-filter, suppose that (X_n^ξ) is a $\preceq^*$-decreasing sequence in $\mathcal{F}$. Let $Y \in \mathcal{F} \cap \mathcal{D}_\xi$. It cannot be the case that $Y \perp X_n^\xi$ for any n, as $\mathcal{F}$ is a filter, so Y must be a diagonalization of (X_n^ξ). Similarly, using the sets $\mathcal{S}_\xi$, $\mathcal{F}$ is strategic.

To see that $\mathcal{F}$ is full, suppose $D_\xi \subseteq E$ and $X_\gamma \in \mathcal{F}$ are such that D_ξ is $\mathcal{F}$-dense below X_γ. Take $Z \in \mathcal{F} \cap \mathcal{F}_{\langle\xi,\gamma\rangle} \neq \emptyset$. By assumption, there is a Y' below both Y and X_γ such that $\langle Y'\rangle \subseteq D_\xi$, but obviously it cannot be that $Y' \perp Y$. Thus, it must be that $\langle Y\rangle \subseteq D_\xi$. □

The next result allows us to obtain (p^+)-filters generically by forcing with $(\mathrm{bb}^\infty(E), \preceq^*)$. Since the dense sets involved are all definable in a simple way from real parameters, they are contained in $\mathbf{L}(\mathbb{R})$. In particular, this establishes (without any large cardinals) the $(\Rightarrow)$ direction of Theorem 1.2.

Lemma 5.4. *For $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ a (p^+)-family forcing with $(\mathcal{H}, \preceq^*)$ adds no new reals and if $\mathcal{G} \subseteq \mathcal{H}$ is $\mathbf{L}(\mathbb{R})$-generic for $(\mathcal{H}, \preceq^*)$, then $\mathcal{G}$ will be a (p^+)-filter. If $\mathcal{H}$ is strategic (has the strong (p)-property, respectively), then $\mathcal{G}$ will also be strategic (have the strong (p)-property, respectively).*

Proof. $\mathcal{H}$ being a (p)-family implies that $(\mathcal{H}, \preceq^*)$ is σ-closed, and, thus, adds no new reals. We use this fact implicitly in what follows. Let $\mathcal{G}$ be as described. To see that $\mathcal{G}$ is full, let $D \subseteq E$ be $\mathcal{G}$-dense below some $X \in \mathcal{G}$. Translating this into the forcing language, there must be an $X' \in \mathcal{G}$, which we may assume is below X, with

$$X' \Vdash_{\mathcal{H}} \forall Y \in \dot{\mathcal{G}} \restriction X \exists Z \preceq Y (\langle Z \rangle \subseteq \check{D}).$$

We claim that the set $\mathcal{D} = \{Z : \langle Z \rangle \subseteq D\}$ is $\preceq$-dense below X' in $\mathcal{H}$. If not, then by fullness of $\mathcal{H}$, D must fail to be $\mathcal{H}$-dense below X'. That is, there is some $Y \in \mathcal{H} \restriction X'$ with no $Z \preceq Y$ such that $\langle Z \rangle \subseteq D$. Then, Y fails to force the statement in the displayed line above, contrary to $Y \preceq X'$. Since $X' \in \mathcal{G}$ and $\mathcal{D}$ is $\preceq$-dense below X' in $\mathcal{H}$, $\mathcal{G} \cap \mathcal{D} \neq \emptyset$, showing that $\mathcal{G}$ is full. The remainder of the proof consists of observing that the relevant $\preceq$-dense sets in Lemmas 2.7, 4.2, and 4.7 are $\preceq$-dense in $\mathcal{H}$ under these hypotheses. $\square$

6. Connections to filters on a countable set

We would like to relate the filters discussed thus far to filters of subsets of a countable set. In our case, the countable set will be $E \setminus \{0\}$, but we will call these filters on E.

Definition 6.1. A filter $\mathcal{F}$ on E is a *block filter* if it has a base consisting of sets of the form $\langle X \rangle$ for $X \in \mathrm{bb}^\infty(E)$.

It is tempting to define a *block ultrafilter* on E to be a block filter on E which is also an ultrafilter. However, unless $|F| = 2$, such objects do not exist: Let $\mathcal{F}$ be a block filter on E. For $A_0, A_1 \subseteq E$ given in Example 2.4, note that $E = A_0 \cup A_0$. But, for every $X \in \mathrm{bb}^\infty(E)$, $\langle X \rangle \cap A_0 \neq \emptyset$ and $\langle X \rangle \cap A_1 \neq \emptyset$, so neither set can be in $\mathcal{F}$.

Let FIN be the set of nonempty finite subsets of $\mathbb{N}$. An ultrafilter $\mathcal{U}$ on FIN is said to be an *ordered union ultrafilter* [11] if it has a base consisting of sets of the form $\langle X \rangle = \{x_{n_0} \cup \cdots \cup x_{n_k} : n_0 < \cdots < n_k\}$, where $X = (x_n)$ is a block sequence in FIN (that is, for all n, $\max(x_n) < \min(x_{n+1})$). The set of infinite block sequences in FIN is denoted by $\mathrm{FIN}^{[\infty]}$. We have, perhaps, overloaded the notation $\langle X \rangle$, but its intended interpretation should be clear from context. If $X = (x_n) \in \mathrm{bb}^\infty(E)$, denote by $\widetilde{X} = (\mathrm{supp}(x_n)) \in \mathrm{FIN}^{[\infty]}$.

If $|F| = 2$, then $E \setminus \{0\}$ can be identified with FIN via each vector's support. Sums of vectors in block position correspond to unions of their supports. As a consequence of Hindman's theorem (Corollary 3.3 in [20]), one can construct (under hypotheses such as CH or MA) ordered union ultrafilters on FIN; these will correspond to block ultrafilters on E.

For the remainder of this section we will consider a general countable field F. The map which takes a vector to its support will provide the connection between this general setting and FIN.

Definition 6.2. Let $\mathcal{F}$ be a block filter on E.

(a) A subset $D \subseteq E$ is *$\mathcal{F}$-dense* if for every $\langle X \rangle \in \mathcal{F}$, there is a $Z \preceq X$ with $\langle Z \rangle \subseteq D$.
(b) $\mathcal{F}$ is *full* if whenever $D \subseteq E$ is $\mathcal{F}$-dense, we have that $D \in \mathcal{F}$.

As in the case for filters in $\mathrm{bb}^\infty(E)$, every full block filter on E is maximal with respect to containment amongst block filters.

The map $s : X \mapsto \langle X \rangle$ takes block sequences to subsets of E. It is straightforward to show that the image of a (full) filter in $\mathrm{bb}^\infty(E)$ under s generates a (full) block filter on E and that the inverse image of a (full) block filter on E is a (full) filter in $\mathrm{bb}^\infty(E)$. By Theorem 5.3 (or Lemma 5.4), it is consistent that such filters exist.

Theorem 6.3. *Suppose that $\mathcal{F}$ is a full block filter on E, and let*

$$\mathrm{supp}(\mathcal{F}) = \{A \subseteq \mathrm{FIN} : \exists F \in \mathcal{F}(A \supseteq \{\mathrm{supp}(v) : v \in F\})\}.$$

Then, $\mathrm{supp}(\mathcal{F})$ *is an ordered union ultrafilter on* FIN.

Proof. Let $A, B \in \mathrm{supp}(\mathcal{F})$, say with $A \supseteq \{\mathrm{supp}(v) : v \in F\}$ and $B \supseteq \{\mathrm{supp}(v) : v \in G\}$, for $F, G \in \mathcal{F}$. Then,

$$\begin{aligned} A \cap B &\supseteq \{s : \exists v \in F \exists w \in G(s = \mathrm{supp}(v) = \mathrm{supp}(w))\} \\ &\supseteq \{\mathrm{supp}(v) : v \in F \cap G\}, \end{aligned}$$

which is in $\mathrm{supp}(\mathcal{F})$, as $F \cap G \in \mathcal{F}$. Since $\mathrm{supp}(\mathcal{F})$ is upwards closed by definition, we have that $\mathrm{supp}(\mathcal{F})$ is a filter on FIN. As $\mathcal{F}$ is a block filter, it follows that $\mathrm{supp}(\mathcal{F})$ has a base consisting of sets $\langle \widetilde{X} \rangle$ for $X \in \mathrm{bb}^\infty(E)$.

It remains to show that $\mathrm{supp}(\mathcal{F})$ is an ultrafilter. Take $A \subseteq \mathrm{FIN}$ such that for all $B \in \mathrm{supp}(\mathcal{F})$, $A \cap B \neq \emptyset$. Let

$$\begin{aligned} D_0 &= \{v \in E : \mathrm{supp}(v) \in A\}, \\ D_1 &= \{v \in E : \mathrm{supp}(v) \notin A\}. \end{aligned}$$

Towards a contradiction, suppose that for all $\langle X \rangle \in \mathcal{F}$, there is a $\langle Z \rangle \subseteq \langle X \rangle$ with $\langle Z \rangle \subseteq D_1$. Since $\mathcal{F}$ is full, there is a $\langle Z \rangle \in \mathcal{F}$ with $\langle Z \rangle \subseteq D_1$. Then, $\langle \widetilde{Z} \rangle \in \mathrm{supp}(\mathcal{F})$, but $A \cap \langle \widetilde{Z} \rangle = \emptyset$, a contradiction.

Thus, there is some $\langle X \rangle \in \mathcal{F}$ such that for no $\langle Z \rangle \subseteq \langle X \rangle$ is $\langle Z \rangle \subseteq D_1$. Take $\langle Y \rangle \in \mathcal{F} \restriction \langle X \rangle$. By Hindman's theorem applied to $\langle \widetilde{Y} \rangle$, there is a $\widetilde{Z} \in \mathrm{FIN}^{[\infty]}$ such that $\langle \widetilde{Z} \rangle \subseteq \langle \widetilde{Y} \rangle$ and either (i) $\langle \widetilde{Z} \rangle \subseteq A$, or (ii) $\langle \widetilde{Z} \rangle \subseteq \langle \widetilde{Y} \rangle \setminus A$.

Take any $Z \preceq Y$ in $\mathrm{bb}^\infty(E)$ whose supports agree with $\widetilde{Z}$, then if (ii) holds, $\langle Z \rangle \subseteq D_1$, contrary to what we know about $\langle X \rangle$. Thus, $\langle \widetilde{Z} \rangle \subseteq A$ and $\langle Z \rangle \subseteq D_0$. Since $\langle Y \rangle \in \mathcal{F} \restriction \langle X \rangle$ was arbitrary, we have that D_0 is $\mathcal{F}$-dense. As $\mathcal{F}$ is full, we can find a $\langle Z \rangle \in \mathcal{F}$ with $\langle Z \rangle \subseteq D_0$. Then, $\langle \widetilde{Z} \rangle \in \mathrm{supp}(\mathcal{F})$ and $\langle \widetilde{Z} \rangle \subseteq A$, so $A \in \mathrm{supp}(\mathcal{F})$. ∎

As a consequence of Theorem 6.3 and the Corollary on p. 87 of [11] we have the following:

Corollary 6.4. *If $\mathcal{F}$ is a full filter on E, then*

$$\min(\mathcal{F}) = \{\{n = \min(\mathrm{supp}(v)) : v \in F\} : F \in \mathcal{F}\},$$
$$\max(\mathcal{F}) = \{\{n = \max(\mathrm{supp}(v)) : v \in F\} : F \in \mathcal{F}\}$$

are selective ultrafilters on $\mathbb{N}$.

As it is consistent that there are no selective ultrafilters [24], we have the following:

Corollary 6.5. *The existence of full block filters on E, and, thus, full filters in* $\mathrm{bb}^\infty(E)$*, is independent of* ZFC.

An ordered union ultrafilter $\mathcal{U}$ on FIN is *stable* [9] if whenever $(\langle X_n \rangle)_{n\in\mathbb{N}}$ is contained in $\mathcal{U}$, for $X_n \in \mathrm{FIN}^{[\infty]}$, there is an $\langle X \rangle \in \mathcal{U}$ with $\langle X \rangle \subseteq^* \langle X_n \rangle$ for all n. Much as selective ultrafilters on $\mathbb{N}$ provide local witnesses to Silver's theorem, selective ultrafilters on FIN witness a theorem of Milliken [32] on analytic partitions of $\mathrm{FIN}^{[\infty]}$. It is easy to see, given Theorem 6.3, that (p^+)-filters in $\mathrm{bb}^\infty(E)$ induce stable ordered union ultrafilters on FIN. See [12], [31], and [42] for (equivalent) alternate definitions of "selective ultrafilter" on FIN.

7. Extending to universally Baire sets and $\mathbf{L}(\mathbb{R})$

In this section, we show that under additional set-theoretic hypotheses, Theorem 1.1 can be extended beyond the analytic sets to obtain Theorems 1.2 and 1.3, provided the families involved are strategic. We begin by noting the following result:

Theorem 7.1 (Rosendal [35])**.** $(\mathsf{MA}(\aleph_1))$ *A union of $\aleph_1$-many strategically Ramsey sets is strategically Ramsey.*

The above theorem, plus existing results in the literature, yields the following:

Theorem 7.2. *Assume that there is a supercompact cardinal.*[5] *Every subset of* $\mathrm{bb}^\infty(E)$ *in* $\mathbf{L}(\mathbb{R})$ *is strategically Ramsey.*[6]

Proof. We follow the proof of Theorem 4 in [28]. The existence of a supercompact cardinal implies that $\mathbf{L}(\mathbb{R})$ is a *Solovay model* in the sense of [13], and Lemma 4.4 of the same reference shows that every set of reals in such a model is a union of $\aleph_1$-many analytic sets. By Theorem 7.1, under $\mathsf{MA}(\aleph_1)$ a union of $\aleph_1$-many strategically Ramsey sets is again strategically Ramsey. Since supercompactness implies (see [37]) that $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ is elementarily equivalent to $\mathbf{L}(\mathbb{R})$ for any set-forcing extension $\mathbf{V}[G]$, and one can force $\mathsf{MA}(\aleph_1)$ in a way which preserves $\aleph_1$, the same is true in $\mathbf{L}(\mathbb{R})$. As analytic sets are strategically Ramsey by Theorem 1.1, every set in $\mathbf{L}(\mathbb{R})$ is as well. $\square$

[5]Throughout this section, the assumption of supercompactness can be weakened to the existence of a proper class of Woodin cardinals; see [27]. We use supercompactness due to its central role in the literature and verbal brevity.

[6]Noé de Rancourt has announced a different proof of this result using methods inspired by determinacy considerations.

Following [33], given a notion of forcing Q and a complete metric space (X, d), we say that a Q-name $\dot{x}$ is a *nice Q-name for an element of $\dot{X}$* if there is a countable collection $\mathcal{D}$ of dense subsets of Q such that $\dot{x}(G)$ (the interpretation of $\dot{x}$ by G) is an element of X whenever G is a $\mathcal{D}$-generic filter for Q. One can show that if $\dot{y}$ is a Q-name and $p \Vdash_Q \dot{y} \in \dot{X}$, then there is a nice Q-name $\dot{x}$ for an element of $\dot{X}$ such that $p \Vdash_Q \dot{y} = \dot{x}$.

A subset $A \subseteq X$ is *universally Baire* if whenever Q is a notion of forcing, there is a Q-name $\dot{A}$ such that for every nice Q-name $\dot{x}$ for an element of $\dot{X}$, there is a countable collection $\mathcal{D}$ of dense subsets of Q such that

(1) $\{q \in Q : q \text{ decides } \dot{x} \in \dot{A}\}$ is in $\mathcal{D}$;
(2) whenever G is $\mathcal{D}$-generic for Q, $\dot{x}(G)$ is in X and $\dot{x}(G)$ is in A if and only if there is a $q \in G$ such that $q \Vdash \dot{x} \in \dot{A}$.

The following result will be the main tool for going beyond the analytic sets.

Theorem 7.3 (Feng, Magidor, and Woodin [16])**.** *Assume that there is a supercompact cardinal. Every set of reals in* $\mathbf{L}(\mathbb{R})$ *is universally Baire.*

Consider the following variant of the infinite asymptotic game: If $A \subseteq E$ is an infinite-dimensional subspace of E, we define $F[A]$ to be the game in which I plays natural numbers n_k, which we assume are increasing, and II plays vectors $y_k \in A$ subject to the constraint $n_k < y_k < y_{k+1}$. By Lemma 2.1, this is well-defined. One can define *outcome*, *strategies*, and the game $F[\vec{x}, A]$ exactly as in §3. Note that the game $F[\vec{x}, \langle X\rangle]$ in this sense, where $X \in \mathrm{bb}^\infty(E)$, coincides with $F[\vec{x}, X]$ from §3, and we will denote it as such.

Suppose that σ is a strategy for I in $F[A]$ and τ a strategy for I in $F[B]$, where $B \subseteq A$ are infinite-dimensional subspaces. We write $\tau \geq \sigma$ if for all $\vec{y}$ in the domain of τ, $\tau(\vec{y}) \geq \sigma(\vec{y})$ ($\sigma(\vec{y})$ is well-defined by induction). Observe that if $\tau \geq \sigma$, then whenever (y_n) is an outcome of $F[B]$ where I follows τ, then it is also an outcome of $F[A]$ where I follows σ. In particular, if σ is a strategy for playing into a set $\mathbb{A}$, then so is τ.

If σ is a strategy for I in $F[A]$ and $B \subseteq A$ as above, then denote by $\sigma \upharpoonright B$ the restriction of σ to the part of its domain contained in B, a strategy for I in $F[B]$. Clearly, $\sigma \upharpoonright B \geq \sigma$. Let ε be the strategy in $F[E]$ where I plays n on the nth move. Then, for all A and strategies σ for I in $F[A]$, we have that $\sigma \geq \varepsilon \upharpoonright A$.

Definition 7.4. Let $\mathbb{P}$ be the set of all triples $(\vec{x}, A, \sigma)$, where $\vec{x} \in \mathrm{bb}^{<\infty}(E)$, A is an infinite-dimensional subspace of E, and σ is a strategy for I in $F[\vec{x}, A]$. We say that $(\vec{y}, B, \tau) \leq (\vec{x}, A, \sigma)$ if

(i) $\vec{y} = \vec{x}^\frown(y_0, \ldots, y_{k-1})$ where $y_0, \ldots, y_{k-1}$ are the first k moves by II in a round of $F[\vec{x}, A]$ where I follows σ;
(ii) $B \subseteq A$;
(iii) $\tau(\cdot) \geq \sigma((y_0, \ldots, y_k)^\frown \cdot)$.

The ordering $\leq$ on $\mathbb{P}$ is reflexive and transitive, though fails to be antisymmetric. We treat $\mathbb{P}$ as a notion of forcing. Note that $\mathbb{P}$ has a maximal element, namely, $(\emptyset, E, \varepsilon)$. If $X \in \mathrm{bb}^\infty(E)$, we write $(\vec{x}, X, \sigma)$ for $(\vec{x}, \langle X\rangle, \sigma)$. If $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a family, let

$$\mathbb{P}(\mathcal{H}) = \{(\vec{x}, A, \sigma) \in \mathbb{P} : \exists X \in \mathcal{H}(\langle X\rangle \subseteq A)\}$$

be a suborder of $\mathbb{P}$. Note that if $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is a family, then the set of conditions $(\vec{x}, X, \sigma)$ where $X \in \mathcal{H}$ is dense in $\mathbb{P}(\mathcal{H})$.

For $(\vec{x}, A, \sigma) \in \mathbb{P}$, let

$$[\vec{x}, A, \sigma] = \{Y \in \mathrm{bb}^\infty(E) : Y \text{ is an outcome of } F[\vec{x}, A] \text{ where I follows } \sigma\}.$$

We collect some basic properties of $\mathbb{P}$ in the following lemma:

Lemma 7.5.

(a) *If $(\vec{y}, B, \tau) \leq (\vec{x}, A, \sigma)$ in $\mathbb{P}$, then $[\vec{y}, B, \tau] \subseteq [\vec{x}, A, \sigma]$. Conversely, if $[\vec{y}, B, \tau] \subseteq [\vec{x}, A, \sigma]$, then $(\vec{y}, B, \tau)$ is below $(\vec{x}, A, \sigma)$ in the separative quotient of $\mathbb{P}$.*

(b) *If $(\vec{x}, A, \sigma) \in \mathbb{P}$, then the set $[\vec{x}, A, \sigma]$ is (topologically) closed.*

(c) *If $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ is a filter, then $\mathbb{P}(\mathcal{F})$ is σ-centered.*

Proof. (a) The first part follows from our observations about the ordering on strategies for I. For the converse, suppose that $[\vec{y}, B, \tau] \subseteq [\vec{x}, A, \sigma]$. Then, every outcome of $F[\vec{y}, B]$ where I follows τ is an outcome of $F[\vec{x}, A]$ where I follows σ. In particular, $\vec{y} = \vec{x}^\frown(y_0, \ldots, y_{k-1})$ where $y_0, \ldots, y_{k-1}$ are the first k moves by II in a round of $F[\vec{x}, A]$ where I follows σ.

We claim that $B/m \subseteq A$, where $m = \max\{\mathrm{supp}(\vec{y}), \tau(\emptyset)\}$ and $B/m = \{y \in B : y > m\}$. To see this, note that for any $y \in B/m$, there is an outcome $\vec{y}^\frown y^\frown Z \in [\vec{y}, B, \tau]$ and, thus, in $[\vec{x}, A, \sigma]$. In particular, $y \in A$.

By our choice of m, $\tau \restriction B/m = \tau$. So, $(\vec{y}, B/m, \tau) \leq (\vec{x}, A, \sigma)$ and the sets of extensions of $(\vec{y}, B/m, \tau)$ and $(\vec{y}, B, \tau)$ coincide. Thus, their images in the separative quotient of $\mathbb{P}$ coincide.

(b) If $Y = (y_n) \notin [\vec{x}, A, \sigma]$, then either $\vec{x} \not\sqsubseteq Y$, or there is some least n such that y_n is not a valid response to $\sigma(y_0, \ldots, y_{n-1})$, i.e., $y_n \notin A$ or $y_n \not> \sigma(y_0, \ldots, y_{n-1})$. As E is discrete, these are open conditions.

(c) Suppose that $(\vec{x}, A, \sigma)$ and $(\vec{x}, B, \tau)$ are both in $\mathbb{P}(\mathcal{F})$. There are $X, Y \in \mathcal{F}$ with $\langle X \rangle \subseteq A$ and $\langle Y \rangle \subseteq B$. Since $\mathcal{F}$ is a filter, there is a $Z \in \mathcal{F}$ below both. Let ρ be the strategy for I in $F[Z]$ given by $\rho(\vec{z}) = \max\{\sigma(\vec{z}), \tau(\vec{z})\}$. Then, $(\vec{x}, Z, \rho) \in \mathbb{P}(\mathcal{F})$ and extends both $(\vec{x}, A, \sigma)$ and $(\vec{x}, B, \tau)$. Since there are only countably many such $\vec{x}$, this shows that $\mathbb{P}(\mathcal{F})$ is σ-centered. $\square$

Given a family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ and a sufficiently generic filter G for $\mathbb{P}(\mathcal{H})$, we denote by $X_{\mathrm{gen}}(G)$ the *generic block sequence* determined by G,

$$X_{\mathrm{gen}}(G) = \bigcup\{\vec{x} : \exists(\vec{x}, A, \sigma) \in G\}.$$

In what follows, G will be $\mathcal{D}$-generic for some countable collection of dense sets $\mathcal{D}$ coming from the definition of universally Baire, and so G can be taken to be in $\mathbf{V}$. Any such $\mathcal{D}$ will ensure that $X_{\mathrm{gen}}(G)$ is infinite. We write $\dot{X}_{\mathrm{gen}}$ to be a nice (as defined above) $\mathbb{P}(\mathcal{H})$-name for this block sequence.

Lemma 7.6. *Let $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ be a filter, let $\mathcal{D}$ be a collection of dense subsets of $\mathbb{P}(\mathcal{F})$, and let G be a $\mathcal{D}$-generic filter for $\mathbb{P}(\mathcal{F})$. For $X = X_{\mathrm{gen}}(G)$, the set*

$$G(X) = \{(\vec{x}, A, \sigma) \in \mathbb{P}(\mathcal{F}) : X \in [\vec{x}, A, \sigma]\}$$

is a $\mathcal{D}$-generic filter for $\mathbb{P}(\mathcal{F})$ which contains G and $X_{\mathrm{gen}}(G(X)) = X$.

Proof. By Lemma 7.5(a), $G(X)$ is closed upwards. If $(\vec{x}, A, \sigma) \in G$, then one can build a decreasing sequence $(\vec{x}_n, A_n, \sigma_n)$ in G with $(\vec{x}_0, A_0, \sigma_0) = (\vec{x}, A, \sigma)$, $|\vec{x}_n| \to \infty$ as $n \to \infty$, and X is the union of the $\vec{x}_n$. By construction, X must be in $[\vec{x}, A, \sigma]$. This shows that $G \subseteq G(X)$, and consequently the latter is $\mathcal{D}$-generic.

It remains to show that $G(X)$ is a filter. Take $(\vec{x}, A, \sigma), (\vec{y}, B, \tau) \in G(X)$. As X has both $\vec{x}$ and $\vec{y}$ as an initial segment, one must be an initial segment of the other, say $\vec{x} \sqsubseteq \vec{y}$, and the part of $\vec{y}$ above $\vec{x}$ is a sequence of moves by II against σ. As $\mathcal{F}$ is a filter, $A \cap B$ is infinite-dimensional. Let ρ be the strategy for I in $F[A \cap B]$ given by $\rho(\vec{v}) = \max\{\sigma(\vec{v}), \tau(\vec{v})\}$ for $\vec{v}$ in its domain. Then, $(\vec{y}, A \cap B, \rho)$ is below both $(\vec{x}, A, \sigma)$ and $(\vec{y}, B, \tau)$. Moreover, $X \in [\vec{y}, A \cap B, \rho]$, and so $(\vec{y}, A \cap B, \rho) \in G(X)$. That $X_{\mathrm{gen}}(G(X)) = X$ is clear. $\square$

A consequence of Lemma 7.6 is that if G is generic for $\mathbb{P}(\mathcal{F})$ over a model of a sufficient fragment of ZFC, then $G(X) = G$, though we will not make use of this here.

Lemma 7.7. *Let $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ be a filter, and let $\mathcal{D}$ be a countable collection of dense open subsets of $\mathbb{P}(\mathcal{F})$.*

(a) *For any $(\vec{x}, A, \sigma) \in \mathbb{P}(\mathcal{F})$, the set*

$$\mathbb{G}_{\mathcal{D},(\vec{x},A,\sigma)} = \{X_{\mathrm{gen}}(G) : G \text{ a } \mathcal{D}\text{-generic filter for } \mathbb{P}(\mathcal{F}) \text{ with } (\vec{x}, A, \sigma) \in G\}$$

is an $F_{\sigma\delta}$ subset of $\mathrm{bb}^\infty(E)$.

(b) *If $X \in \mathcal{F}$, then for no $Y \in \mathcal{F} \upharpoonright X$ does I have a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{G}_{\mathcal{D},(\vec{x},X,\sigma)})^c$.*

(c) *If $\mathcal{F}$ is a (p^+)-filter and $X \in \mathcal{F}$, then there is a $Y \in \mathcal{F} \upharpoonright X$ for which II has a strategy in $G[\vec{x}, Y]$ for playing into $\mathbb{G}_{\mathcal{D},(\vec{x},X,\sigma)}$.*

Proof. (a) Enumerate $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$. Since $\mathbb{P}(\mathcal{F})$ is ccc by Lemma 7.5(c), each D_n contains a countable maximal antichain A_n below $(\vec{x}, A, \sigma)$. We claim that

$$\mathbb{G}_{\mathcal{D},(\vec{x},A,\sigma)} = \bigcap_{n \in \mathbb{N}} \bigcup \{[\vec{y}, B, \tau] : (\vec{y}, B, \tau) \in A_n\},$$

which is $F_{\sigma\delta}$, as each set $[\vec{y}, Y, \tau]$ is closed by Lemma 7.5(b).

If $X = X_{\mathrm{gen}}(G)$ where G is a $\mathcal{D}$-generic filter with $(\vec{x}, A, \sigma) \in G$, then for each n, $G \cap A_n \neq \emptyset$, say with $(\vec{y}_n, B_n, \tau_n) \in G \cap A_n$. By Lemma 7.6, for each n, $X \in [\vec{y}_n, B_n, \tau_n]$, and so X is in the set on right-hand side of the above-displayed line. For the reverse inclusion, suppose that X is in set on the right-hand side. Then, by Lemma 7.6, $G(X)$ is a $\mathcal{D}$-generic filter containing $(\vec{x}, A, \sigma)$ for which $X_{\mathrm{gen}}(G(X)) = X$, and so $X \in \mathbb{G}_{\mathcal{D},(\vec{x},A,\sigma)}$.

(b) Let $X \in \mathcal{F}$ and $Y \in \mathcal{F} \upharpoonright X$ be given. Towards a contradiction, suppose that ρ is a strategy for I in $F[\vec{x}, Y]$ for playing into $(\mathbb{G}_{\mathcal{D},(\vec{x},X,\sigma)})^c$. We may assume $\rho \geq \sigma \upharpoonright Y$. Consider the following play of $F[\vec{x}, Y]$: I plays $\rho(\emptyset) = n_0$. Pick

$$p_0 = (\vec{x}^\frown(y_0^0, \ldots, y_{k_0}^0), B^0, \rho^0) \leq (\vec{x}, Y, \rho) \leq (\vec{x}, X, \sigma)$$

in D_0, and let II play $y_0 = y_0^0$. Note that this is a valid move by definition of $\leq$ in $\mathbb{P}(\mathcal{F})$. Next, I plays $\rho(y_0) = n_1$. Pick

$$p_1 = (\vec{x}^\frown(y_0^0, \ldots, y_{k_0}^0)^\frown(y_0^1, \ldots, y_{k_1}^1), B^1, \rho^1) \leq (\vec{x}^\frown(y_0^0, \ldots, y_{k_0}^0), B^0, \rho^0)$$

in D_1, and let II play $y_1 = y_1^0$ if $k_0 \geq 1$, and $y_1 = y_0^1$ otherwise. Continuing in this fashion, we build an outcome (y_n). Observe that (y_n) must be in $\mathbb{G}_{\mathcal{D},(\vec{x},X,\sigma)}$: the conditions p_n picked in D_n above form a $\mathcal{D}$-generic chain in $\mathbb{P}(\mathcal{F})$ below $(\vec{x}, X, \sigma)$, thus, generate a $\mathcal{D}$-generic filter G with $X_{\mathrm{gen}}(G) = (y_n)$ and $(\vec{x}, X, \sigma) \in G$. This contradicts our choice of ρ.

(c) follows from (a) and (b) by an application of Theorem 1.1. $\square$

Lemma 7.8. *Let $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ be a (p^+)-filter. If $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is universally Baire, then for any $\vec{x} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathcal{F}$, there is a $Y \in \mathcal{F} \upharpoonright X$ such that II has a strategy in $G[\vec{x}, Y]$ for playing into one of $\mathbb{A}$ or $\mathbb{A}^c$.*

Proof. Let $X \in \mathcal{F}$ be given. We may assume that $\vec{x}$ is $\emptyset$. Recall, for $\vec{y} \in \mathrm{bb}^{<\infty}(E)$ and $Y \in \mathcal{F}$, Definition 3.2 of $(\vec{y}, Y)$ being good/bad/worse (for the set $\mathbb{A}$). By Lemma 3.3, there is a $Y \in \mathcal{F} \upharpoonright X$ such that either $(\emptyset, Y)$ is good or I has a strategy σ in $F[Y]$ to play into the set

$$\{(z_n) : \forall n (z_0, \ldots, z_n, Y) \text{ is worse}\}.$$

In the former case we are done, so we assume the latter.

Since $\mathbb{A}$ is universally Baire, we may let $\dot{\mathbb{A}}$ be a $\mathbb{P}(\mathcal{F})$-name for $\mathbb{A}$, and let $\mathcal{D}$ be a countable collection of dense open subsets of $\mathbb{P}(\mathcal{F})$ such that

(i) $\{q \in \mathbb{P}(\mathcal{F}) : q \text{ decides } \dot{X}_{\mathrm{gen}} \in \dot{\mathbb{A}}\}$ is in $\mathcal{D}$, and
(ii) whenever G is $\mathcal{D}$-generic in $\mathbb{P}(\mathcal{F})$, $X_{\mathrm{gen}}(G)$ is in $\mathrm{bb}^\infty(E)$ and $X_{\mathrm{gen}}(G)$ is in $\mathbb{A}$ if and only if there is a $q \in G$ such that $q \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \in \dot{\mathbb{A}}$.

Thus, if G is $\mathcal{D}$-generic for $\mathbb{P}(\mathcal{F})$, contains $(\emptyset, Y, \sigma)$, and $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \notin \dot{\mathbb{A}}$, then $X_{\mathrm{gen}}(G) \notin \mathbb{A}$. We claim that $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \notin \dot{\mathbb{A}}$.

Suppose not, then there is a $(\vec{y}, Z, \tau) \leq (\emptyset, Y, \sigma)$, with $Z \in \mathcal{F}$ such that $(\vec{y}, Z, \tau) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \in \dot{\mathbb{A}}$. Applying Lemma 7.7(c), take $W \in \mathcal{F} \upharpoonright Z$ such that II has a strategy α in $G[\vec{y}, W]$ for playing into $\mathbb{G}_{\mathcal{D},(\vec{y},Z,\tau)}$. We claim that $\mathbb{G}_{\mathcal{D},(\vec{y},Z,\tau)} \subseteq \mathbb{A}$. Let (z_n) be in $\mathbb{G}_{\mathcal{D},(\vec{y},Z,\tau)}$. Take G a $\mathcal{D}$-generic filter for which $(z_n) = X_{\mathrm{gen}}(G)$ and $(\vec{y}, Z, \tau) \in G$. Since $(\vec{y}, Z, \tau) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \in \dot{\mathbb{A}}$, we have that $(z_n) \in \mathbb{A}$. Thus, α is a strategy for II in $G[\vec{y}, W]$ for playing into $\mathbb{A}$. This, however, contradicts the fact that σ ensures that $(\vec{y}, Z)$ is bad.

Thus, $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \notin \dot{\mathbb{A}}$. Then, exactly as in the preceding paragraph, we may find $W \in \mathcal{F} \upharpoonright Y$ such that II has a strategy in $G[W]$ for playing into $\mathbb{G}_{\mathcal{D},(\emptyset,Y,\sigma)}$, and, thus, into $\mathbb{A}^c$. ☐

While the symmetric result in Lemma 7.8 is appealing on its own and applies to all analytic sets (being universally Baire [16]) in ZFC, it is not a true "dichotomy" as II can easily have strategies for playing into both $\mathbb{A}$ and $\mathbb{A}^c$.

One consequence of Lemma 7.7 and the proof of Lemma 7.8 is that, given (p^+)-filter $\mathcal{F}$ and a universally Baire set $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$, there is always an $X \in \mathcal{F}$ such that one of $\mathbb{A}$ or $\mathbb{A}^c$ contains an $F_{\sigma\delta}$ set $\preceq$-dense below X.

We can now complete the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We have already proven the $(\Rightarrow)$ direction in Lemma 5.4. For the remaining direction, let $\mathbb{D} \subseteq \mathrm{bb}^\infty(E)$ be a $\preceq$-dense open set which is in $\mathbf{L}(\mathbb{R})$, and, thus, universally Baire by Theorem 7.3. By Lemma 7.8, there is an $X \in \mathcal{F}$ such that II has a strategy in $G[X]$ for playing into either $\mathbb{D}$ or $\mathbb{D}^c$. By Lemma 4.4, the latter can never occur. Thus, II has a strategy in $G[X]$ for playing into $\mathbb{D}$. Since $\mathcal{F}$ is strategic, there is a play by this strategy, say Z, with $Z \in \mathbb{D} \cap \mathcal{F} \neq \emptyset$. ☐

Lemma 7.9. *Assume that there is a supercompact cardinal. Let $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ be a strategic (p^+)-filter. Every subset of $\mathrm{bb}^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is $\mathcal{F}$-strategically Ramsey.*

Proof. Let $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ be in $\mathbf{L}(\mathbb{R})$, and fix $\vec{x} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathcal{F}$. By Theorem 7.2, the set of all $Y \preceq X$ witnessing that $\mathbb{A}$ is strategically Ramsey is $\preceq$-dense below X, and is clearly in $\mathbf{L}(\mathbb{R})$. Since $\mathcal{F}$ is $\mathbf{L}(\mathbb{R})$-generic, $\mathcal{F}$ must contain such a Y. ☐

Proof of Theorem 1.3. Let $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ be in $\mathbf{L}(\mathbb{R})$, and fix $\vec{x} \in \mathrm{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$. Let $\mathcal{G}$ be $\mathbf{V}$-generic for $(\mathcal{H}, \preceq^*)$ and contain X. By Lemma 5.4, $\mathcal{G}$ is a strategic (p^+)-filter in $\mathbf{V}[\mathcal{G}]$. By Lemma 7.9, there is a $Y \in \mathcal{G} \upharpoonright X$ witnessing that $\mathbb{A}$ is strategically Ramsey in $\mathbf{V}[\mathcal{G}]$. Since forcing with $(\mathcal{H}, \preceq^*)$ adds no new reals, Y witnesses that $\mathbb{A}$ is $\mathcal{H}$-strategically Ramsey in $\mathbf{V}$. $\square$

8. Normed spaces and a local Gowers dichotomy

We now consider the case when E is a countably infinite-dimensional normed vector space, with *normalized* basis (e_n) (that is, $\|e_n\| = 1$ for all n), over a countable subfield F of $\mathbb{C}$ such that the norm takes values in F. If V is a subspace of E, let $S(V) = \{x \in V : \|x\| = 1\}$.

Let $\mathrm{bb}_1^\infty(E) = \{(x_n) \in \mathrm{bb}^\infty(E) : \forall n(\|x_n\| = 1)\}$, and let $\mathrm{bb}_1^{<\infty}(E) = \{\vec{x} \in \mathrm{bb}^{<\infty}(E) : \forall n < |\vec{x}|(\|x_n\| = 1)\}$. For $X \in \mathrm{bb}^\infty(E)$, let $[X] = \{Y \in \mathrm{bb}_1^\infty(E) : Y \preceq X\}$. Taking E discrete, $\mathrm{bb}_1^\infty(E)$ is a closed subset of the Polish space $\mathrm{bb}^\infty(E)$, thus, itself Polish.

For $X = (x_n), Y = (y_n) \in \mathrm{bb}_1^\infty(E)$, and $\Delta = (\delta_n)$ a sequence of positive real numbers written $\Delta > 0$, we write $d(X, Y) \le \Delta$ if for all n, $\|x_n - y_n\| \le \delta_n$. Given $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ and $\Delta > 0$, let

$$\mathbb{A}_\Delta = \{Y \in \mathrm{bb}_1^\infty(E) : \exists X \in \mathbb{A}(d(X,Y) \le \Delta)\},$$

the Δ-*expansion* of $\mathbb{A}$. We collect a few useful properties of Δ-expansions in a lemma which will be used tacitly in what follows. The proof is left to the reader.

Lemma 8.1. *Let* $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ *and* $\Delta > 0$.

(a) *If* $\mathbb{A} = \bigcup_{i \in I} \mathbb{A}_i$, *then* $\mathbb{A}_\Delta = \bigcup_{i \in I} (\mathbb{A}_i)_\Delta$.
(b) *If* $\mathbb{A}$ *is analytic, then so is* $\mathbb{A}_\Delta$.
(c) $(\mathbb{A}_\Delta)^c \subseteq ((\mathbb{A}_\Delta)^c)_\Delta \subseteq \mathbb{A}^c$.
(d) *If* $0 < \Gamma \le \Delta/2$, *then* $((\mathbb{A}_\Delta)^c)_\Gamma \subseteq (\mathbb{A}_\Gamma)^c$.

$\square$

The notions of *family*, *filter*, *fullness*, (p)-*property*, etc., in $\mathrm{bb}_1^\infty(E)$ are defined exactly as for $\mathrm{bb}^\infty(E)$ in §2. Moreover, all of the results established in the previous sections could have been carried out in $\mathrm{bb}_1^\infty(E)$ in the event that E is normed. The only necessary modification is that in the games $G[\vec{x}, X]$ and $F[\vec{x}, X]$, the two players must play normalized block sequences and vectors, respectively. This will be assumed in what follows.

For $D \subseteq S(E)$, let

$$D_\epsilon = \{x \in S(E) : \exists y \in D(\|x - y\| \le \epsilon)\}.$$

We weaken the notion of fullness to the following approximate version.[7]

Definition 8.2. A family $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ is *almost full* if whenever $D \subseteq S(E)$ and $X \in \mathcal{H}$ are such that D is $\mathcal{H}$-dense below X (that is, for all $Y \in \mathcal{H} \upharpoonright X$, there is a $Z \preceq Y$ with $S(\langle Z \rangle) \subseteq D$), then for any $\epsilon > 0$, there is a $Z \in \mathcal{H} \upharpoonright X$ with $S(\langle Z \rangle) \subseteq D_\epsilon$.

Definition 8.3. If a family has the (p)-property and is almost full, we call it a (p^*)-*family*. Likewise for (p^*)-*filter*, *strategic* (p^*)-*family*, etc.

[7] While this hampers our ability to reuse results of §3 and §7, we hope that it will enable further applications. An elementary proof of Proposition 8.21, without the hypothesis of being "strategic", would greatly simplify the situation in the cases of interest.

The following is a discrete version of Gowers's weakly Ramsey property [19] relativized to a family $\mathcal{H}$.

Definition 8.4. Given a family $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$, a set $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ is *$\mathcal{H}$-weakly Ramsey* if for every $\Delta > 0$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

(i) $[Y] \subseteq \mathbb{A}^c$, or
(ii) II has a strategy in $G[Y]$ for playing into $\mathbb{A}_\Delta$.

The first goal of this section is to show that for certain (p^*)-families $\mathcal{H}$, analytic sets in $\mathrm{bb}_1^\infty(E)$ are $\mathcal{H}$-weakly Ramsey. We begin with variants of Lemmas 3.3 and 3.4, and Theorem 1.1, for (p^*)-families. Since dealing with both families and Δ-expansions requires some care, we include proofs of these results. As in §3, they are very similar to those in [35].

Definition 8.5. Given a family $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$, $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(H)$, and $\Delta > 0$ for $\vec{y} \in \mathrm{bb}_1^{<\infty}(E)$ and $Y \in \mathcal{H}$, we say the pair $(\vec{y}, Y)$ is *Δ-good*/*Δ-bad*/*Δ-worse* if it is good/bad/worse for the set $\mathbb{A}_\Delta$ (in the sense of Definition 3.2). Further,

(1) $(\vec{y}, Y)$ is *Δ^*-good* if it is $\Delta(|\vec{y}|)$-good;
(2) $(\vec{y}, Y)$ is *Δ^*-bad* if it is $\Delta(|\vec{y}|)$-bad;
(3) $(\vec{y}, Y)$ is *Δ^*-worse* if it is Δ^*-bad and there is an n such that for all $v \in S(\langle Y/n\rangle)$, $(\vec{y}^\frown v, Y)$ is Δ^*-bad.

Here, $\Delta(m) = (\delta_0/2, \delta_1/2, \ldots, \delta_{m-1}/2, \delta_m, \delta_{m+1}, \ldots)$.

Note that Δ^*-good implies Δ-good and Δ^*-bad implies $\Delta/2$-bad.

Lemma 8.6. *If $\mathcal{H}$ is a (p^*)-family and $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$, then for every $\vec{x} \in \mathrm{bb}^{<\infty}(E)$, $X \in \mathcal{H}$ and $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either*

(i) *$(\vec{x}, Y)$ is Δ-good, or*
(ii) *I has a strategy in $F[\vec{x}, Y]$ for playing into*

$$\{(z_n) : \forall n(\vec{x}^\frown(z_0, \ldots, z_n), Y) \text{ is } \Delta/2\text{-bad}\}.$$

Proof. Let $\mathcal{H}$, $\mathbb{A}$, $X \in \mathcal{H}$, and $\Delta > 0$ be given. As in the proof of Lemma 3.3, for any $\vec{y}$ and $\Gamma > 0$, the set

$$\mathcal{D}_{\vec{y}}^\Gamma = \{Y : (\vec{y}, Y) \text{ is } \Gamma\text{-good or } \Gamma\text{-bad}\}$$

is $\preceq$-dense open in $\mathcal{H}$, and if $(\vec{y}, Y)$ is Γ-bad, then for every $V \in \mathcal{H} \upharpoonright Y$, there is a $Z \preceq V$ such that for all $x \in S(\langle Z\rangle)$, $(\vec{y}^\frown x, Y)$ is not Γ-good.

Claim. For any $\vec{y} \in \mathrm{bb}_1^{<\infty}(E)$, the set

$$\mathcal{E}_{\vec{y}} = \{Y : (\vec{y}, Y) \text{ is } \Delta^*\text{-good or } \Delta^*\text{-worse}\}$$

is $\preceq$-dense open in $\mathcal{H}$.

Proof of claim. Let $Y \in \mathcal{H}$. By diagonalizing over the sets $\mathcal{D}_{\vec{z}}^{\Delta(|\vec{z}|)}$, we may assume that for all $\vec{z}$, $(\vec{z}, Y)$ is Δ^*-good or Δ^*-bad. Assume that $(\vec{y}, Y)$ is Δ^*-bad. Let

$$D = \{x \in S(E) : (\vec{y}^\frown x, Y) \text{ is not } \Delta(|\vec{y}|)\text{-good}\}.$$

Take $\epsilon = \delta_{|\vec{y}|}/2$. By almost fullness, there is a $Z \in \mathcal{H} \upharpoonright Y$ such that $S(\langle Z\rangle) \subseteq D_\epsilon$. Given $z \in S(\langle Z\rangle)$, pick $z' \in D$ with $\|z - z'\| < \epsilon$. If $(\vec{y}^\frown z, Z)$ is Δ^*-good, then there is a strategy α for II in $G[\vec{y}^\frown z, Z]$ for playing into $\mathbb{A}_{\Delta(|\vec{y}|+1)}$. We may assume that all plays according to α are above z and z', so we can treat α as a strategy α' for II in $G[\vec{y}^\frown z', Z]$. If $\vec{y}^\frown z'^\frown W$ is an outcome of α', then $\vec{y}^\frown z^\frown W$ is an outcome of α, and, thus, in $\mathbb{A}_{\Delta(|\vec{y}|+1)}$. By our choice of ϵ, it follows that $\vec{y}^\frown z'^\frown W$ is in $\mathbb{A}_{\Delta(|\vec{y}|)}$. Then, $(\vec{y}^\frown z', Z)$ is $\Delta(|\vec{y}|)$-good, contradicting that $z' \in D$. Thus, $(\vec{y}^\frown z, Z)$ is Δ^*-bad, and $(\vec{y}, Z)$ is Δ^*-worse. $\square_{\text{(claim.)}}$

Returning to the proof of the lemma, assume that $\vec{x} = \emptyset$. By the claim, we can find $Y \in \mathcal{H} \upharpoonright X$ such that for all $\vec{y}$, $(\vec{y}, Y)$ is either Δ^*-good or Δ^*-worse. If $(\emptyset, Y)$ is Δ^*-good, we are done, so assume that it is Δ^*-worse. In this case, we define a strategy for I in $F[Y]$ for playing into $\{(z_n) : \forall n (z_0, \ldots, z_n, Y) \text{ is } \Delta^*\text{-worse}\}$ exactly as in the proof of Lemma 3.3. $\square$

Lemma 8.7 (cf. Lemma 2 in [35])**.** *Let $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ be a (p^*)-family. Given $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ open, $x \in \mathrm{bb}_1^{<\infty}(E)$, $X \in \mathcal{H}$, and $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright Y$ such that either*

(i) *I has a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or*
(ii) *II has a strategy in $G[\vec{x}, Y]$ for playing into $\mathbb{A}_\Delta$.*

Proof. The proof is similar to Lemma 3.4, using Lemma 8.6. $\square$

Lemma 8.8 (cf. Lemma 4 in [35])**.** *Let $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ be a (p^*)-family. Suppose that $\mathbb{A} = \bigcup_{n\in\mathbb{N}} \mathbb{A}_n$, each $\mathbb{A}_n \subseteq \mathrm{bb}_1^\infty(E)$. Let $\vec{x}$, $X \in \mathcal{H}$, and $\Delta > 0$ be given. Then, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either*

(i) *I has a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or*
(ii) *II has a strategy in $G[Y]$ for playing into*

$$\{(z_k) : \exists n \forall V \in \mathcal{H} \upharpoonright Y (\textit{I has no strategy in } F[\vec{x}^\frown(z_0, \ldots, z_n), V] \\ \textit{for playing into } ((\mathbb{A}_n)_\Delta)^c)\}.$$

Proof. For $Y \in \mathcal{H}$, $\vec{y} \in \mathrm{bb}^{<\infty}(E)$, and $n \in \mathbb{N}$, we say $(\vec{y}, n)$ Γ-*accepts* Y if I has a strategy in $F[\vec{y}, Y]$ for playing into $((\mathbb{A}_n)_\Gamma)^c$ and $(\vec{y}, n)$ Γ-*rejects* Y if for all $Z \in \mathcal{H} \upharpoonright Y$, $(\vec{y}, n)$ does not Γ-accept Z. Both acceptance and rejection are $\preceq^*$-hereditary in $\mathcal{H}$, and the sets

$$\mathcal{D}^\Gamma_{\vec{y},n} = \{Y : (\vec{y}, n) \ \Gamma\text{-accepts or } \Gamma\text{-rejects } Y\}$$

are clearly $\preceq$-dense open in $\mathcal{H}$. By the (p)-property, we can find $Y \in \mathcal{H} \upharpoonright X$ such that for all $\vec{y}$ and n, $(\vec{y}, n)$ either $\Delta/2$-accepts or $\Delta/2$-rejects Y. Put

$$R = \{(z_k) : \exists n (\vec{x}^\frown(z_0, \ldots, z_n), n) \ \Delta/2\text{-rejects } Y\},$$

and notice that R is open in $\mathrm{bb}_1^\infty(E)$. By Lemma 8.7, there is $Y' \in \mathcal{H} \upharpoonright Y$ such that either II has a strategy in $G[Y']$ for playing into $R_{\Delta/2}$, or I has a strategy in $F[Y']$ for playing into $(R_{\Delta/4})^c \subseteq R^c$. In the first case, suppose that (z_k) is an outcome of II's strategy. Then, there is (z'_k) with $\|z_k - z'_k\| \leq \delta_k/2$ for all k, and an n such that $(\vec{x}^\frown(z'_0, \ldots, z'_n), n)$ $\Delta/2$-rejects Y. We claim $(\vec{x}^\frown(z_0, \ldots, z_n), n)$ Δ-rejects Y. If not, then for some $Z \in \mathcal{H} \upharpoonright Y$, I has a strategy in $F[\vec{x}^\frown(z_0, \ldots, z_n), Z]$ for playing into $((\mathbb{A}_n)_\Delta)^c$. This yields a strategy for I in $F[\vec{x}^\frown(z'_0, \ldots, z'_n), Z]$ for playing into $(((\mathbb{A}_n)_\Delta)^c)_{\Delta/2}$. By Lemma 8.1(d), $(((\mathbb{A}_n)_\Delta)^c)_{\Delta/2} \subseteq ((\mathbb{A}_n)_{\Delta/2})^c$, and so $(\vec{x}^\frown(z'_0, \ldots, z'_n), n)$ fails to $\Delta/2$-reject Y, a contradiction. Thus, (z_k) is as desired for (ii).

Suppose that I has a strategy σ in $F[Y]$ for playing into $(R_{\Delta/4})^c \subseteq R^c$. In particular, I plays (z_k) such that for all n, I has a strategy $\sigma_{(z_0,\ldots,z_n)}$ in $F[\vec{x}^\frown(z_0, \ldots, z_n), Y]$ to play into $((\mathbb{A}_n)_{\Delta/2})^c$. As in the proof of Lemma 4 in [35], we successively put more strategies for I into play, and obtain a strategy for playing into $\bigcap_n((\mathbb{A}_n)_{\Delta/2})^c = (\mathbb{A}_{\Delta/2})^c$. $\square$

Theorem 8.9 (cf. Theorem 5 in [35])**.** *Let $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ be a (p^*)-family. If $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ is analytic, $\Delta > 0$, $\vec{x} \in \mathrm{bb}_1^{<\infty}(E)$, and $X \in \mathcal{H}$, then there is a $Y \in \mathcal{H} \restriction Y$ such that either*

(i) *I has a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or*
(ii) *II has a strategy in $G[\vec{x}, Y]$ for playing into $\mathbb{A}_\Delta$.*

Proof. We consider the case when $\vec{x} = \emptyset$. Let $F : \mathbb{N}^\mathbb{N} \to \mathbb{A}$ be a continuous surjection and for each $s \in \mathbb{N}^{<\mathbb{N}}$, let $\mathbb{A}_s = F''(N_s)$ where $N_s = \{\alpha \in \mathbb{N}^\mathbb{N} : s \subseteq \alpha\}$. Note that $\mathbb{A}_s = \bigcup_n \mathbb{A}_{s^\frown n}$.

Let $R(s, \vec{x}, Y)$ (for $Y \in \mathcal{H}$) be the set of all (z_k) for which there is an n such that for all $Z \in \mathcal{H} \restriction Y$, I has no strategy in $F[\vec{x}^\frown(z_0, \ldots, z_n), Z]$ for playing into $((\mathbb{A}_{s^\frown n})_\Delta)^c$. By Lemma 8.8 and the (p)-property, there is a $Y \in \mathcal{H} \restriction X$ such that for all $\vec{x}$ and $s \in \mathbb{N}^{<\mathbb{N}}$, either

(i) I has a strategy in $F[\vec{x}, Y]$ for playing into $((\mathbb{A}_s)_{\Delta/2})^c$, or
(ii) II has a strategy in $G[Y]$ for playing into $R(s, \vec{x}, X)$.

Suppose I has no strategy in $F[Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c = ((\mathbb{A}_\emptyset)_{\Delta/2})^c$. We will describe a strategy for II in $G[Y]$ for playing into $\mathbb{A}_\Delta$: As II has a strategy in $G[Y]$ for playing into $R(\emptyset, \emptyset, Y)$, they follow this strategy until $(z_0, \ldots, z_{n_0})$ has been played such that I has no strategy in $F[(z_0, \ldots, z_{n_0}), Y]$ for playing into $((\mathbb{A}_{s^\frown n_0})_\Delta)^c$. By the assumption on Y, II must have a strategy in $G[Y]$ to play in $R((n_0), (z_0, \ldots, z_{n_0}), Y)$. II follows this until a further $(z_{n_0+1}, \ldots, z_{n_0+n_1+1})$ has been played so that I has no strategy in $F[(z_0, \ldots, z_{n_0}, \ldots, z_{n_0+n_1+1}), Y]$ for playing into $((\mathbb{A}_{s^\frown n_0 {}^\frown n_1})_\Delta)^c$.

We continue in this fashion, exactly as in the proof of Theorem 5 in [34], so that the outcome $Z = (z_n)$ satisfies that for all k, with $m_k = (\sum_{j \leq k} n_k) + k$, there is some $Z^k \sqsupseteq (z_0, \ldots, z_{m_k})$ in $(\mathbb{A}_{(n_0, \ldots, n_k)})_\Delta = (F''(N_{(n_0, \ldots, n_k)}))_\Delta$. Continuity of F ensures that, for $\alpha = (n_0, n_1, \ldots)$, $d(F(\alpha), Z) \leq \Delta$. $\square$

The following result provides the link between strategically Ramsey sets and weakly Ramsey sets:

Theorem 8.10 (Rosendal [34], [35])**.** *Suppose that, for some $X \in \mathrm{bb}_1^\infty(E)$, I has a strategy in $F[X]$ to play into some set $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$. Then, for any $\Delta > 0$, there is a sequence of finite intervals $I_0 < I_1 < \cdots$ in $\mathbb{N}$ such whenever $Y = (y_n) \preceq X$ and $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$, we have that $Y \in \mathbb{A}_\Delta$.*

Inspired by this theorem, we define the following:

Definition 8.11. A family $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ is *spread* if whenever $X = (x_n) \in \mathcal{H}$ and $I_0 < I_1 < \cdots$ is a sequence of intervals in $\mathbb{N}$, there is a $Y = (y_n) \in \mathcal{H} \restriction X$ such that $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$.

This property is analogous to the "(q)-property" (see Lemma 7.4 of [40]) for coideals on $\mathbb{N}$; one can show that a coideal $\mathcal{H}$ on $\mathbb{N}$ has the (q)-property if and only if for every $x \in \mathcal{H}$ and sequence of finite intervals $I_0 < I_1 < \cdots$, there is a $y \in \mathcal{H} \restriction x$ such that $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$.

By appropriately thinning down a block sequence, we see the following.

Lemma 8.12. *Given a sequence of intervals $I_0 < I_1 < \cdots$ in $\mathbb{N}$, the set*

$$\{(y_n) : \forall n \exists m (I_0 < y_n < I_m < y_{n+1})\}$$

is $\preceq$-dense open in $\mathrm{bb}_1^\infty(E)$. $\square$

Clearly, $\mathrm{bb}_1^\infty(E)$ itself is spread. As in §5, one can build spread filters (which are full, almost full, strategic, etc.) under additional set-theoretic hypotheses or by forcing. We note that the strong (p)-property suffices:

Lemma 8.13. *If $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ is a strong (p)-family, then it is spread. In particular, strategic families are spread.*

Proof. Fix $X \in \mathcal{H}$, and let $I_0 < I_1 < \cdots$ be an increasing sequence of intervals in $\mathbb{N}$. Consider the following strategy σ for I in $F[X]$: $\sigma(\emptyset) = \max(I_0)$. If II responds with some $y_0 > \sigma(\emptyset)$, then let $\sigma(y_0) = \max(I_m)$, where I_m is the first interval entirely above $\mathrm{supp}(y_0)$. Continue in this fashion. Any outcome (y_n) will satisfy $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$. Since $\mathcal{H}$ is a strong (p)-family, Theorem 4.3 implies that some outcome is in $\mathcal{H}$. $\square$

Theorem 8.14. *Let $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ be a spread (p^*)-family. Then, every analytic set is $\mathcal{H}$-weakly Ramsey.*

Proof. Let $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ be analytic. Fix $X \in \mathcal{H}$ and $\Delta > 0$. By Theorem 8.9, there is $Y \in \mathcal{H} \upharpoonright X$ such that either I has a strategy in $F[Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or II has a strategy in $G[Y]$ for playing into $\mathbb{A}_\Delta$. In the latter case, we are done, so assume the former. Theorem 8.10 and $\mathcal{H}$ being spread implies that there is some $Z \in \mathcal{H} \upharpoonright Y$ with $[Z] \subseteq ((\mathbb{A}_{\Delta/2})^c)_{\Delta/2} \subseteq \mathbb{A}^c$. $\square$

In order to extend to sets in $\mathbf{L}(\mathbb{R})$, we will use the following analogue of Lemma 7.8:

Lemma 8.15. *Let $\mathcal{F} \subseteq \mathrm{bb}_1^\infty(E)$ be a (p^*)-filter. If $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ is such that continuous images of $\mathbb{A}$ are universally Baire, then for any $X \in \mathcal{F}$ and $\Delta > 0$, there is a $Y \in \mathcal{F} \upharpoonright X$ for which II has a strategy in $G[Y]$ for playing into one of $(\mathbb{A}_{\Delta/8})^c$ or $\mathbb{A}_\Delta$.*

Proof. Let $X \in \mathcal{F}$ and $\Delta > 0$. By Lemma 8.6, there is a $Y \in \mathcal{F} \upharpoonright X$ such that either $(\emptyset, Y)$ is Δ-good or I has a strategy σ in $F[Y]$ for playing into

$$\{(z_n) : \forall n (z_0, \ldots, z_n, Y) \text{ is } \Delta/2\text{-bad}\}.$$

In the former case, we are done, so assume the latter.

By hypothesis, $\mathbb{A}_\Gamma$ is universally Baire for all Γ. In particular, we may let $\dot{\mathbb{A}}_{\Delta/4}$ be a $\mathbb{P}(\mathcal{F})$-name for $\mathbb{A}_{\Delta/4}$ and $\mathcal{D}$ a countable collection of dense open subsets of $\mathbb{P}(\mathcal{F})$ such that

(i) $\{q \in \mathbb{P}(\mathcal{F}) : q \text{ decides } \dot{X}_{\mathrm{gen}} \in \dot{A}\}$ is in $\mathcal{D}$, and
(ii) whenever G is $\mathcal{D}$-generic in $\mathbb{P}(\mathcal{F})$, $\dot{X}_{\mathrm{gen}}$ is in $\mathrm{bb}_1^\infty(E)$ and $\dot{X}_{\mathrm{gen}}(G)$ is in $\mathbb{A}_{\Delta/4}$ if and only if there is a $q \in G$ such that $q \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \in \dot{\mathbb{A}}_{\Delta/4}$.

We claim that $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \notin \dot{\mathbb{A}}_{\Delta/4}$.

Suppose not. Then, there is a $(\vec{y}, Z, \tau) \leq (\emptyset, Y, \sigma)$, with $Z \in \mathcal{F}$, such that $(\dot{y}, Z, \tau) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\mathrm{gen}} \in \dot{\mathbb{A}}_{\Delta/4}$. Applying Lemma 7.7(b) and Theorem 8.9, there is a $W \in \mathcal{F} \upharpoonright Z$ such that II has a strategy α in $G[\vec{y}, W]$ for playing into $(\mathbb{G}_{\mathcal{D},(\vec{y},Z,\tau)})_{\Delta/4}$. As in the proof of Lemma 7.8, $\mathbb{G}_{\mathcal{D},(\vec{y},Z,\tau)} \subseteq \mathbb{A}_{\Delta/4}$, so α is a strategy for II in $G[\vec{y}, W]$ for playing into $\mathbb{A}_{\Delta/2}$. This, however, contradicts the fact that σ ensures $(\vec{y}, Z)$ is $\Delta/2$-bad.

Thus, $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \notin \dot{\mathbb{A}}_{\Delta/4}$. But then, exactly as in the preceding paragraph, we may find $W \in \mathcal{F} \upharpoonright Y$ such that II has a strategy in $G[W]$ for playing into $(\mathbb{G}_{\mathcal{D},(\emptyset,Y,\sigma)})_{\Delta/8}$, and, thus, into $((\mathbb{A}_{\Delta/4})^c)_{\Delta/8} \subseteq (\mathbb{A}_{\Delta/8})^c$, where the last containment follows from Lemma 8.1(d). □

In what follows, we strengthen the hypotheses on the basis (e_n), asserting that there is some $K > 0$ such that for all $m \leq n$ and scalars (a_k),

$$\| \sum_{k \leq m} a_k e_k \| \leq K \| \sum_{k \leq n} a_k e_k \|.$$

This is equivalent to (e_n) being a Schauder basis of the completion $\overline{E}$ of E; cf. Proposition 1.1.9 [2]. The infimum of all such K as above is called the *basis constant* of (e_n). The following lemma about perturbations of blocks sequences appears to be well known:

Lemma 8.16. *For any $\Delta > 0$, there is a $\Gamma > 0$ such that whenever $X = (x_n), X' = (x'_n) \in \mathrm{bb}_1^\infty(E)$ satisfy $d(X', X) \leq \Gamma$, then $[X'] \subseteq [X]_\Delta$. In fact, if $Y' \in [X']$, then $\widetilde{Y} \in [X]$ and $d(Y', \widetilde{Y}) \leq \Delta$, where $\widetilde{Y}$ is the normalization of the image of Y' under the linear map extending $x'_n \mapsto x_n$.*

Proof. Let $\Delta > 0$. If K is the basis constant of (e_n), then by Lemma 1.3.5 in [2], the basis constant of X is $\leq K$. Pick $\Gamma > 0$ with $\sum_{n \geq m} \gamma_n \leq \min\{1/6K, \delta_m/8K\}$. For $X' = (x'_n)$ with $d(X', X) \leq \Gamma$, consider the map on the completions $T : \overline{\langle X \rangle} \to \overline{\langle X' \rangle}$ extending $x_n \mapsto x'_n$. T is a bounded linear isomorphism, as whenever $v = \sum a_n x_n \in \overline{\langle X \rangle}$,

$$\begin{aligned} \|Tv\| - \|v\| \leq \|Tv - v\| \leq \| \sum a_n x'_n - \sum a_n x_n \| &\leq \sup_n |a_n| \sum \|x'_n - x_n\| \\ &\leq 2K \|v\| \sum \|x'_n - x_n\| \leq 1/3 \|v\|, \end{aligned}$$

and so $\|T\| \leq 4/3$. Using $1/\|T^{-1}\| = \inf_{\|v\|=1} \|Tv\|$, we have $\|T^{-1}\| \leq 3/2$.

As the basis constant for X' is also $\leq K$, for $v' = \sum_{n \geq m} a_n x'_n \in \overline{\langle X' \rangle}$, we have that $\|T^{-1}v' - v'\| \leq \delta_m/4 \|v'\|$ by a similar argument as above.

If v' is a unit vector, then we also have that

$$|1 - \frac{1}{\|T^{-1}v'\|}| \leq \|T\| \|T^{-1}v' - v'\| \leq (4/3)(\delta_m/4) \leq \delta_m/3.$$

For $Y' = (y'_m) \in [X']$, we claim $d(Y', \widetilde{Y}) \leq \Delta$, where $\widetilde{Y}$ is the normalization of $Y = (y_m) = (T^{-1}(y'_m))$. Observe that

$$\|y_m - \frac{1}{\|y_m\|} y_m\| \leq \left| 1 - \frac{1}{\|T^{-1}(y'_m)\|} \right| \|T^{-1}(y'_m)\| \leq (\delta_m/3)(3/2) = \delta_m/2.$$

Thus, for all m,

$$\|y'_m - \frac{1}{\|y_m\|} y_m\| = \|y'_m - y_m\| + \|y_m - \frac{1}{\|y_m\|} y_m\| \leq \delta_m.$$

□

The following lemma expresses the uniform continuity of the games $F[X]$ and $G[X]$:

Lemma 8.17. *Let $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ and $\Delta > 0$. There is a $\Gamma > 0$ such that whenever $X \in \mathrm{bb}_1^\infty(E)$ is such that I (II, respectively) has a strategy in $F[X]$ ($G[X]$, respectively) for playing into $\mathbb{A}$ and $d(X, X') \leq \Gamma$, then I (II, respectively) has a strategy in $F[X']$ ($G[X']$, respectively) for playing into $\mathbb{A}_\Delta$.*

Proof. Take $\Gamma > 0$ as in Lemma 8.16. Suppose I has a strategy σ in $F[X]$ for playing into $\mathbb{A}$ and $d(X, X') \leq \Gamma$. We define a strategy σ' for I in $F[X']$. Let $\sigma'(\emptyset) = \sigma(\emptyset)$. Inductively, suppose that $\sigma'(y'_0, \dots, y'_k)$ has been defined and is equal to $\sigma(y_0, \dots, y_k)$, where $y_0, \dots, y_k$ is a valid play by II in $F[X]$ against σ, and $\|y'_i - y_i\| \leq \gamma_i$ for $0 \leq i \leq k$. Suppose that $y'_{k+1} > \sigma'(y'_0, \dots, y'_k)$ in $S(\langle X' \rangle)$. By our choice of Γ, there is a $y_{k+1} > \sigma'(y'_0, \dots, y'_k) = \sigma(y_0, \dots, y_k)$ in $S(\langle X \rangle)$ with $\|y'_{k+1} - y_{k+1}\| \leq \gamma_{k+1}$. Let $\sigma'(y'_0, \dots, y'_k, y'_{k+1}) = \sigma(y_0, \dots, y_k, y_{k+1})$. It follows that σ' is a strategy for playing into $\mathbb{A}_\Delta$.

Suppose that II has a strategy α in $G[X]$ for playing into $\mathbb{A}$, and $d(X, X') \leq \Gamma$. Let $T : \langle X \rangle \to \langle X' \rangle$ be as in the proof of Lemma 8.16. We define a strategy α' for II in $G[X']$. Suppose that I begins by playing $Y'_0 \in [X']$. Let $\alpha'(Y'_0) = \widetilde{T}(\alpha(\widetilde{T}^{-1}(Y'_0)))$, where $\widetilde{T}$ and $\widetilde{T^{-1}}$ indicate taking normalizations. Continue in this fashion. Then, α is a strategy for playing into $\mathbb{A}_\Delta$. $\square$

Theorem 8.18. *Assume that there is a supercompact cardinal. Let $\mathcal{F} \subseteq \mathrm{bb}_1^\infty(E)$ be a strategic (p^*)-filter. Then, every set $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is $\mathcal{F}$-weakly Ramsey.*

Proof. Let $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ be in $\mathbf{L}(\mathbb{R})$, $X \in \mathcal{F}$, and $\Delta > 0$. By Theorem 7.2, the set $\mathbb{D}$ of all $Y \preceq X$ such that either I has a strategy in $F[Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or II has a strategy in $G[Y]$ for playing into $\mathbb{A}_{\Delta/2}$, is $\preceq$-dense open, and is clearly in $\mathbf{L}(\mathbb{R})$. By Lemmas 4.4 and 8.15, there is a $Y \in \mathcal{F} \upharpoonright X$ such that II has a strategy for playing into $\mathbb{D}_\Gamma$, where Γ is as in Lemma 8.17 applied to $\Delta/4$. Since $\mathcal{F}$ is strategic, there is a $Z \in \mathcal{F} \upharpoonright Y$ which is in $\mathbb{D}_\Gamma$. By our choice of Γ, then either I has a strategy in $F[Z]$ for playing into $((\mathbb{A}_{\Delta/2})^c)_{\Delta/4} \subseteq (\mathbb{A}_{\Delta/2})^c$, or II has a strategy in $G[Z]$ for playing into $\mathbb{A}_\Delta$. In the latter case, we are done, and in the former case, we need only apply Theorem 8.10 and Lemma 8.13. $\square$

We will use the following analogue of Lemma 5.4, whose proof is similar and left to the reader:

Lemma 8.19. *For $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ a (p^*)-family, forcing with $(\mathcal{H}, \preceq^*)$ adds no new reals and if $\mathcal{G} \subseteq \mathcal{H}$ is $\mathbf{L}(\mathbb{R})$-generic for $(\mathcal{H}, \preceq^*)$, $\mathcal{G}$ will be a (p^*)-filter. If $\mathcal{H}$ is strategic (spread, respectively), then $\mathcal{G}$ will also be strategic (spread, respectively).*

Theorem 8.20. *Assume that there is a supercompact cardinal. Let $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E)$ be a strategic (p^*)-family. Then, every set $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is $\mathcal{H}$-weakly Ramsey.*

Proof. The proof is similar to that of Theorem 1.3, using Lemma 8.19 and Theorem 8.18. $\square$

Some of the above can be simplified in the case when the family $\mathcal{H}$ in question is *invariant under small perturbations*; that is, there is some $\Delta > 0$ so that $\mathcal{H}_\Delta = \mathcal{H}$. The reason lies in the following fact:

Proposition 8.21. *If $\mathcal{H}$ is a strategic (p^*)-family which is invariant under small perturbations, then $\mathcal{H}$ is a (p^+)-family as well.*

Proof.[8] Let $D \subseteq S(E)$ be $\mathcal{H}$-dense below some $X \in \mathcal{H}$ and put $\mathbb{D} = \{Y \preceq X : S(\langle Y \rangle) \subseteq D\}$. Take $\Delta > 0$ so that $\mathcal{H}_\Delta = \mathcal{H}$. Note that $\mathbb{D}$ is closed and, thus, it and its continuous images are universally Baire. Let $\mathcal{G}$ be a $\mathbf{V}$-generic filter for $(\mathcal{H}, \preceq^*)$ which contains X, so that by Lemma 8.19, $\mathcal{G}$ is a strategic (p^*)-filter in $\mathbf{V}[\mathcal{G}]$. By Lemma 8.15 in $\mathbf{V}[\mathcal{G}]$, there is a $Y \in \mathcal{G} \upharpoonright X$ so that II has a strategy in $G[Y]$ for playing into one of $(\mathbb{D}_{\Delta/8})^c$ or $\mathbb{D}_\Delta$. However, as I has a strategy in $G[Y]$ for playing into $\mathbb{D}$, and $(\mathbb{D}_{\Delta/8})^c \subseteq \mathbb{D}^c$ by Lemma 8.1(c), II's strategy must be for playing into $\mathbb{D}_\Delta$. Since forcing with $(\mathcal{H}, \preceq^*)$ added no new reals, such a strategy must exist in $\mathbf{V}$ (we are using Lemma 4.7 implicitly here). As $\mathcal{H}$ is strategic and $\mathcal{H}_\Delta = \mathcal{H}$, we have that $\mathcal{H} \cap \mathbb{D} \neq \emptyset$, showing that $\mathcal{H}$ is full. $\square$

We now extend these principles to Banach spaces. In what follows, B is a (separable) Banach space with normalized Schauder basis (e_n). We say that a countable field F is *suitable* if the norm on E_F, the F-span of E, takes values in F. Let $\langle X \rangle_F$ the F-span of $X \in \mathrm{bb}^\infty(E_F)$. If V is a subspace of B, let $S(V) = \{v \in V : \|v\| = 1\}$.

Let $\mathrm{bb}_1^\infty(B)$ be the set of all infinite block sequences (with respect to (e_n)) in B, which we endow with the Polish topology inherited from $B^{\mathbb{N}}$. The relations $\preceq$ and $\preceq^*$ extend to $\mathrm{bb}_1^\infty(B)$. For $Y \in \mathrm{bb}_1^\infty(B)$, let $[Y]^* = \{Z \in \mathrm{bb}_1^\infty(B) : Z \preceq Y\}$. We denote by $G^*[Y]$ the *Gowers game* defined as before, except that the players may now play real (complex) block sequences and block vectors. The notions of *family*, (p)-*family*, *spread*, and *strategic* are defined as before, with appropriate modifications for real (complex) scalars.

Strategic families in $\mathrm{bb}_1^\infty(B)$ arise naturally from strategic families in $\mathrm{bb}_1^\infty(E_F)$: Given a strategic $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(E_F)$, if $\widehat{\mathcal{H}}$ is invariant under small perturbations and equal to the $\preceq$ upwards closure of $\mathcal{H}_\Delta$ (taken in $\mathrm{bb}_1^\infty(B)$) for some small $\Delta > 0$, then $\widehat{\mathcal{H}}$ is strategic. This follows from the fact that Lemma 8.16 and the proof of Lemma 8.17 can be carried out in B.

Definition 8.22. We say that $\mathcal{H}$ is *almost full* if whenever $D \subseteq S(B)$ is closed and $\mathcal{H}$-dense below some $X \in \mathcal{H}$ (that is, for all $Y \in \mathcal{H} \upharpoonright X$, there is a $Z \preceq Y$ with $S(\langle Z \rangle) \subseteq D$), then for any $\epsilon > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ with $S(\overline{\langle Y \rangle}) \subseteq D_\epsilon$.

Definition 8.23. An almost full (p)-family in $\mathrm{bb}_1^\infty(B)$ is called a (p^*)-*family.*

While we have reused this terminology, the meaning should be clear from context. The following is the relativized version of Gowers's weakly Ramsey property [19].

Definition 8.24. Given a family $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(B)$, a set $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(B)$ is $\mathcal{H}$-*weakly Ramsey* if for every $\Delta > 0$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

(i) $[Y]^* \subseteq \mathbb{A}^c$, or
(ii) II has a strategy in $G^*[Y]$ for playing into $\mathbb{A}_\Delta$.

Proving Theorems 1.4 and 1.5 amounts to showing that for spread (strategic) (p^*)-families $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(B)$ which are invariant under small perturbations, analytic ($\mathbf{L}(\mathbb{R})$) sets are $\mathcal{H}$-weakly Ramsey.

Lemma 8.25. *Let F be suitable. If $X_0 \succeq X_1 \succeq X_2 \succeq \cdots$ is a $\preceq$-decreasing sequence in $\mathrm{bb}_1^\infty(E_F)$, $X \in \mathrm{bb}_1^\infty(B)$ is such that $X \preceq X_n$ for all n, and $\Delta > 0$, then there is an $X' \in \mathrm{bb}_1^\infty(E_F)$ with $X' \in [X]_\Delta$, and $X' \preceq^* X_n$ for all n.*

[8]We suspect that an elementary proof of this result can be found and that "strategic" can be relaxed to "spread".

Proof. Let (X_n), X, and $\Delta > 0$ be as described, say with $X = (x_n)$. We construct $X' = (x'_n)$ as follows: There is an $M_0 \in \mathbb{N}$ so that $\overline{\langle X/M_0\rangle_F} \subseteq \overline{\langle X_0\rangle_F}$. Let x_{n_0} be the first entry of X/M_0. Pick a unit vector $x'_0 \in \langle X_0\rangle_F$ such that $d(x_{n_0}, x'_0) \leq \delta_0$. Continue inductively. At stage k, we have chosen $M_0 < \cdots < M_k$ and $x'_0 < \cdots < x'_k$ so that if x_{n_i} is the first entry of X/M_i, then $x'_i \in \langle X_i\rangle_F$ and $d(x_{n_i}, x'_i) \leq \delta_i$, for $i \leq k$. By construction, $X'/n \preceq X_n$ for all n, and $X' \in [X]_\Delta$. □

Lemma 8.26. *If $\mathcal{H} \subseteq \mathrm{bb}_1^\infty(B)$ is a (p^*)-family which is invariant under small perturbations, then $\mathcal{H} \cap \mathrm{bb}_1^\infty(E_F)$ is a (p^*)-family for any suitable subfield F of $\mathbb{R}$ (or $\mathbb{C}$). If $\mathcal{H}$ is spread (strategic, respectively), then so is $\mathcal{H} \cap \mathrm{bb}_1^\infty(E_F)$.*

Proof. Let $\mathcal{H}$ and F be as described and put $\widetilde{\mathcal{H}} = \mathcal{H} \cap \mathrm{bb}_1^\infty(E_F)$. Lemma 8.25 implies that $\widetilde{\mathcal{H}}$ is a (p)-family. To see that $\widetilde{\mathcal{H}}$ is almost full, let $D \subseteq S(E_F)$ be $\widetilde{\mathcal{H}}$-dense below $X \in \widetilde{\mathcal{H}}$, and take $\epsilon > 0$. Consider $\overline{D_{\epsilon/3}} \subseteq S(B)$. For $\Delta = (\epsilon/3, \epsilon/3, \ldots)$, let Γ be as in Lemma 8.16. For any $Y \in \mathcal{H} \upharpoonright X$, there is a $Y' \in \widetilde{\mathcal{H}} \upharpoonright X$, with $d(Y, Y') \leq \Gamma$ and $Z' \preceq Y'$ with $S(\langle Z\rangle) \subseteq D$. By our choice of Γ, there is a $Z \in [Y]^*$ with $S(\langle Z\rangle) \subseteq D_{\epsilon/3}$, and so $S(\overline{\langle Z\rangle}) \subseteq \overline{D_{\epsilon/3}}$. Thus, $\overline{D_{\epsilon/3}}$ is $\mathcal{H}$-dense below X. By almost fullness of $\mathcal{H}$, there is a $W \in \mathcal{H} \upharpoonright X$ with $S(\overline{\langle W\rangle}) \subseteq (\overline{D_{\epsilon/3}})_{\epsilon/3}$. Then, one can find a $W' \in \widetilde{\mathcal{H}} \upharpoonright X$ with $S(\langle W'\rangle) \subseteq D_\epsilon$, showing that $\widetilde{\mathcal{H}}$ is almost full.

To see that $\mathcal{H}$ being strategic implies that $\widehat{\mathcal{H}}$ is strategic, let α be a strategy for II in $G[X]$, with $X \in \widehat{\mathcal{H}}$. Define a strategy α' in $G^*[X]$ which is equal to α on their shared domain, and otherwise plays so that the outcomes are sufficiently small (using Lemma 8.16 and our assumption about $\mathcal{H}$) perturbations of outcomes of α. Then, if any outcome of α' is in $\mathcal{H}$, an outcome of α must be in $\widehat{\mathcal{H}}$. The proof for being spread is left to the reader. □

Proof of Theorem 1.4. Suppose that $\mathbb{A} \subseteq \mathrm{bb}_1^\infty(B)$ is analytic, $\Delta > 0$, and $X \in \mathcal{H}$ is such that for no $Y \in \mathcal{H} \upharpoonright X$ is $[Y]^* \subseteq \mathbb{A}^c$. Let F be a suitable field for (e_n). Let $\widetilde{\mathcal{H}} = \mathcal{H} \cap \mathrm{bb}_1^\infty(E_F)$. If there was some $Y \in \widetilde{\mathcal{H}} \upharpoonright X$ with $[Y] \subseteq (\mathbb{A}_{\Delta/3})^c \cap \mathrm{bb}_1^\infty(E_F)$, then $[Y]^* \subseteq ((\mathbb{A}_{\Delta/3})^c)_{\Delta/3} \subseteq \mathbb{A}^c$, contrary to our assumption. Thus, by Lemma 8.26 and Theorem 8.14, there is a $Y \in \widetilde{\mathcal{H}} \upharpoonright X$ such that II has a strategy in $G[Y]$ for playing into $\mathbb{A}_{\Delta/2} \cap \mathrm{bb}_1^\infty(E_F)$. Easy perturbation arguments show that II has a strategy in $G^*[Y]$ for playing into $\mathbb{A}_\Delta$. □

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.4, using Theorem 8.20, or, alternatively Proposition 8.21 and Theorem 1.3. □

The following is an analytical example of a strategic (p^*)-family, which, though trivial in the sense that is $\preceq$-downwards closed, we hope suggests further applications:

Example 8.27. Given B as above, suppose that B contains a normalized block sequence X equivalent to the standard basis of c_0 or ℓ^p for $1 \leq p < \infty$. Let $\mathcal{H}$ be the set of all block sequences in B which have a further block subsequence equivalent to X. Then, $\mathcal{H}$ is a strategic (p^*)-family which is invariant under small perturbations. These facts follow from the block homogeneity characterization of the standard bases of c_0 and ℓ^p, Lemma 2.1.1 in [2].

9. Projections in the Calkin algebra

Given a Banach space with a Schauder basis, one might wish to develop a notion of forcing with block sequences "modulo small perturbation" and then prove an analogue of Theorem 1.2 characterizing $\mathbf{L}(\mathbb{R})$-generic filters.[9] We focus on a particular variant of this which is of significant interest.

Let H be a complex infinite-dimensional separable Hilbert space with orthonormal basis (e_n). Note that any normalized block sequence (with respect to (e_n)) is necessarily orthonormal. Throughout, E will denote the $\overline{\mathbb{Q}}$-linear span of (e_n) in H, $\mathrm{bb}_1^\infty(E)$ the space of infinite normalized block sequences in E, and for $X \in \mathrm{bb}_1^\infty(E)$, $\langle X \rangle$ is the $\overline{\mathbb{Q}}$-span of X.

For $X \in \mathrm{bb}_1^\infty(E)$, let P_X be the orthogonal projection onto $\overline{\langle X \rangle}$. Note that, for $X, Y \in \mathrm{bb}_1^\infty(E)$, $X \preceq Y$ if and only if $P_X \leq P_Y$ in the usual ordering of projections (that is, $P \leq Q$ if $\mathrm{ran}(P) \subseteq \mathrm{ran}(Q)$, or equivalently $PQ = P$). We call such projections *block projections*.

Let $\mathcal{B}(H)$ be the C*-algebra of bounded operators on H and $\mathcal{K}(H)$ the ideal of compact operators on H. The quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is also a C*-algebra called the *Calkin algebra*. We write $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ for the quotient map.

Denote by $\mathcal{P}(H)$ ($\mathcal{P}_\infty(H)$, respectively) the set (infinite-rank, respectively) projections in $\mathcal{B}(H)$, and $\mathcal{P}(\mathcal{C}(H))$ ($\mathcal{P}(\mathcal{C}(H))^+$, respectively) the set of (nonzero, respectively) projections, i.e., self-adjoint idempotents, in $\mathcal{C}(H)$. By Proposition 3.1 in [41], $\mathcal{P}(\mathcal{C}(H)) = \pi(\mathcal{P}(H))$. The ordering $\leq$ on $\mathcal{P}(\mathcal{C}(H))$ is inherited from the ordering on $\mathcal{P}(H)$.

Definition 9.1.

(a) For projections $P, Q \in \mathcal{P}(H)$, we write $P \leq_{\mathrm{ess}} Q$ if $\pi(P) \leq \pi(Q)$ in $\mathcal{P}(\mathcal{C}(H))$ and $P \equiv_{\mathrm{ess}} Q$ if $\pi(P) = \pi(Q)$.

(b) For $X, Y \in \mathrm{bb}_1^\infty(E)$, we write $X \leq_{\mathrm{ess}} Y$ if $P_X \leq_{\mathrm{ess}} P_Y$ and $X \equiv_{\mathrm{ess}} Y$ if $P_X \equiv_{\mathrm{ess}} P_Y$.

The last sentence of the following lemma requires a slight modification of the original proof and is left to the reader:

Lemma 9.2 (Proposition 3.3 in [41]). *For P and Q projections on H, the following are equivalent:*

(i) $P \leq_{\mathrm{ess}} Q$.

(ii) *For every $\epsilon > 0$, there is a finite-codimensional subspace V of* $\mathrm{ran}(P)$ *such that every unit vector $v \in V$ satisfies $d(v, \mathrm{ran}(Q)) \leq \epsilon$.*

In the event that $P = P_X$ and $Q = P_Y$ for $X, Y \in \mathrm{bb}_1^\infty(E)$, one can replace "finite-codimensional subspace" in (ii) *with "tail subspace".*

The following lemma is well known:

Lemma 9.3. *Suppose that $\Delta = (\delta_n) > 0$ is summable and P and Q are projections on H whose ranges have orthonormal bases (x_n) and (y_n), respectively. If for all n, $\|x_n - y_n\| \leq \delta_n$, then $P \equiv_{\mathrm{ess}} Q$.*

Proof. Assuming that for all n, $\|x_n - y_n\| \leq \delta_n$, we will show that $P \leq_{\mathrm{ess}} Q$. The result follows by symmetry. Let $\epsilon > 0$ and choose an N such that $\sum_{n \geq N} \delta_n \leq \epsilon$.

[9] There are obstacles to this being a meaningful endeavor in general; e.g., in a hereditarily indecomposable Banach space, the collection of *all* infinite-dimensional subspaces modulo small perturbations forms a filter (cf. (iii) on p. 820 of [19]), and is, thus, trivial as a forcing notion.

Let $V = \overline{\langle (x_n)_{n\geq N}\rangle}$, a finite-codimensional subspace of $\mathrm{ran}(P)$. If $v \in V$ is a unit vector, say with $v = \sum_{n\geq N} a_n x_n$, then for $y = \sum_{n\geq N} a_n y_n \in \mathrm{ran}(Q)$, we have

$$\|v - y\| = \| \sum_{n\geq N} a_n(x_n - y_n)\| \leq \sum_{n\geq N} \|x_n - y_n\| \leq \epsilon.$$

The claim follows by Lemma 9.2. ☐

In particular, $\equiv_{\mathrm{ess}}$-invariant families in $\mathrm{bb}_1^\infty(E)$ or $\mathrm{bb}_1^\infty(H)$ are invariant under small perturbations. The following observation can be proved using Lemma 9.3 and standard manipulations with basic sequences (cf. Proposition 1.3.10 in [2]):

Lemma 9.4. *The set of block projections is dense in* $(\mathcal{P}_\infty(H), \leq_{\mathrm{ess}})$.

It follows that $(\mathcal{P}(\mathcal{C}(H))^+, \leq)$, $(\mathcal{P}_\infty(H), \leq_{\mathrm{ess}})$, and $(\mathrm{bb}_1^\infty(E), \leq_{\mathrm{ess}})$ are equivalent as notions of forcing. It is for this reason that we focus on $(\mathrm{bb}_1^\infty(E), \leq_{\mathrm{ess}})$.

Lemma 9.5. *If* $X_0 \succeq X_1 \succeq X_2 \succeq \cdots$ *is a* $\preceq$*-decreasing sequence in* $\mathrm{bb}_1^\infty(E)$ *and* $X \in \mathrm{bb}_1^\infty(E)$ *is such that* $X \leq_{\mathrm{ess}} X_n$ *for all* n*, then there is an* $X' \leq_{\mathrm{ess}} X$ *such that* $X' \preceq^* X_n$ *for all* n*.*

Proof. This can be proved using Lemmas 9.2 and 9.3 in a way similar to Lemma 8.25. ☐

Clearly, any $\preceq$-dense subset of $\mathrm{bb}_1^\infty(E)$ is also $\leq_{\mathrm{ess}}$-dense. The following lemma is a converse to this:

Lemma 9.6. *If* $\mathcal{D} \subseteq \mathrm{bb}_1^\infty(E)$ *is* $\leq_{\mathrm{ess}}$*-dense open, then it is* $\preceq$*-dense open.*

Proof. Suppose $\mathcal{D} \subseteq \mathrm{bb}_1^\infty(E)$ is $\leq_{\mathrm{ess}}$-dense open. Given any $X \in \mathrm{bb}_1^\infty(E)$, there is a $Y \in \mathcal{D}$ with $Y \leq_{\mathrm{ess}} X$. Applying Lemma 9.5 (with $X_n = X$ for all n), there is a $Y' \leq_{\mathrm{ess}} Y$ with $Y' \preceq X$. Then, $Y' \in \mathcal{D}$. ☐

We can now establish Theorem 1.6, an analogue of Theorem 1.2 for projections in the Calkin algebra. We first prove a more general result.

Theorem 9.7.

(a) *If* $\mathcal{G}$ *is an* $\mathbf{L}(\mathbb{R})$*-generic filter for* $(\mathrm{bb}_1^\infty(E), \leq_{\mathrm{ess}})$*, then* $\mathcal{G}$ *is a strategic* (p^+)*-family.*

(b) *Assume that there is a supercompact cardinal. If* $\mathcal{G} \subseteq \mathrm{bb}_1^\infty(E)$ *is a strategic* (p^*)*-family which is also a* $\leq_{\mathrm{ess}}$*-filter, then* $\mathcal{G}$ *is* $\mathbf{L}(\mathbb{R})$*-generic for* $(\mathrm{bb}_1^\infty(E), \leq_{\mathrm{ess}})$*.*

Proof. (a) Let $\mathcal{G}$ be as described. Clearly, it is a family. To see that it is full, suppose that $D \subseteq S(E)$ is $\mathcal{G}$-dense below some $X \in \mathcal{G}$. Let

$$\mathcal{D}_0 = \{Z : \langle Z\rangle \subseteq D \text{ or } \forall V \preceq X(\langle V\rangle \subseteq D \to V \perp Z)\},$$

where $\perp$ denotes incompatibility with respect to $\preceq$. $\mathcal{D}_0$ is $\preceq$-dense open by Lemma 2.7, thus, $\leq_{\mathrm{ess}}$-dense as well, and clearly in $\mathbf{L}(\mathbb{R})$, so there is a $Z \in \mathcal{D}_0 \cap (\mathcal{G} \upharpoonright X)$. Then, there is a $Z' \preceq Z \preceq X$ with $S(\langle Z'\rangle) \subseteq D$, so we have that $\langle Z\rangle \subseteq D$, showing that $\mathcal{G}$ is full.

To see that $\mathcal{G}$ is a (p)-family, let $X_0 \succeq X_1 \succeq X_2 \succeq \cdots$ in $\mathcal{G}$. Let

$$\mathcal{D}_1 = \{Y : \forall n(Y \preceq^* X_n) \text{ or } \exists n(Y \perp_{\mathrm{ess}} X_n)\},$$

where $\perp_{\mathrm{ess}}$ denotes incompatibility with respect to $\leq_{\mathrm{ess}}$. We want to show that $\mathcal{D}_1$ is $\leq_{\mathrm{ess}}$-dense. The set

$$\mathcal{D}_1' = \{Y : \forall n(Y \leq_{\mathrm{ess}} X_n) \text{ or } \exists n(Y \perp_{\mathrm{ess}} X_n)\}$$

is $\leq_{\mathrm{ess}}$-dense open. Then, given any X, we can find a $Y \in \mathcal{D}_1'$ below X. If $Y \perp_{\mathrm{ess}} X_n$ for some n, we are done. Otherwise, $Y \leq_{\mathrm{ess}} X_n$ for all n, and we can apply Lemma 9.5 to find a $Y' \leq_{\mathrm{ess}} Y$ with $Y \preceq^* X_n$ for all n. Such a Y' is in $\mathcal{D}_1$, verifying that this set is $\leq_{\mathrm{ess}}$-dense. As $\mathcal{D}_1$ is in $\mathbf{L}(\mathbb{R})$, $\mathcal{G} \cap \mathcal{D}_1 \neq \emptyset$, and anything in this intersection must be a diagonalization of (X_n). It is likewise easy to see that $\mathcal{G}$ must be strategic.

(b) Let $\mathcal{D} \subseteq \mathrm{bb}_1^\infty(H)$ be $\leq_{\mathrm{ess}}$-dense open and in $\mathbf{L}(\mathbb{R})$. By Lemma 9.6, $\mathcal{D}$ is also $\preceq$-dense open. For $\Delta > 0$ summable, $\mathcal{D}_\Delta = \mathcal{D}$ by Lemma 9.3. Thus, by Theorem 8.14, there is an $X \in \mathcal{H}$ such II has a strategy for playing into $\mathcal{D}$. Since $\mathcal{G}$ is strategic, it follows that $\mathcal{G} \cap \mathcal{D} \neq \emptyset$. ☐

Proof of Theorem 1.6. The ($\Rightarrow$) direction is proved by a straightforward verification of the relevant sets being $\preceq$-dense open, thus, $\leq_{\mathrm{ess}}$-dense by Lemma 9.6. The ($\Leftarrow$) direction follows from Theorem 9.7(b) or Theorem 1.5. ☐

We conclude this section by describing a hoped-for application of our machinery and its limitations. A *state* τ on $\mathcal{B}(H)$ is a linear functional on $\mathcal{B}(H)$ which is *positive*; that is, $\tau(T^*T) \geq 0$ for all T, and it satisfies $\tau(I) = 1$, where I is the identity operator. The set of states forms a weak*-compact convex subset of the dual of $\mathcal{B}(H)$ and, thus, has extreme points called *pure states.* These definitions generalize to any unital C*-algebra, including $\mathcal{C}(H)$.

A state on $\mathcal{B}(H)$ is *singular* if it vanishes on $\mathcal{K}(H)$. Composing with the quotient map $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ yields a bijective correspondence between singular pure states on $\mathcal{B}(H)$ and pure states on $\mathcal{C}(H)$.

For any choice of orthonormal basis (f_k) for H, and any ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the functional defined by $\tau_{\mathcal{U}}(T) = \lim_{k\to\mathcal{U}}\langle Tf_k, f_k\rangle$ is a pure state which is singular if and only if $\mathcal{U}$ is nonprincipal (cf. Theorem 4.21 and Example 6.1 in [15]). Such pure states are said to be *diagonalizable.* On an abelian C*-algebra, pure states coincide with characters, so the aforementioned $\tau_{\mathcal{U}}$ restricts to a pure state on the atomic maximal abelian self-adjoint subalgebra (or *masa*) generated by the rank-one projections corresponding to the f_k. The following problem asks to what extent this is true of all pure states:

Problem (Kadison and Singer [21])**.** Does every pure state on $\mathcal{B}(H)$ restrict to a pure state on some (atomic or continuous) masa?

Anderson conjectured that not only is the answer to this question "yes", but that every pure state is of the form $\tau_{\mathcal{U}}$ for some choice of orthonormal basis (f_k) and ultrafilter $\mathcal{U}$:

Conjecture (Anderson [3])**.** *Every pure state on* $\mathcal{B}(H)$ *is diagonalizable.*

Akemann and Weaver [1] showed that the above problem of Kadison and Singer has a negative answer, and, thus, Anderson's conjecture is false, assuming CH. It remains an open question whether Anderson's conjecture is consistent with ZFC.

By the recent positive solution [29] to the Kadison–Singer problem regarding *extensions* of pure states (which differs from the above), Anderson's conjecture is equivalent to saying that every pure state on $\mathcal{B}(H)$ restricts to a pure state on some atomic masa.

Following [8], we say that a subset $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))^+$ is *centered*[10] if for every finite subset of $\mathcal{F}$ has a lower bound in $\mathcal{P}(\mathcal{C}(H))^+$. $\mathcal{F}$ is *linked* if every pair of elements in $\mathcal{F}$ has a lower bound in $\mathcal{P}(\mathcal{C}(H))^+$. *Maximal centered* has the obvious meaning. Similarly, we define $\leq_{\text{ess}}$-*centered*, $\leq_{\text{ess}}$-*linked*, and *maximal* $\leq_{\text{ess}}$-*centered* in $\text{bb}_1^\infty(E)$.

Theorem 9.8 (Farah and Weaver, Theorem 6.42 in [15])**.** *There is a bijective correspondence between singular pure states τ on $\mathcal{B}(H)$ and maximal centered subsets of $\mathcal{P}(\mathcal{C}(H))^+$ via $\tau \mapsto \mathcal{F}_\tau = \{p \in \mathcal{P}(\mathcal{C}(H))^+ : \tau(p) = 1\}$.*

If $\mathcal{F} = \mathcal{F}_\tau$ as above and τ fails to restrict to a pure state on any atomic masa, we say that $\mathcal{F}$ *yields a counterexample to Anderson's conjecture.*

Theorem 9.9 (essentially Farah and Weaver, cf. Theorem 6.46 in [15])**.** *If $\mathcal{G}$ is $\mathbf{V}$-generic for $\mathcal{P}(\mathcal{C}(H))^+$, then $\mathcal{G}$ is a maximal centered set which yields a counterexample to Anderson's conjecture.*

In fact, this result uses much less than full genericity, or even genericity over $\mathbf{L}(\mathbb{R})$. By considering the complexity of the dense sets involved in the proof, we obtain Theorem 1.7:

Proof of Theorem 1.7. Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be spread (p^*)-family which is $\leq_{\text{ess}}$-centered and $\widehat{\mathcal{H}}$ the upwards closure of $\pi(\mathcal{H})$ in $\mathcal{P}(\mathcal{C}(H))^+$. First, we claim that $\widehat{\mathcal{H}}$ is a maximal centered set. Clearly, $\widehat{\mathcal{H}}$ is centered. For maximality, let $p \in \mathcal{P}(\mathcal{C}(H))^+$ be such that p is compatible with every finite subset of $\widehat{\mathcal{H}}$. Let $P \in \mathcal{P}(H)$ be such that $\pi(P) = p$, and define

$$\mathcal{D}_P = \{X : P_X \leq_{\text{ess}} P \text{ or } P_X \perp_{\text{ess}} P\},$$

which is a coanalytic and $\leq_{\text{ess}}$-dense open subset of $\text{bb}_1^\infty(H)$. By Lemma 9.6, $\mathcal{D}_P$ is $\preceq$-dense open, so by Theorem 8.14, we can find a $Y \in \mathcal{H} \restriction X$ with $Y \in \mathcal{D}_P$. It must then be the case that $P_Y \leq_{\text{ess}} P$ and so $p \in \widehat{\mathcal{H}}$.

To see that $\widehat{\mathcal{H}}$ yields a counterexample to Anderson's conjecture, we refer to the proof of Theorem 6.46 in [15] and omit the details except to note that it suffices to show that $\mathcal{H}$ meets the $\leq_{\text{ess}}$-dense open sets

$$\mathcal{D}_{\vec{J}} = \{X \in \text{bb}_1^\infty(E) : \forall n(\|P^{(f_k)}_{J_n \cup J_{n+1}} P_X\| < 1/2)\},$$

where $\vec{J} = (J_n)$ is a partition of $\mathbb{N}$ into finite intervals J_n and $P^{(f_k)}_J$ denotes the orthogonal projection onto $\overline{\text{span}}\{f_k : k \in J\}$, for (f_k) an orthonormal basis of H. These sets are easily seen to be Borel, and meeting them with $\mathcal{H}$ uses the combination of Lemma 9.6 and Theorem 8.14 as before. $\square$

For spread (p^*)-families being $\leq_{\text{ess}}$-linked implies being a $\leq_{\text{ess}}$-filter:

Lemma 9.10. *Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a spread (p^*)-family which is, moreover, $\leq_{\text{ess}}$-linked. Then, $\mathcal{H}$ is a $\leq_{\text{ess}}$-filter.*

Proof. Let $X, Y \in \mathcal{H}$, and consider the set

$$\mathcal{D} = \{Z : (Z \leq_{\text{ess}} X \text{ and } Z \leq_{\text{ess}} Y) \text{ or } (Z \perp_{\text{ess}} X \text{ or } Z \perp_{\text{ess}} Y)\}.$$

[10] These were called *quantum filters* by Farah and Weaver [15].

It is easy to check that $\mathcal{D}$ is coanalytic. Clearly, $\mathcal{D}$ is $\leq_{\mathrm{ess}}$-dense open, thus, $\preceq$-dense open by Lemma 9.6. By Theorem 8.14 applied to the analytic set $\mathbb{A} = \mathcal{D}^c$, there is a $Z \in \mathcal{H}$ with $[Z]_1 \subseteq \mathcal{D}$. In particular, $Z \in \mathcal{D}$. Since $\mathcal{D}$ is $\leq_{\mathrm{ess}}$-linked, we must have that $Z \leq_{\mathrm{ess}} X$ and $Z \leq_{\mathrm{ess}} Y$. $\square$

By Lemma 9.10, the maximal centered sets in Theorem 1.7 are also filters in $\mathcal{P}(\mathcal{C}(H))^+$. The following result of Bice, using Shelah's model without p-points (VI. §4 in [36]) presents an obstacle to ZFC constructions.

Theorem 9.11 (Bice [8])**.** *It is consistent with* ZFC *that no maximal centered set in $\mathcal{P}(\mathcal{C}(H))^+$ is a filter.*

Consequently, we have the following:

Corollary 9.12. *It is consistent with* ZFC *that no spread (p^*)-family in* $\mathrm{bb}_1^\infty(E)$ *can be $\leq_{\mathrm{ess}}$-linked, and in particular, that there are no spread (p^*)-filters.*

10. Further questions

Despite our constructions, under additional hypotheses, of (p^+)-filters, there remains a lack of examples of interesting, purely analytical (p^+)- and (p^*)-families, Example 8.27 notwithstanding.

Question. Are there naturally occurring nontrivial (ZFC) examples of (p^+)- or (p^*)-families of block sequences?

While Theorem 1.6 does give a criterion for $\mathbf{L}(\mathbb{R})$-genericity for filters of projections in the Calkin algebra, it would be desirable to have a such criterion expressed in the language of C*-algebras.

Question. Can the (local) Ramsey theory of block sequences in a separable infinite-dimensional Hilbert space be described in C*-algebraic terms? Under large cardinals, is there a C*-algebraic characterization of $\mathbf{L}(\mathbb{R})$-generic filters in the projections in the Calkin algebra?

Lastly, as the sufficient conditions described in Theorem 1.7 for producing a counterexample to Anderson's conjecture cannot be satisfied in Shelah's model without p-points, the status of Anderson's conjecture in that model appears to be a natural test question.

Question. Does Anderson's conjecture hold in Shelah's model without p-points?[11]

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[11] Added in proof: This question has a negative answer. It follows from the fact that Anderson's conjecture fails whenever $\mathfrak{d} \leq \mathfrak{t}^*$ [15], which holds in Shelah's model.

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Department of Mathematics, Rutgers, The State University of New Jersey, Piscataway, New Jersey 08854

Email address: `i.smythe@rutgers.edu`