The Bieberbach Conjecture

Sheng Gong
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The Bieberbach Conjecture

Sheng Gong
This book is a revised translation of The Bieberbach Conjecture, Science Press, 1989, in Chinese. Permission has been granted by Science Press to reuse material from the original book translated into English and incorporated into this new volume.
To my wife Huiyi
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FOREWORD

If $f(z)$ is a univalent holomorphic function on the unit disc, $D = \{ z : |z| < 1 \}$, in the complex plane, we may add normalization conditions, $f(0) = 0$ and $f'(0) = 1$. Thus $f(z)$ has the Taylor expansion $f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n + \cdots$, on $D$. The set of all such functions forms a normal family $S$.

In 1916, Bieberbach conjectured: If $f \in S$, then $|a_n| \leq n$ holds true for $n = 2, 3, \cdots$. The equality holds if and only if $f(z)$ is the Koebe function $\frac{z}{(1-z)^2}$ or one of its rotations. The conjecture was not completely solved until 1984 by de Branges. That is, mathematicians spent 68 years solving this simple-looking conjecture.

During these 68 years, there were a huge number of papers discussing this conjecture and its related problems. For example, when S. D. Bernardi listed the bibliography of univalent functions, 4282 papers had been published up to 1981. No doubt, a high percentage of these papers are related to this conjecture. Moreover, during this period, many very nice books were published that systematically presented the known theory of univalent functions. Among those books are four especially nice ones by the following authors: Duren, Goluzin, Hayman and Pommerenke. These are listed in the references.

After de Branges proved this famous conjecture, I wrote and published in 1989 a small book in Chinese titled “The Bieberbach Conjecture,” presenting the history of related coefficient problems and de Branges’ proof. This is the English translation of my small book with many changes. In particular, it includes some results related to several complex variables. Anybody who
FOREWORD

has completed the standard material in a one year graduate complex analysis course can easily understand this small book.

Several people have been very helpful in publishing the English edition of this book. I am greatly indebted to Professor S. T. Yau for encouraging me to translate the Chinese edition of this book to English. Also I am deeply indebted to Professor Carl H. FitzGerald for writing a wonderful preface and giving me lots of important suggestions. It is a great pleasure to thank Dr. Carolyn Thomas and Dr. Weigi Gao who made many useful suggestions for mathematics and for improving the English throughout the text.

Finally, I would like to take this opportunity to express my sincere thanks to the Department of Mathematics, University of California, San Diego, for their hospitality in providing me with a stimulating environment, where I was able to complete both the Chinese edition and the English edition of this small book.

Sheng Gong Feb. 1998
The dramatic story of the Bieberbach Conjecture illustrates the creation of mathematics. Made in 1916, this conjecture stood as a challenge to complex analysis for sixty-eight years. During that time, many mathematicians made contributions to mathematics of complex variables in their efforts to solve this problem. For example, M. Schiffer brought calculus of variation technique into complex analysis. C. Löwner used some of Lie’s ideas to find a way to represent the functions involved as solutions to certain partial differential equations. W. Kaplan brought attention to the class of close-to-convex mappings; and M. Reade showed that the conjecture was true for this large class. And many others made impressive advances in complex analysis in their efforts to solve the problem. When the final winning assault was made on the conjecture, it was clearly manifest that a magnificent piece of mathematics had been discovered; and it was clear that earlier work had laid a foundation for that success. Thus, this history of the Bieberbach Conjecture shows some ways in which mathematicians continue to build the science of mathematics.

The initial interest in the Bieberbach Conjecture came from the completion of an earlier program. In the first decade of the twentieth century, mathematicians had studied the analytic functions $p(z) = 1 + 2c_1z + 2c_2z^2 + \cdots$ on the unit disk $\{z : |z| < 1\}$ such that the real part of $p(z)$ is positive. A very satisfactory theory was developed. In particular, the bounds $|c_n| \leq 1$ were proved for all positive integers $n$. These bounds are sharp since for
each positive \( n \),

\[
p(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + \cdots
\]

shows that the upper bound is reached. More generally, a characterization of the coefficients of positive real part functions was found.

With the successful analysis of the class of positive real part functions, it was natural to consider other classes of analytic functions. One obvious candidate was the class \( S \) of functions \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) which are analytic and one to one on the unit disk. (The letter \( S \) is used for the German \( Schlicht \) since the Riemann surface is “simple”.) The Koebe function is an interesting example of a function in \( S \). The function is

\[
K(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots.
\]

It takes the unit disk onto the plane minus the negative real axis from \(-\frac{1}{4}\) to minus infinity. Bieberbach showed that \( |a_2| \leq 2 \). In a footnote, he indicated the general expectation that \( |a_n| \leq n \) for \( n = 2, 3, 4, \cdots \); and furthermore, for each \( n \), the only the functions which attain the upper bound are the Koebe function and its rotations \( K_\theta(z) = e^{-i\theta}K(e^{i\theta}z) \).

The problem quickly became a focus of complex analysis. When in 1923 Löwner presented his proof that \( |a_3| \leq 3 \), Bieberbach shook his hand and assured him that he had joined the “realm of the immortals”. Also Bieberbach suggested that Löwner put a “one” at the end of the title of the paper; the next installment would include the solution for all \( n \). But, of course, much happened after first paper before Löwner’s theory became a tool in de Branges’ proof of the Bieberbach Conjecture.

The eminent mathematician, Professor Sheng Gong, tells this story of the Bieberbach Conjecture by presenting a large sample of the mathematical results it inspired. In particular, his survey includes de Branges’ proof of the conjecture. To his original Chinese version of this book, Professor Gong has added a presentation of L. Weinstein’s simplification of the de Branges’ proof, H. Wilf’s comments on Weinstein’s proof and some others.

Professor Sheng Gong has had a dynamic career. As a student he studied with the internationally respected mathematician, Hua Lou-keng. Through the years, Gong’s principal employer has been the important University of Science and Technology of China. (There was a hiatus during the Cultural Revolution to acquire first hand knowledge of rural agriculture.) He held many administrative positions; in particular, he became the vice president
in charge of foreign affairs and personnel at USTC. Also Professor Gong has visited several American universities, including the University of California at San Diego.

The mathematical interests of Professor Gong have been in one and several complex variables. Indeed, he is one of the founders of modern complex analysis in China. Four of his Chinese books include *Harmonic Analysis on Classical Groups*, *The Integral of Cauchy Type on the Ball*, *Convex and star-like mappings in several complex variables* and *The Bieberbach Conjecture*. Each of these books has been translated into English and published for the benefit of mathematicians in the West.

Professor Gong has used his expertise as a writer, a teacher and a research mathematician to create an attractive, readable monograph. This work is accessible to those who know the standard material in a one year graduate complex analysis course. Care has been taken to present the work in as self-contained a form as possible. Each theorem presented is worthwhile in itself. And, as a collection, these results have the additional interest of being a case study in the development of mathematics.

Carl H. FitzGerald

July 1994 at UCSD
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Weinstein, L.

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Ye, Z. Q.

LIST OF SYMBOLS

$\text{Aut}(\Omega)$ group of holomorphic automorphisms on domain $\Omega$, 1,165
$B^n$ unit ball in $\mathbb{C}^n$, 162
$B_p$ Reinhardt domain \( \{ z = (z_1, \cdots, z_n) | \|z\|_p = (\sum_{i=1}^n |z_i|^p)^{\frac{1}{p}} < 1 \}, p > 1, 161 \)
$D$ unit disk, 1
$\Delta$ exterior of unit disk, 7
$\Gamma$ connection of Poincaré metric, 132
$\nabla$ covariant derivative, 132
$\delta f \overline{\delta s}$ intrinsic derivative, 133
$P_n(x)$ Legendre polynomial of degree $n$, 137
$P^k_n(x)$ Ferrer associated Legendre function of degree $n$ and order $k$, 138-9
$P^{(\alpha,\beta)}_n(x)$ Jacobi polynomial, 105
$P^{(\lambda)}_n(x)$ ultraspherical polynomial, 110
$\mathbf{2F}_1(a, b; c; t)$ hypergeometric function, 108
$\mathbf{3F}_2(a, b, c; d, e; t)$ hypergeometric function, 108
$S$ normalized univalent functions, 2
$S_0$ linear invariant family, 2
$\Sigma$ functions univalent in $\Delta$, 7
$\Sigma'$ non-vanishing functions in $\Sigma$, 7
$\tilde{\Sigma}$ full mappings, 7
$J_f(z)$ Jacobian of a mapping $f$ at $z$, 150
$K(z)$ Koebe function, 2
$K(z, \zeta)$ Bergman kernel function, 1
$f \prec g$ subordination, 29
$f \ast g$ convolution, 30

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<td>$M_{\infty}(r, f)$</td>
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<td>$M_p(r, f)$</td>
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