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Anomaly and Obstruction, Part I

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Preface

With the advent of the method of pseudo-holomorphic curves developed by Gromov in the 80's and the subsequent Floer's invention of elliptic Morse theory resulted in Floer cohomology, the landscape of symplectic geometry has changed drastically. Many previously intractable problems in symplectic geometry were solved by the techniques of pseudo-holomorphic curves, and the concept of symplectic topology gradually began to take shape. This progress was accompanied by parallel developments in physics most notably in closed string theory.

In 1993, partially motivated by Donaldson's pants product construction in Floer cohomology, the first named author introduced the structure of an A_∞-category in symplectic geometry whose objects are Lagrangian submanifolds and whose morphisms are the Floer cohomologies (or complexes). Based on this algebraic framework, Kontsevich proposed the celebrated homological mirror symmetry between the derived category of coherent sheaves and the Fukaya category of Lagrangian submanifolds in his 1994 ICM talk in Zürich. Enhanced by the later development in open string theory of D-branes, this homological mirror symmetry has been a source of many new insights and progresses in both algebraic geometry and symplectic geometry as well as in physics. However the rigorous formulation of homological mirror symmetry has not been made, largely due to lack of understanding the Floer theory of Lagrangian submanifolds itself.

In this book, we explain how the obstruction to and anomaly in the construction of Floer cohomology arise, provide a precise formulation of the obstructions and then carry out detailed algebraic and analytic study of deformation theory of Floer cohomology. It turns out that even a description of such an obstruction (in a mathematically precise way) requires new homological algebra of filtered A_∞-algebras. In addition, verification of existence of such an algebraic structure in the geometric context of Lagrangian submanifolds requires non-trivial analytic study of the corresponding moduli space of pseudo-holomorphic discs. We also provide various immediate applications of the so constructed Floer cohomology to problems in symplectic topology. Many of these improve the previously known results obtained via Floer theory and some firsthand applications to homological mirror symmetry are new. We expect more nontrivial applications of the theory will soon follow as its true potential is unveiled and then realized.

While we have been preparing this book, there have been several important developments in symplectic geometry and in related fields. The relationship between topological strings, D-branes and pseudo-holomorphic curves and symplectic Floer theory is now more clearly understood. The usage of higher algebraic structures in Floer theory, which we have been promoting while writing this book, has now become a popular and essential area of research. Furthermore advances of the techniques handling various moduli spaces of solutions to nonlinear PDE's, intertwined with the formalism of higher algebraic structures, has now made the geometric picture more transparent. This will help facilitate the further progression of the geometric theory. In this book we take full advantage of these developments and provide the Floer theory of Lagrangian submanifolds in the most general form available at this time. We hope that this book will be a stepping stone for future advancements in symplectic geometry and homological mirror symmetry.

Our collaboration which has culminated in completion of this book started during the 1996 (8 July–12 July) conference held in Ascona, Switzerland. We hardly imagined then that our project would continue to span more than 10 years. We have greatly enjoyed this collaboration and hope to continue it into the coming decades. In fact our second journey into newly landscaped field of symplectic topology and mirror symmetry has already begun, and we hope to garner more fruits of collaboration: The scene in front looks very different and much more exciting than the one we left behind 13 years ago!

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This is a two-volume series research monograph on the general Lagrangian Floer theory and on the accompanying homological algebra of filtered A_∞-algebras. This book provides the most important step towards a rigorous foundation of the Fukaya category in general context. In Volume I, general deformation theory of the Floer cohomology is developed in both algebraic and geometric contexts. An essentially self-contained homotopy theory of filtered A_∞ algebras and A_∞ bimodules and applications of their obstruction-deformation theory to the Lagrangian Floer theory are presented. Volume II contains detailed studies of two of the main points of the foundation of the theory: transversality and orientation. The study of transversality is based on the virtual fundamental chain techniques (the theory of Kuranishi structures and their multisections) and chain level intersection theories. A detailed analysis comparing the orientations of the moduli spaces and their fiber products is carried out. A self-contained account of the general theory of Kuranishi structures is also included in the appendix of this volume.

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