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The Generalized Riemann Integral

Robert M. McLeod

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THE GENERALIZED RIEMANN INTEGRAL

By

ROBERT M. McLEOD

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THE GENERALIZED RIEMANN INTEGRAL

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ROBERT M. McLEOD

Kenyon College

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PREFACE

In calculus courses we learn what integrals are and how to use them to compute areas, volumes, work and other quantities which are useful and interesting. The calculus sequence, and frequently the whole of the undergraduate mathematics program, does not reach the most powerful theorems of integration theory. I believe that the generalized Riemann integral can be used to bring the full power of the integral within the reach of many who, up to now, get no glimpse of such results as monotone and dominated convergence theorems. As its name hints, the generalized Riemann integral is defined in terms of Riemann sums. It reaches a higher level of generality because a more general limit process is applied to the Riemann sums than the one familiar from calculus. This limit process is, all the same, a natural one which can be introduced through the problem of approximating the area under a function graph by sums of areas of rectangles. The path from the definition to theorems exhibiting the full power of the integral is direct and short.

I address myself in this book to persons who already have an acquaintance with integrals which they wish to extend and to the teachers of generations of students to come. To the first of these groups, I express the hope that the organization of the work will make it possible for you to extract the principal results without struggling through technical details which you find formidable or extraneous to your purposes. The technical level starts low at the opening of each chapter. Thus you are invited to follow each chapter as far as you wish and then to skip to the beginning of the next. To readers who do wish to see all the details of the arguments, let me say that they are given. It was a virtual necessity to include them. There are no works to refer you to which are generally available and compatible with this one in approach to integration.

I first learned of the generalized Riemann integral from the pioneering work of Ralph Henstock. I am in his debt for the formulation of the basic concept and for many important methods of proof. Nevertheless, my presentation of the subject differs considerably from his. In particular, I chose to use only a part of his technical vocabulary and to supplement the part I selected with terms from E. J. McShane and other terms of my own devising.

I wish to express my appreciation to the members of the Subcommittee on Carus Monographs for their encouragement. I am particularly indebted to D. T. Finkbeiner. His support enabled me to persevere through the years since this writing project began. He and Helene Shapiro also worked through an earlier version of the book and provided helpful comments.

I thank the Department of Mathematical Sciences of New Mexico State University for its generous hospitality during a sabbatical leave year devoted in large part to the writing of the first version of the book. Finally, I thank Jackie Hancock, Joy Krog, and Hope Weir for expert typing.

Gambier, Ohio
January, 1980

ROBERT M. MCLEOD

LIST OF SYMBOLS

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APPENDIX

SOLUTIONS OF IN-TEXT EXERCISES

Chapter 1

1. Begin by choosing an integer m so that $1/m \leqslant \epsilon$. Enclose each number $1/j$ in an interval $[1/j - d_j, 1/j + d_j]$ which is contained in $\gamma(1/j)$ for $j = 1, 2, \ldots, m$. These can also be chosen so that $1/j + d_j < 1/(j-1) - d_{j-1}$ for $j = 2, 3, \ldots, m$ and $0 < 1/m - d_m$.

Tag $[1/j - d_j, 1/j + d_j]$ with $1/j$ for $2 \leqslant j \leqslant m$. Tag $[1 - d_1, 1]$ with 1. The rest of $[0, 1]$ is $[0, 1/m - d_m]$ and each of the intervals $[1/j + d_j, 1/(j-1) - d_{j-1}]$ for $2 \leqslant j \leqslant m$. Tag the former with 0 and the latter with any of its points, say $1/j + d_j$. Now we have a division of $[0, 1]$ with $2m$ intervals tagged in such a way as to be γ-fine.

2. (a) Let $\mathfrak{D} \in R_\delta$ and let $zJ \in \mathfrak{D}$. Since J has length less than δ it is contained in any open interval of length 2δ centered on a point of J. Thus $J \subseteq (z - \delta, z + \delta)$. In consequence $\mathfrak{D} \in GR_\delta$.

When $\mathfrak{D} \in GR_\delta$ and $zJ \in \mathfrak{D}$, $J \subseteq (z - \delta, z + \delta)$. Since the length of $(z - \delta, z + \delta)$ is 2δ and the endpoints of J are between $z - \delta$ and $z + \delta$, $L(J) < 2\delta$. Consequently $\mathfrak{D} \in R_{2\delta}$.

(b) Suppose the Riemann integral of f on $[a, b]$ exists. Choose δ so that $|\int_a^b f - fL(\mathfrak{D})| < \epsilon$ when $L(J) < \delta$ for all

zJ in $\mathcal{D}$. That is, the inequality holds for all $\mathcal{D}$ in R_δ. Let $\gamma(z) = (z - \delta/2, z + \delta/2)$ for all z in $[a, b]$. Then the set of all γ-fine $\mathcal{D}$ is $GR_{\delta/2}$. Since $GR_{\delta/2} \subseteq R_\delta$, the generalized Riemann integral of f exists and is the same as its Riemann integral.

Now let f be a generalized Riemann integrable function with the added property that $|\int_a^b f - fL(\mathcal{D})| < \epsilon$ for all $\mathcal{D}$ in GR_δ for some positive δ. Since $R_\delta \subseteq GR_\delta$ the Riemann integral of f also exists.

Now we have characterized the Riemann integrable functions among the generalized Riemann integrable functions as those for which there is a gauge $\gamma(z) = (z - \delta, z + \delta)$ with constant δ such that $|\int_a^b f - fL(\mathcal{D})| < \epsilon$ for all γ-fine $\mathcal{D}$.

3. Let $f(x) = 0$ when x is irrational. Let p/q be a fraction in lowest terms. Set $f(p/q) = q$. Since the rationals are countable this function is integrable on any interval $[a, b]$ and its integral is zero, according to Example 4, p. 19. Also f is unbounded on every interval $[c, d]$ for the following reasons. Let M be given. Select a prime number q such that $q > M$ and $2/q < d - c$. There is an integer k such that $k/q \leqslant c < (k+1)/q$. Then $(k+2)/q < d$. The prime q divides at most one of $k + 1$ and $k + 2$. Thus $f((k+1)/q) = q$ or $f((k+2)/q) = q$.

4. There are elements a and b in $\overline{\mathbf{R}}$ such that I is one of (a, b), $[a, b)$, $(a, b]$, and $[a, b]$. Fix d in (a, b). There is a function f on I such that F and G are primitives of f on I, hence on any *closed* subinterval of I. For any x in I such that $x < d$, $\int_x^d f = F(d) - F(x)$ and $\int_x^d f = G(d) - G(x)$. Thus $F(x) = G(x) + K$ where $K = F(d) - G(d)$ when $x \in I$ with $x < d$. When $x \in I$ and $d < x$, $\int_d^x f = F(x) - F(d) = G(x) - G(d)$. Again $F(x) = G(x) + K$. Trivially $F(d) = G(d) + K$. Thus $F(x) = G(x) + K$ for all x in I.

5. There is a gauge γ on $[a, b]$ such that $|\int_a^b f - fL(\mathcal{D})| < \epsilon$ when $\mathcal{D}$ is a γ-fine division of $[a, b]$. Let $s \in (a, b)$.

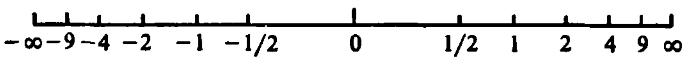
Since f is also integrable on $[s, b]$, there is a gauge γ_s such that $|\int_s^b f - fL(\mathcal{E})| < \epsilon$ when $\mathcal{E}$ is a γ_s-fine division of $[s, b]$. It is possible to choose γ_s so that $\gamma_s(z) \subseteq \gamma(z)$, too. Choose c so that $c \in \gamma(a)$ and $|f(a)|L([a, c]) < \epsilon$. Let $s \in (a, c)$. Let $\mathcal{E}$ be a γ_s-fine division of $[s, b]$. Let $\mathfrak{D} = \{a[a, s]\} \cup \mathcal{E}$. Then $\mathfrak{D}$ is a γ-fine division of $[a, b]$. Now

$$\left|\int_a^b f - \int_s^b f\right| \leqslant \left|\int_a^b f - fL(\mathfrak{D})\right| + \left|fL(\mathcal{E}) - \int_s^b f\right| + |f(a)L([a, s])|.$$

Each term on the right is less than ϵ. Hence $\lim_{s\to a}\int_s^b f = \int_a^b f$.

6. We know that $\int_0^\infty f$ exists if and only if $\lim_{t\to\infty}\int_0^t f$ exists. Also $\int_0^\infty f = \lim_{t\to\infty}\int_0^t f$. Using a primitive of f on $[0, t]$ we get $\int_0^t f = \sum_{k=1}^m a_k + (t - m)a_{m+1}$ where $m \leqslant t < m + 1$. When $\int_0^\infty f$ exists we specialize t to integer values and get $\int_0^\infty f = \lim_{m\to\infty}\int_0^m f = \sum_{k=1}^\infty a_k$. Conversely, when the series converges, $\lim_{m\to\infty} a_{m+1} = 0$. Thus $\lim_{t\to\infty}\int_0^t f = \lim_{m\to\infty}\sum_{k=1}^m a_k = \sum_{k=1}^\infty a_k$.

7. (a) This is the scale:



(b) Clearly h is strictly increasing and maps $[-\infty, \infty]$ onto $[-1, 1]$. Given x and y in $\overline{\mathbf{R}}^p$ with $x \neq y$, there is some coordinate where they differ, say $x_i \neq y_i$. Then $h(x_i) \neq h(y_i)$ so that $H(x) \neq H(y)$. We have shown H is one-to-one. To show that it is onto, take y such that $-1 \leqslant y_i \leqslant 1$ for $1 \leqslant i \leqslant p$. There is x_i in $\overline{\mathbf{R}}$ such that $h(x_i) = y_i$. Hence H maps the point x with these components x_i onto y.

Since h is increasing it maps $[u, v]$ onto $[h(u), h(v)]$. Consequently H maps $[u_1, v_1] \times \cdots \times [u_p, v_p]$ onto the Cartesian product of the intervals $[h(u_i), h(v_i)]$. The same is true of intervals other than closed intervals.

(c) Draw a square. Put on its edges the scale shown in part (a).

Chapter 2

1. Fix γ_1 and γ_2 so that $|fM(\mathcal{D}) - \int_I f| < \epsilon$ when $\mathcal{D}$ is γ_1-fine and $|gM(\mathcal{D}) - \int_I g| < \epsilon$ when $\mathcal{D}$ is γ_2-fine. Let $\gamma(z) \subseteq \gamma_1(z) \cap \gamma_2(z)$ for all z in $[a, b]$. Then $\mathcal{D}$ is γ_1-fine and γ_2-fine whenever it is γ-fine. Thus

$$\left|(f+g)M(\mathcal{D}) - \int_I f - \int_I g\right|$$

$$\leqslant \left|fM(\mathcal{D}) - \int_I f\right| + \left|gM(\mathcal{D}) - \int_I g\right| < 2\epsilon$$

when $\mathcal{D}$ is γ-fine since $(f+g)M(\mathcal{D}) = fM(\mathcal{D}) + gM(\mathcal{D})$. This shows that $f + g$ is integrable and that $\int_I f + \int_I g$ is its integral.

Since $(cf)M(\mathcal{D}) = c(fM(\mathcal{D}))$ we also have

$$\left|(cf)M(\mathcal{D}) - c\int_I f\right| \leqslant |c|\left|fM(\mathcal{D}) - \int_I f\right| \leqslant |c|\epsilon$$

when $\mathcal{D}$ is γ-fine. Thus $c\int_I f$ is the integral of cf.

A standard induction argument shows that $\sum_{k=1}^n c_k f_k$ is integrable when each f_k is integrable and that $\sum_{k=1}^n c_k \int_I f_k$ is its integral.

2. Let γ be a gauge on I such that $|\int_I f - fM(\mathcal{D})| < \epsilon$ and $|\int_I g - gM(\mathcal{D})| < \epsilon$ when $\mathcal{D}$ is γ-fine. Since g is real valued, $gM(\mathcal{D}) < \int_I g + \epsilon$. Also $|fM(\mathcal{D})| \leqslant gM(\mathcal{D})$ for all

$\mathfrak{D}$ since $|f| \leqslant g$ and M takes nonnegative values. Thus

$$\left|\int_I f\right| \leqslant |fM(\mathfrak{D})| + \left|\int_I f - fM(\mathfrak{D})\right|$$

$$\leqslant gM(\mathfrak{D}) + \epsilon < \int_I g + 2\epsilon.$$

Since ϵ is arbitrary, $|\int_I f| \leqslant \int_I g$, as claimed.

3. By hypothesis, for each whole number n there is a gauge γ_n such that $|fM(\mathfrak{D}) - fM(\mathfrak{E})| < 1/n$ when $\mathfrak{D}$ and $\mathfrak{E}$ are γ_n-fine divisions of I. We may also suppose $\gamma_j(z) \subseteq \gamma_i(z)$ when $i < j$. (Replace $\gamma_n(z)$ by $\gamma_1(z) \cap \gamma_2(z) \cap \cdots \cap \gamma_n(z)$.) For each n fix a γ_n-fine division $\mathfrak{D}_n$.

Consider the sequence of elements $fM(\mathfrak{D}_n)$ in $\mathbf{R}^q$. Let's show that this is a Cauchy sequence. Suppose $i < j$. Then $\mathfrak{D}_j$ is not only γ_j-fine it is also γ_i-fine since γ_j is stricter than γ_i. Hence

$$|fM(\mathfrak{D}_j) - fM(\mathfrak{D}_i)| < 1/i.$$

It follows easily that $(fM(\mathfrak{D}_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbf{R}^q$. Consequently it converges to a limit A in $\mathbf{R}^q$.

To show that A is the integral of f, fix N so that $1/N < \epsilon/2$ and $|A - fM(\mathfrak{D}_N)| < \epsilon/2$. Let $\mathfrak{D}$ be γ_N-fine. Then

$$|fM(\mathfrak{D}) - A| \leqslant |fM(\mathfrak{D}) - fM(\mathfrak{D}_N)| + |fM(\mathfrak{D}_N) - A|$$

$$< 1/N + \epsilon/2 < \epsilon.$$

Thus $A = \int_I f$.

4. The techniques needed for the solution of this exercise anticipate the discussion on p. 75. There it is shown that $\mathfrak{E}$ is a subset of a division $\mathfrak{D}$ of I. Let $\mathfrak{F} = \mathfrak{D} - \mathfrak{E}$. Then $\nu(\mathfrak{E}) + \nu(\mathfrak{F}) = \nu(\mathfrak{D})$. Since $0 \leqslant \nu(J)$ for all J, $0 \leqslant \nu(\mathfrak{F})$. Moreover $\nu(\mathfrak{D}) = \nu(I)$ since ν is finitely additive. Consequently $\nu(\mathfrak{E}) \leqslant \nu(I)$.

5. Let $E_t = \{x \in [a, b] : \sigma(x) = t\}$. Then $E_t \subseteq C$ when $t = \infty$ and when $t = -\infty$. When t is finite we need to show that $E_t - C$ is countable. Let $x \in E_t - C$. Then $\sigma'(x) \neq 0$, consequently there is δ_x such that $\sigma(y) \neq \sigma(x)$ when $0 < |y - x| < \delta_x$. In other words, the set $E_t - C$ is made up of isolated points. It must be countable. To see this, let $F_n = \{x \in [-n, n] : \delta_x > 1/n\}$. Then $E_t - C = \bigcup_{n=1}^{\infty} F_n$. It suffices to show that each F_n is finite. Form a division of $[-n, n]$ into subintervals of length $1/n$. Each subinterval contains at most one point of F_n. Thus F_n is finite.

6. There are gauges γ_1 and γ_2 on $[a, c]$ and $[d, b]$ such that $|\int_a^c f - fL(\mathfrak{D}_1)| < \epsilon/2$ when $\mathfrak{D}_1$ is a γ_1-fine division of $[a, c]$ and $|\int_d^b f - fL(\mathfrak{D}_2)| < \epsilon/2$ when $\mathfrak{D}_2$ is a γ_2-fine division of $[d, b]$. We may also suppose $\gamma_1(z) \subseteq \gamma(z)$ for all z in $[a, c]$ and $\gamma_2(z) \subseteq \gamma(z)$ for all z in $[d, b]$. Fix $\mathfrak{D}_1$ and $\mathfrak{D}_2$. Let $\mathfrak{S}$ be a γ-fine division of $[c, d]$. Set $\mathfrak{D} = \mathfrak{D}_1 \cup \mathfrak{S} \cup \mathfrak{D}_2$. Then $\mathfrak{D}$ is a γ-fine division of $[a, b]$. Consequently,

$$\left|\int_c^d f - fL(\mathfrak{S})\right| \leq \left|\int_a^b f - fL(\mathfrak{D})\right| + \left|fL(\mathfrak{D}_1) - \int_a^c f\right| + \left|fL(\mathfrak{D}_2) - \int_d^b f\right|.$$

This uses the finite additivity of the integral. Now these terms are less than ϵ, $\epsilon/2$, and $\epsilon/2$. Hence the conclusion follows.

(If $a = c$ or $b = d$, some terms become zero but otherwise the argument goes the same.)

Note: A sharpening of the argument above shows that $|\int_c^d f - fL(\mathfrak{S})| \leq \epsilon$. This latter inequality is a special case of Henstock's lemma. (See p. 74.)

7. (a) When $f \geq 0$ the function $s \to \int_s^b f$ is decreasing on the interval (a, b) and nonnegative. Such a function is bounded above if and only if $\lim_{s \to a} \int_s^b f$ exists. Existence of this limit is exactly the criterion for existence of $\int_a^b f$.

(b) The function $F : s \to \int_s^b f$ has a limit at a provided the Cauchy criterion is satisfied. That is, it suffices to find c such that $|F(t) - F(s)| < \epsilon$ when $a < s < t < c$. Since $\int_a^b g$ exists, $\lim_{s \to a} \int_s^b g$ does exist. Hence there is c such that $|\int_s^b g - \int_t^b g| < \epsilon$ when $a < s < t < c$. But now

$$|F(t) - F(s)| = \left|\int_s^t f\right| \leqslant \int_s^t g = \int_s^b g - \int_t^b g < \epsilon.$$

Chapter 3

1. Since $\lim_{t \to \infty} \int_1^t 1/\sqrt{x}\, dx = \infty$, we know that $\int_1^\infty 1/\sqrt{x}\, dx$ does not exist. Set $f_n(x) = 1/\sqrt{x}$ when $1 \leqslant x \leqslant n$ and $f_n(x) = 0$ when $n < x$. Then f_n converges uniformly to $1/\sqrt{x}$ on $[1, \infty)$.

2. The function f is integrable on $[a, b]$. Set $F(c) = \lim_{n \to \infty} F_n(c)$ and $F(x) = F(c) + \int_c^x f$ elsewhere in $[a, b]$. (We are using the oriented integral here.)

We are going to get an inequality from which two separate deductions can be made. By hypothesis there is n_ϵ such that $|f_n(x) - f(x)| < \epsilon$ for all x in $[a, b]$ and all $n \geqslant n_\epsilon$. Let u and v be in $[a, b]$. Then

$$\begin{aligned} |F_n(v) - F_n(u) - (F(v) - F(u))| &= \left|\int_u^v (f_n - f)\right| \\ &\leqslant \epsilon|v - u| \end{aligned}$$

when $n \geqslant n_\epsilon$.

The first conclusion is uniform convergence of F_n to F. To get it let $u = c$. Then one more application of the triangle inequality yields

$$|F_n(v) - F(v)| \leqslant |F_n(c) - F(c)| + \epsilon(b - a).$$

Uniform convergence follows easily.

The next conclusion is uniform convergence of difference quotients. Let $u = x$ and $v = x + t$ with $t \neq 0$. Divide by $|t|$. Then

$$\left| \frac{F_n(x+t) - F_n(x)}{t} - \frac{F(x+t) - F(x)}{t} \right| \leqslant \epsilon$$

for all x and t such that x and $x + t$ are in $[a, b]$ and all $n \geqslant n_\epsilon$.

Choose x so that $F_n'(x) = f_n(x)$ for every integer n. Select δ_ϵ such that

$$\left| \frac{F_n(x+t) - F_n(x)}{t} - f_n(x) \right| < \epsilon$$

when $n = n_\epsilon$ and $|t| < \delta_\epsilon$. Now

$$\left| \frac{F(x+t) - F(x)}{t} - f(x) \right|$$

$$\leqslant \left| \frac{F(x+t) - F(x)}{t} - \frac{F_n(x+t) - F_n(x)}{t} \right|$$

$$+ \left| \frac{F_n(x+t) - F_n(x)}{t} - f_n(x) \right| + |f_n(x) - f(x)|.$$

Each term is less than ϵ when $n \geqslant n_\epsilon$ and $|t| < \delta_\epsilon$. Consequently $F'(x) = f(x)$.

Let C_n be the set of values of x for which we cannot claim that $F_n'(x) = f_n(x)$. Then C_n is countable and $\bigcup_{n=1}^{\infty} C_n$ is also countable. Outside this set we know that

$F' = f$. Since F is continuous, it is a primitive of f on $[a, b]$.

(This argument can be greatly abbreviated by use of the iterated limits theorem given in Section S3.9.)

3. It is enough to consider increasing functions. We may apply the proposition proved in the preceding exercise. Thus it is our goal to define functions f_n which converge uniformly to f and have primitives. Step functions are the right choice. Since f is increasing, for any number K in $[f(a), f(b)]$ there is at least one number c in $[a, b]$ such that $f(x) \leqslant K$ when $x \leqslant c$ and $K \leqslant f(x)$ when $c < x$. Let the integer n be given and set $d_n = (f(b) - f(a))/n$. For $i = 1, 2, \ldots, n-1$, let $K_i = f(a) + id_n$. Choose a corresponding c_i. Set $c_0 = a$ and $c_n = b$. Define f_n so that $f_n(c_i) = f(c_i)$ for $i = 0, \ldots, n$ and $f_n(x) = K_i$ when $c_{i-1} < x < c_i$. It is immediate that $|f(x) - f_n(x)| < d_n$ for all x in $[a, b]$. Consequently f_n converges uniformly to f on $[a, b]$.

Chapter 4

1. Recall that E is gfM-null whenever E is fM-null. Now apply this with the constant function with value 1 in place of f and f in place of g. Thus E is fM-null when E is M-null. By definition this means that $|f|M(\mathfrak{S}) < \epsilon$ for every γ-fine partial division $\mathfrak{S}$ whose tags are in E. When $\mathfrak{D}$ is a division of I and $\mathfrak{S}$ is the subset of $\mathfrak{D}$ whose tags are in E, $|f|M(\mathfrak{D}) = |f|M(\mathfrak{S})$ since f vanishes outside E. Thus $\int_I |f| = 0$.

2. There is a gauge γ on I such that $|f|M(\mathfrak{D}) < \epsilon$ when $\mathfrak{D}$ is a γ-fine division of I. Let $E = \{x \in I : f(x) \neq 0\}$. Let $\mathfrak{S}$ be a γ-fine partial division of I whose tags are in E. There is a γ-fine division $\mathfrak{D}$ such that $\mathfrak{S} \subseteq \mathfrak{D}$. Then

$|f|M(\mathcal{S}) = |f|M(\mathcal{D}) < \epsilon$. Thus E is $|f|M$-null. Let $g(x) = 1/|f(x)|$ when $x \in E$. Then E is $g|f|M$-null, i.e., M-null.

3. Let M' and M be the interval measures in $\mathbf{R}^{p-1}$ and $\mathbf{R}^p$, respectively. For a given ϵ we must define γ on E so that $M(\mathcal{S}) < \epsilon$ when $\mathcal{S}$ is a γ-fine partial division with tags in E.

Using the continuity of f choose $\gamma_1(z)$ so that $|f(x) - f(z)| < \epsilon'$ when $x \in I \cap \gamma_1(z)$. Fix a γ_1-fine division $\mathcal{D}_1$ of I. For each zJ in $\mathcal{D}_1$ let H_J be the interior of J, i.e., the open interval obtained by peeling the faces from J. Let B be the union of all faces of all J of $\mathcal{D}_1$. Then $B \times \mathbf{R}$ is an M-null subset of $\mathbf{R}^p$ since it is a finite union of degenerate intervals. Thus there is γ defined on $B \times \mathbf{R}$ such that $M(\mathcal{S}) < \epsilon/2$ when $\mathcal{S}$ is γ-fine with tags in $B \times \mathbf{R}$.

When $x \in I - B$ there is a unique J such that $x \in H_J$. Set $\gamma(x, f(x)) = H_J \times (f(z) - \epsilon', f(z) + \epsilon')$. Let $\mathcal{S}$ be γ-fine with tags in E. Separate $\mathcal{S}$ into subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ having tags in $B \times \mathbf{R}$ and $(I - B) \times \mathbf{R}$. Then $M(\mathcal{S}_1) < \epsilon/2$ and $M(\mathcal{S}_2) \leqslant \sum_J M'(J)2\epsilon' \leqslant 2\epsilon' M'(I)$. (The first inequality on $M(\mathcal{S}_2)$ results from grouping together all terms associated with a single J of $\mathcal{D}_1$.) The choice $\epsilon' = \epsilon/(4M'(I))$ gives $M(\mathcal{S}) < \epsilon$, as desired.

4. According to Example 5, p. 86, the functions $g \wedge (i^{-1}f_n)$ are integrable. There is a bound A for $\int_I f_n$. Since $g \wedge (i^{-1}f_n) \leqslant i^{-1}f_n$ for all n, $\int_I (g \wedge (i^{-1}f_n)) \leqslant i^{-1}A$. Since $f_n \leqslant f_{n+1}$ the sequence $g \wedge (i^{-1}f_n)$ increases for fixed i. Thus $\int_I h_i = \lim_{n\to\infty} \int_I (g \wedge (i^{-1}f_n)) \leqslant i^{-1}A$, too. To see that $\lim_{i\to\infty} h_i = h$, first let $x \in E$. Then $g \wedge (i^{-1}h_n)$ has value 1 at x for all large n. Hence $h_i(x) = 1$ and $\lim_{i\to\infty} h_i(x) = 1$. When $x \in I - E$, $(g \wedge (i^{-1}f_n))(x) \leqslant i^{-1}f(x)$, hence $h_i(x) \leqslant i^{-1}f(x)$. Consequently, $\lim_{i\to\infty} h_i(x) = 0$. A second use of monotone convergence gives $\int_I h = \lim_{i\to\infty} \int_I h_i = 0$.

5. The essentials of the solution can be conveyed by an example in $\mathbf{N}$. Consider the sequence $1, -1, 1, 1/2, -1/2, 1/2, 1/3, -1/3, 1/3, \ldots$. This is our function $f: \mathbf{N} \to \mathbf{R}$. Note that the terms go in groups of three. Let's replace every third member by zero. That means integrate f over $E = \{1, 2, 4, 5, 7, 8, \ldots\}$. It is easy to see that $\int_E f$ exists. Next replace the first member of each group by zero; i.e., integrate f over $F = \{2, 3, 5, 6, 8, 9, \ldots\}$. Integration over $E \cap F$ is the same as summing the middle terms of each group. These terms are negatives of terms of the harmonic series. Thus $\int_{E \cap F} f$ does not exist. Neither does $\int_{E \cup F} f$ for much the same reason.

6. For each positive integer n, partition I into a finite collection $\mathfrak{F}_n$ of pairwise disjoint intervals K such that the diameter of K is no more than $1/n$. Also make the choices so that each interval of $\mathfrak{F}_{n+1}$ is contained in an interval of $\mathfrak{F}_n$; i.e., $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$.

Define f_n from $\mathfrak{F}_n$ as follows. For each x in I, there is just one interval K of $\mathfrak{F}_n$ to which x belongs. Let $f_n(x) = \text{lub}\{f(x) : x \in K\}$. This is meaningful everywhere since f is bounded above. (Recall that f has a maximum on E and f is constant on $I - E$.)

The sequence $(f_n)_{n=1}^{\infty}$ decreases since $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$.

Now let $x \in I - E$. There is an open interval $\gamma(x)$ which does not intersect E since E is closed. Every interval of sufficiently small diameter containing x is a subset of $\gamma(x)$. Since f vanishes on $\gamma(x)$ so does $f_n(x) = 0$ for sufficiently large n.

When $x \in E$ there is an open interval $\gamma(x)$ such that $|f(y) - f(x)| < \epsilon$ for all $y \in E \cap \gamma(x)$. There may be points in $\gamma(x)$ which are not in E. But f is zero at those points. Since $0 \leqslant f(x)$, $f(y) \leqslant f(x) + \epsilon$ for all y in $\gamma(x)$. For large enough n the interval of $\mathfrak{F}_n$ which contains x is a subset of $\gamma(x)$. Thus $f_n(x) \leqslant f(x) + \epsilon$ for all sufficiently large n. The definition of f_n also insures that $f(x) \leq f_n(x)$. Hence $f(x) \leq f_n(x) \leq f(x) + \epsilon$ for sufficiently large n.

We have shown that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in I$, as required.

7. Set $E_1 = H_1$ and $E_n = H_n - H_{n-1}$ for $n \geq 2$. Note that $H_{n-1} \subseteq H_n$. Thus f is integrable on E_n. Since f is also bounded on each E_n, it is absolutely integrable on E_n. Moreover $\int_{E_n}|f| \leq b_n(M(H_n) - M(H_{n-1}))$ for $n \geq 2$. But $M(H_n) - M(H_{n-1}) = (2n)^p - (2(n-1))^p \leq p2^p n^{p-1}$. Thus $\sum_{n=1}^{\infty}\int_{E_n}|f|$ is convergent since $\sum_{n=1}^{\infty} b_n n^{p-1}$ is assumed convergent. This implies absolute integrability of f.

8. We know that $|f|$ is integrable on E_n for all n since $|f| \leq g$ and g is integrable on I. By induction f is absolutely integrable on $\bigcup_{i=1}^{n} E_i$ for all n since absolute integrability carries over to unions of two sets. Let $F_1 = E_1$ and $F_n = \bigcup_{i=1}^{n} E_i - (\bigcup_{i=1}^{n-1} E_i)$ when $n \geq 2$. Now f is absolutely integrable on each member of the pairwise disjoint sequence $(F_n)_{n=1}^{\infty}$. Moreover $\sum_{i=1}^{n}\int_{F_n}|f| \leq \int_I g$ for all n. Thus $\sum_{n=1}^{\infty}\int_{F_n}|f|$ is finite. By the countable additivity proposition f is absolutely integrable on $\bigcup_{n=1}^{\infty} F_n$, i.e., on E.

9. The characteristic functions of $\bigcap_{i=1}^{n} E_i$ decrease to the characteristic function of $\bigcap_{i=1}^{\infty} E_i$. Thus the monotone convergence theorem yields the integrability of the intersection of the sequence.

Now $\bigcup_{i=1}^{n} E_i$ are integrable sets whose characteristic functions increase to the characteristic function of the union of the sequence. Since $\mu(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mu(E_i)$ and $\mu(\bigcup_{i=1}^{n} E_i) \leq \mu(F)$ when $\bigcup_{i=1}^{\infty} E_i$ is contained in the integrable set F, the monotone convergence theorem implies integrability of $\bigcup_{i=1}^{\infty} E_i$ under each of the conditions which have been given.

10. The conclusions in (a) follow from known properties of integrable sets, since the intersection with a bounded interval J distributes over the other operations. For instance, $(E \cup F) \cap J = (E \cap J) \cup (F \cap J)$. Thus $(E \cup F) \cap J$ is the union of integrable sets and is integrable itself. The others go similarly.

Measurability of $\bigcap_{n=1}^{\infty} E_n$ and $\bigcup_{n=1}^{\infty} E_n$ follow similarly. The inequality is trivial when the right-hand side equals ∞. When it is finite, every set E_n is integrable and we know from the previous exercise that $\bigcup_{n=1}^{\infty} E_n$ is integrable. Moreover $\mu(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mu(E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$ for all n. Monotone convergence tells us that $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

When the sets are nonoverlapping and $\bigcup_{n=1}^{\infty} E_n$ is not integrable the inequality just proved becomes equality. When $\bigcup_{n=1}^{\infty} E_n$ is integrable, we are back to countable additivity of integrable sets since every E_n must also be integrable.

Chapter 5

1. We prove that (i) implies (ii) implies (iii) implies (iv) implies (i).

Assume (i). Let $G = (-\infty, a]$ and $G' = (a, \infty)$. Then $f^{-1}(G) = I - f^{-1}(G')$. Since $f^{-1}(G')$ is measurable according to (i) the set $f^{-1}(G)$ is also measurable. This establishes (ii).

Assume (ii). Let $G = (-\infty, a)$ and set $G_n = (-\infty, a - 1/n]$. Then $\bigcup_{n=1}^{\infty} G_n = G$ and $f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(G_n)$. By (ii) each $f^{-1}(G_n)$ is measurable. Thus $f^{-1}(G)$ is also measurable and (iii) holds.

Prove (iv) from (iii) like (ii) from (i).

Deduce (i) from (iv) on the model of (iii) from (ii).

Measurability of f obviously implies (i), hence all the others. Assume (i) through (iv). Then $f^{-1}(G)$ is measurable for every unbounded open interval. Since

$(a, b) = (a, \infty) - [b, \infty)$ and $f^{-1}((a, b)) = f^{-1}((a, \infty)) - f^{-1}([b, \infty))$, it follows from (i) and (ii) that $f^{-1}(G)$ is measurable for bounded open intervals, too. Thus f is measurable.

2. In $\mathbf{R}^q$ let $J_n = [-n, n) \times \cdots \times [-n, n)$. Partition each factor $[-n, n)$ into $n2^n$ intervals of length $1/2^{n-1}$. Let each subinterval be closed on the left and open on the right. Form a partition $\mathcal{G}_n$ of J_n by taking Cartesian products. (See Fig. 3, p. 144.)

For each $G \in \mathcal{G}_n$ the set $I_n \cap f^{-1}(G)$ is a bounded measurable subset of I. Let K be the closed interval having the same faces as G. Then K has on its boundary a unique point y which is nearest to the origin. Set $f_n(x) = y$ for all x in $I_n \cap f^{-1}(G)$. Do this for all G in $\mathcal{G}_n$. This defines f_n on $I_n \cap f^{-1}(J_n)$. Set $f_n(x) = 0$ elsewhere. Then $|f_n(x)| \leqslant |f(x)|$ for all x in I. Since $\mathcal{G}_{n+1}$ is a refinement of $\mathcal{G}_n$ and $I_n \subseteq I_{n+1}$, $|f_n(x)| \leqslant |f_{n+1}(x)|$ for all x. Moreover this construction produces a nonnegative function f_n when f is nonnegative.

The convergence of $f_n(x)$ to $f(x)$ is proved as follows. Let $x \in I \cap \mathbf{R}^p$. There is n_x such that $x \in I_n$ and $f(x) \in J_n$ when $n \geqslant n_x$. Then $f(x)$ is in some G of $\mathcal{G}_n$. Since $f_n(x)$ belongs to the closed interval having the same faces as G, $|f(x) - f_n(x)| \leqslant \sqrt{q}/2^{n-1}$. Consequently, $\lim_{n\to\infty} f_n(x) = f(x)$.

3. Let $E = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f_k^{-1}(G_n)$. Let $x \in E$. Then $x \in \bigcap_{k=m}^{\infty} f_k^{-1}(G_n)$ for some m and n. Consequently, $f_k(x) > a + 1/n$ when $k \geqslant m$. Thus $\lim_{k\to\infty} f_k(x) \geqslant a + 1/n$. From this we see that $x \in f^{-1}(G)$. Thus $E \subseteq f^{-1}(G)$.

Conversely, let $x \in f^{-1}(G)$. Fix n so that $a + 1/n < f(x)$. There is m such that $f_k(x) > a + 1/n$ when $k \geqslant m$. Then $x \in f_k^{-1}(G)$ when $k \geqslant m$; i.e., $x \in \bigcap_{k=m}^{\infty} f_k^{-1}(G_n)$. Consequently, $x \in E$. This completes the proof that $f^{-1}(G) \subseteq E$. Thus the sets are equal.

4. If one of f and g is a null function the same is true of fg. Then both sides of the inequality are zero.

Suppose neither f nor g is a null function. Then $\int_I |f|^s > 0$ and $\int_I |g|^t > 0$. We can choose a positive constant c so that $\int_I |cf|^s = 1$. In fact we need only have $c^{-s} = \int_I |f|^s$ or $c^{-1} = (\int_I |f|^s)^{1/s}$. Similarly, $\int_I |kg|^t = 1$ when $k^{-1} = (\int_I |g|^t)^{1/t}$. Integration on both sides of

$$|(cf)(kg)| \leqslant |cf|^s/s + |kg|^t/t$$

yields

$$ck\int_I |fg| \leqslant 1/s + 1/t = 1.$$

Hence $\int_I |fg| \leqslant c^{-1}k^{-1}$, as desired.

Chapter 6

1. Some notation is needed. Suppose $I \subseteq \overline{\mathbf{R}}^p$. Let the r-fold integration be accomplished by expressing $\overline{\mathbf{R}}^p$ as $P_1 \times P_2 \times \cdots \times P_r$ with $r \leqslant p$. Write the intervals as products like this: $I = I_1 \times I_2 \times \cdots \times I_r$ where $I_j \subseteq P_j$ for $1 \leqslant j \leqslant r$.

We may assume $I = G \cup H$. Since G and H do not overlap, there is an integer k such that $I_j = G_j = H_j$ when $j \neq k$ and $G_k \cup H_k = I_k$ with G_k and H_k nonoverlapping.

For convenience denote iterated integrals as follows. When $i < r$ let the result of the first i integrations over an interval J be

$$\nu(J; x_{i+1}, \ldots, x_r)$$
$$= \int_{J_i} \cdots \int_{J_2}\int_{J_1} f(x_1, x_2, \ldots, x_r)\, dx_1\, dx_2 \cdots dx_i.$$

Let $\nu(J)$ be the result of the r-fold integration.

The equalities $I_j = G_j = H_j$ when $j < k$ imply

$$\nu(I; x_{i+1}, \ldots, x_r) = \nu(G; x_{i+1}, \ldots, x_r)$$
$$= \nu(H; x_{i+1}, \ldots, x_r)$$

when $i < k$. The additivity of integrals allows us to replace the second equality by summation when $i = k$. The result is either

$$\nu(I; x_{k+1}, \ldots, x_r) = \nu(G; x_{k+1}, \ldots, x_r)$$
$$+ \nu(H; x_{k+1}, \ldots, x_r)$$

when $k < r$ or, of course, $\nu(I) = \nu(G) + \nu(H)$ if it happens that $k = r$. When $k < r$ the linearity of integration applies in the last $r - k$ integrations to yield the desired final conclusion that $\nu(I) = \nu(G) + \nu(H)$. The equalities $I_j = G_j = H_j$ for $j = k + 1, \ldots, r$ must be used in these last $r - k$ integrations.

2. Since U can be expressed as a countable union of bounded sets, it is enough to give the solution for a bounded interval I. It is enough to show that there is a gauge γ on U such that

$$|f(z)M(J) - \phi(J)| < \epsilon M(J)$$

when $z \in J$ and $J \subseteq \gamma(z)$.

When $z \in U$, the continuity of f at z implies that there is $\gamma(z)$ such that $|f(x) - f(z)| < \epsilon$ when $x \in I \cap \gamma(z)$. Let $J \subseteq I \cap \gamma(z)$. On J consider the constant function g such that $g(x) = f(z)$. The iterated integral of g over J has the value $f(z)M(J)$. Then $\phi(J) - f(z)M(J)$ is the iterated integral of $f - g$. Since $|f(x) - g(x)| < \epsilon$ on J, repeated application of integral inequalities yields $|\phi(J) - f(z) \cdot M(J)| < \epsilon M(J)$.

3. The first step is to snow that the extension of f which is zero outside E has an iterated integral over any bounded interval. Actually, it is enough to work within $I = [a, b] \times [c, d]$ where $c \leq g(x) \leq h(x) \leq d$ for all x in $[a, b]$. The order of integration will always be the same as this:

$$\phi(I) = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

The inner integral exists because, for fixed x, $f(x, y)$ vanishes on $[c, g(x))$ and $(h(x), d]$ and is continuous when restricted to $[g(x), h(x)]$.

Let $[r, s] \subseteq [c, d]$. The outer integral exists if we show $\int_r^s f(x, y)\, dy$ is a continuous function of x. Thus it is appropriate to consider $\int_r^s f(x, y)\, dy - \int_r^s f(u, y)\, dy$, i.e., $\int_r^s [f(x, y) - f(u, y)]\, dy$. It is important to estimate this integrand.

Let $K_x = [r, s] \cap [g(x), h(x)]$ for each x in $[a, b]$. The uniform continuity of f on E implies that $|f(x, y) - f(u, y)| < \epsilon$ when $|x - u| < \delta$ and $y \in K_x \cap K_u$. Since f is bounded, there is a constant A such that $|f(x, y) - f(u, y)| \leq A$ for all x, u, and y. Next note that $f(x, y) - f(u, y) = 0$ when y is outside $K_x \cup K_u$. Finally, note that the total length of the intervals making up $(K_x \cup K_y) - (K_x \cap K_y)$ is no more than $|g(x) - g(u)| + |h(x) - h(u)|$.

Now suppose $|x - u| < \delta$. The estimates of the preceding paragraph give

$$\left|\int_r^s [f(x, y) - f(u, y)]\, dy\right| \leq \epsilon L(K_x \cap K_u) + A(|g(x) - g(u)| + |h(x) - h(u)|).$$

Since $L(K_x \cap K_u) \leq d - c$ and g and h are continuous, the desired continuity of the inner integral follows.

Now $\phi(J)$ is meaningful on every J contained in I. Let G and H be the graphs of g and h. Then f is continuous on $I-(G \cup H)$ and this set is $(fM-\phi)$-null. We know $G \cup H$ is M-null. It is also ϕ-null since $|\phi(J)| \leqslant AM(J)$ for every J. It follows that f is integrable on E and $\int_E f = \phi(I)$.

Chapter 7

1. Let A be the $\mathfrak{N}$-limit of $f\Delta\alpha(\mathfrak{D})$. Choose δ so that $|A - f\Delta\alpha(\mathfrak{D})| < \epsilon$ when $\|\mathfrak{D}\| < \delta$. Fix a division $\mathfrak{F}$ with $\|\mathfrak{F}\| < \delta$. Let $\mathfrak{D}$ be a refinement of $\mathfrak{F}$. Then $\|\mathfrak{D}\| \leqslant \|\mathfrak{F}\| < \delta$. Hence $|A - f\Delta\alpha(\mathfrak{D})| < \epsilon$. Therefore A is also the $\mathfrak{R}$-limit of $f\Delta\alpha(\mathfrak{D})$.

Now assume A is the $\mathfrak{R}$-limit. To show that A is the $\mathcal{G}$-limit requires the use of a special gauge associated with a division. Fix a division $\mathfrak{F}$ for which $|A - f\Delta\alpha(\mathfrak{D})| < \epsilon$ when $\mathfrak{D}$ is a refinement of $\mathfrak{F}$. Let $\gamma_{\mathfrak{F}}$ be defined so that $\gamma_{\mathfrak{F}}(z)$ contains no endpoint of $\mathfrak{F}$ distinct from z. Let $\mathfrak{D}$ be $\gamma_{\mathfrak{F}}$-fine. Then every endpoint of $\mathfrak{F}$ appears as tag of each interval of $\mathfrak{D}$ which contains that endpoint. Form $\mathfrak{S}$ from $\mathfrak{D}$ by replacing $z[u, v]$ by $z[u, z]$ and $z[z, v]$ when $u < z < v$. Then $\mathfrak{S}$ is a refinement of $\mathfrak{F}$. Moreover $f\Delta\alpha(\mathfrak{D}) = f\Delta\alpha(\mathfrak{S})$. Thus $|A - f\Delta\alpha(\mathfrak{D})| = |A - f\Delta\alpha(\mathfrak{S})| < \epsilon$ because of the choice of $\mathfrak{F}$.

2. It is enough to find γ for which $|f\Delta\alpha(\mathfrak{D}) - fg\Delta\beta(\mathfrak{D})| \leqslant \epsilon$ for all γ-fine $\mathfrak{D}$. From this statement and the triangle inequality we can deduce existence of both integrals from existence of either of them.

Let $E_n = \{x \in [a, b] : n-1 \leqslant |f(x)| < n\}$. These sets E_n cover $[a, b]$ and are pairwise disjoint. Any division $\mathfrak{D}$ falls into subsets $\mathfrak{D}_n$ having tags in E_n. Now $|f\Delta\alpha(\mathfrak{D}_n) - fg\Delta\beta(\mathfrak{D}_n)| \leqslant n|\Delta\alpha - g\Delta\beta|(\mathfrak{D}_n)$. Thus it is our goal to define γ so that $\sum_n n|\Delta\alpha - g\Delta\beta|(\mathfrak{D}_n) < \epsilon$ when $\mathfrak{D}$ is γ-fine. Henstock's lemma gets us to our goal.

Begin with gauges γ_n on $[a, b]$ so that $|\int_a^b g d\beta - g\Delta\beta(\mathcal{D})| < \epsilon/(nq2^{n+1})$ when $\mathcal{D}$ is γ_n-fine. When $z \in E_n$ set $\gamma(z) = \gamma_n(z)$. Then $\mathcal{D}_n$ is γ_n-fine provided $\mathcal{D}$ is γ-fine. Thus $|\Delta\alpha - g\Delta\beta|(\mathcal{D}_n) \leq 2q\epsilon/(nq2^{n+1})$ and $|f\Delta\alpha(\mathcal{D}_n) - fg\Delta\beta(\mathcal{D}_n)| \leq \epsilon/2^n$. Summation on n does the rest.

3. The step function f can be expressed as a linear combination of functions of the types considered in Example 3, p. 190, and Example 4, p. 190. The n open intervals give us functions g_j such that $g_j(x) = 1$ when $x \in (x_{j-1}, x_j)$ and $g_j(x) = 0$ elsewhere. The $n+1$ endpoints give us functions h_j such that $h_j(x_j) = 1$ and $h_j(x) = 0$ elsewhere. Then $f = \sum_{j=1}^n F_j g_j + \sum_{j=0}^n f(x_j)h_j$.

Part (a) follows immediately from linearity in the integrand and the results given in Examples 3 and 4.

Part (b) can be done most easily from the definition. When $\mathcal{D}$ has every x_j as a tag and at least one endpoint in each interval (x_{j-1}, x_j), the value of $\alpha\Delta f(\mathcal{D})$ is precisely the expression given in (b). It suffices to define γ so that $\gamma(z)$ contains no x_j distinct from z. Then any γ-fine $\mathcal{D}$ has the properties named and the conclusion is immediate.

4. The jumps in f occur on the left-hand side of the integer points. Consequently, from Exercise 3(b), $\int_0^t \alpha\, df = \sum_{j=1}^n \alpha(x_j)$ when $n \leq t < n+1$. Now $\int_0^\infty \alpha\, df$ exists if and only if $\lim_{t\to\infty} \int_0^t \alpha\, df$ exists. Consequently, the existence of the integral is equivalent to convergence of $\sum_{j=1}^\infty \alpha(x_j)$. Moreover $\int_0^\infty \alpha\, df = \sum_{j=1}^\infty \alpha(x_j)$.

5. Let α be the primitive F of Example 2, p. 79. Since this function is continuous, it is a regulated function. It was constructed so that it is not a function of bounded variation.

Since α is a primitive of α', $\alpha^2/2$ is also a primitive of $\alpha\alpha'$ and $\int_a^b \alpha\alpha'$ exists. It can be converted into $\int_a^b \alpha\, d\alpha$.

The failure of $\int_a^b f\, d\alpha$ to exist is the same as failure of existence of $\int_a^b f\alpha'$. Clearly, it is desirable that $f(x)$ and

$\alpha'(x)$ have the same sign. The sign of $\alpha'(x)$ is alternately positive and negative in intervals $(c_0, c_1), (c_1, c_2), \ldots,$ beginning with a positive value in (c_0, c_1). The value of α' on (c_{n-1}, c_n) is $a_n/(c_n - c_{n-1})$ where $(a_n)_{n=1}^{\infty}$ is the sequence $1, -1/2, 1/2, -1/3, 1/3, \ldots$. On $[c_{n-1}, c_n)$ let $f(x) = b_n$ where $(b_n)_{n=1}^{\infty}$ is the sequence $1, -1/\ln 2, 1/\ln 2, -1/\ln 3, 1/\ln 3, \ldots$. Then $\sum_{n=1}^{\infty} a_n b_n$ is divergent. Since $\int_a^{c_n} f\alpha' = \sum_{k=1}^{n} a_k b_k$, it follows that $\lim_{t\to b}\int_a^t f\alpha'$ does not exist. Consequently, $\int_a^b f\alpha'$ and $\int_a^b f\,d\alpha$ do not exist. Finally, observe that f is regulated. It clearly has one-sided limits at each point in $[a, b)$. Since $\lim_{n\to\infty} b_n = 0$ the left-hand limit of f at b is also zero.

6. Let $C = \{c_1, c_2, c_3, \ldots\}$ include all left-hand discontinuities of α, whether $\phi(c_n) \neq 0$ or not. Then $\sum_{n=1}^{\infty} \phi(c_n)$ is absolutely convergent and there is m such that $\sum_{k=m+1}^{\infty} |\phi(c_k)| < \epsilon$. (We may assume $a \notin C$.)

When $z = c_n$ choose $\gamma(z)$ so that

$$|\phi(z) - (\alpha(z) - \alpha(u))(f(z) - f(u))| < \epsilon/2^n$$

for all u such that $u < z$ and $u \in \gamma(z)$.

When $z \notin C$ the function α is left-hand continuous at z. Thus there is $\gamma(z)$ such that

$$|(\alpha(z) - \alpha(u))(f(z) - f(u))| < \epsilon|f(z) - f(u)|$$

when $u < z$ and $u \in \gamma(z)$.

When $z \notin \{c_1, c_2, \ldots, c_m\}$ restrict $\gamma(z)$ further so that $\gamma(z)$ contains none of $c_1, c_2, \ldots, c_m$.

Let $\mathfrak{D}$ be γ-fine. Break $\mathfrak{D}$ into three subsets. Put $z[u, v]$ into $\mathfrak{E}$ when $z \notin C$, into $\mathfrak{F}$ when $z \in C$ and $u < z$, and into $\mathfrak{G}$ when $z \in C$ but $u = z$. Let K be the set of all n such that c_n is a tag in $\mathfrak{F}$. Then $\{1, 2, \ldots, m\} \subseteq K$. Recall that the third restriction on γ implies that each interval of $\mathfrak{D}$ which contains c_n, $n \leq m$, has c_n as its tag. Whether $\mathfrak{D}$

contains one or two intervals to which c_n belongs, one of them does not have c_n as its left endpoint since $a \neq c_n$.

The definition of Φ implies that $\Phi(\mathcal{G}) = 0$. Thus

$$\left|\Phi(\mathcal{D}) - \sum_{n=1}^{\infty} \phi(c_n)\right| \leq \left|\Phi(\mathcal{F}) - \sum_{n\in K} \phi(c_n)\right| + |\Phi(\mathcal{E})| + \left|\sum_{n\notin K} \phi(c_n)\right|.$$

For each $n \in K$ there is exactly one $z[u, v]$ of $\mathcal{F}$ with $z = c_n$. Thus $|\Phi(\mathcal{F}) - \sum_{n\in K}\phi(c_n)| \leq \sum_{n\in K}\epsilon/2^n < \epsilon$. Recall that f is a function of bounded variation. The choice of $\gamma(z)$ when $z \notin C$ allows us to say that $|\Phi(\mathcal{E})| \leq \epsilon|\Delta f|(\mathcal{E}) \leq \epsilon V_a^b f$. Finally $|\sum_{n\notin K}\phi(c_n)| \leq \sum_{n=m+1}^{\infty}|\phi(c_n)| < \epsilon$. In summary

$$\left|\Phi(\mathcal{D}) - \sum_{n=1}^{\infty} \phi(c_n)\right| < \epsilon(2 + V_a^b f).$$

One more observation is needed to complete the proof. The set C was chosen to include all points where α is not left-hand continuous. The sum of the series $\sum_{n=1}^{\infty}\phi(c_n)$ is the same if some other sequence $(c_n)_{n=1}^{\infty}$ is used so long as all points where ϕ is nonzero are included. Thus the proposition is true as stated.

7. Suppose F has bounded variation on $[a, b]$. According to p. 223 the variation of F satisfies

$$V_a^b F = \lim_{t\to b} V_a^t F + \lim_{t\to b} |F(b) - F(t)|.$$

Since α has bounded variation on $[a, t]$, we already know that $V_a^t F = \int_a^t |f(x)|\, dV_a^x\alpha$. Moreover, $\lim_{t\to b}|F(b) - F(t)| = \lim_{t\to b}|\int_t^b f\, d\alpha|$. But from p. 187 we see that this last limit is zero since $\Delta\alpha([t, b]) = 0$. From the same source $\int_a^b|f(x)|\, dV_a^x\alpha = \lim_{t\to b}\int_a^t|f(x)|\, dV_a^x\alpha$ since the variation of α, like α, is defined on $[a, b)$. Now $\int_a^b |f(x)|\, dV_a^x\alpha = V_a^b F$, as required.

8. We know that $\int_c^v f \circ \tau\, d(\alpha \circ \tau) = \int_{\tau(c)}^{\tau(v)} f\, d\alpha$ when $v < d$. For convenience set $g = f \circ \tau$ and $\beta = \alpha \circ \tau$. Then β is continuous at d and $\lim_{v \to d} \Delta\beta([v, d]) = 0$. Consequently, existence of $\int_c^d g\, d\beta$ is equivalent to existence of $\lim_{v \to d} \int_c^v g\, d\beta$ and $\int_c^d g\, d\beta = \lim_{v \to d} \int_c^v g\, d\beta$. The existence of this limit can be determined by examining $\lim_{v \to d} \int_{\tau(c)}^{\tau(v)} f\, d\alpha$. Set $F(x) = \int_{\tau(c)}^x f\, d\alpha$. Then F is continuous at $x = \tau(d)$ since α is continuous there. The next point to note is that $F \circ \tau$ is continuous at d. Thus

$$\int_{\tau(c)}^{\tau(d)} f\, d\alpha = \lim_{v \to d} \int_{\tau(c)}^{\tau(v)} f\, d\alpha = \lim_{v \to d} \int_c^v g\, d\beta = \int_c^d g\, d\beta.$$

9. Let $a = b_1 < c_1 < b_2 < c_2 < \cdots$ with $\lim_{n \to \infty} b_n = b$. For every n let $\alpha(b_n) = 0$ and $\alpha(c_n) = 2/\sqrt{n}$. Let α be linear on the intervals $[b_n, c_n]$ and $[c_n, b_{n+1}]$. Finally, let $\alpha(b) = 0$.

On the intervals $[b_n, c_n]$ where α is increasing we will assign f positive values and on $[c_n, b_{n+1}]$ where α is decreasing f will be negative. To be specific, let $f(b_n) = f(c_n) = 0$ for all n. At the midpoint of $[b_n, c_n]$ let f have the value $1/\sqrt{n}$. Midway between c_n and b_{n+1} let f have the value $-1/\sqrt{n}$. Between these points let f be linear. Let $f(b) = 0$. Then f is continuous on $[a, b]$. Moreover $\int_{b_n}^{c_n} f\, d\alpha = \int_{b_n}^{c_n} f\alpha'$. On (b_n, c_n) the derivative α' is constant with value $\alpha(c_n)/(c_n - b_n)$. Moreover $\int_{b_n}^{c_n} f$ is the area of a triangle with altitude $1/\sqrt{n}$ and base $c_n - b_n$. Thus $\int_{b_n}^{c_n} f\, d\alpha = 1/n$. A similar analysis shows that $\int_{c_n}^{b_{n+1}} f\, d\alpha = 1/n$.

Let $\alpha_n(x) = \alpha(x)$ when $0 \leqslant x \leqslant b_n$ and $\alpha_n(x) = 0$ when $b_n < x \leqslant b$. Then α_n converges uniformly to α on $[a, b]$. Moreover $\int_a^b f\, d\alpha_n = \int_a^{b_n} f\, d\alpha = \sum_{k=1}^{n-1} 2/k$. Consequently, $\lim_{n \to \infty} \int_a^b f\, d\alpha_n$ does not exist.

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The Generalized Riemann Integral

The Generalized Riemann Integral is addressed to persons who already have an acquaintance with integrals they wish to extend and to the teachers of generations of students to come. The organization of the work will make it possible for the first group to extract the principal results without struggling through technical details which they may find formidable or extraneous to their purposes. The technical level starts low at the opening of each chapter. Thus, readers may follow each chapter as far as they wish and then skip to the beginning of the next. To readers who do wish to see all the details of the arguments, they are given.

The generalized Riemann integral can be used to bring the full power of the integral within the reach of many who, up to now, haven't gotten a glimpse of such results as monotone and dominated convergence theorems. As its name hints, the generalized Riemann integral is defined in terms of Riemann sums. The path from the definition to theorems exhibiting the full power of the integral is direct and short.

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