Inequalities from Complex Analysis
Inequalities from Complex Analysis

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Preface

This book discusses inequalities and positivity conditions for various mathematical objects arising in complex analysis. The inequalities range from standard elementary results such as the Cauchy-Schwarz inequality and the triangle inequality to recent results such as characterizing bihomogeneous polynomials in several variables that are positive away from the origin. The inequalities and positivity conditions in this book form the foundation for a small but beautiful part of complex analysis.

I begin by defining the complex numbers in terms of the real numbers. The prerequisites for starting the book are therefore no more than the elementary undergraduate courses in real analysis and algebra; in order to get something meaningful from the text the reader should also know some linear algebra and elementary complex variable theory. Such readers will find herein complete details of the proofs of numerous useful and interesting theorems in complex analysis and elementary Hilbert space theory. Copious examples and discussions of geometric reasoning aid the reader. The development culminates in complete proofs of recent results in the research literature.

This book is accessible to a wide audience. I have written the first five chapters so that an undergraduate mathematics major who has seen elementary real analysis can read them. Complex variable theory is not a strict prerequisite, although a reader who has never seen complex
numbers will probably not get past Chapter I. Many physicists and engineers can read this book; positivity conditions for polynomials often arise in applied mathematics. Detailed discussion of some geometric arguments that have broad scope for application will help readers in this audience. In order to enliven the text, I sometimes interrupt the development with a delightful or novel application of the ideas. I have written the complete text at a level I hope to be accessible to second year graduate students in mathematics.

The starting point for this book is the existence and properties of the real number system $\mathbb{R}$. From there, I define the complex number field $\mathbb{C}$ in Chapter I and prove many results about it. For several reasons the treatment differs considerably from a review of the standard treatment of “one complex variable”. One reason is that many texts treat basic complex analysis well, and hence there is no need to repeat certain things here. A second reason is that I wish to anticipate arguments that will arise later in the book. For example, I discuss the absolute value function in great detail, both to prepare for our discussion of the norm on Hilbert space, and to get the reader used to the emphasis on inequalities. I discuss certain simple inequalities in one variable to anticipate more difficult inequalities used later, so that these later inequalities do not jar the reader. This evokes a basic principle from music; a chord containing a C sharp may be used to anticipate a C sharp (appearing as an accidental) that arises later in the melody. Surprises should be pleasant.

Chapter II provides an introduction to complex Euclidean space $\mathbb{C}^n$ and Hilbert spaces. Proofs of the Riesz representation lemma and related results about orthogonal projection appear here. The chapter closes with a discussion of how to use generating functions to verify the orthonormality of a given collection of vectors. This enables us to introduce Laguerre and Hermite polynomials.

Chapter III provides an introduction to functions of several complex variables. I discuss enough theory of holomorphic functions of several variables to introduce the Bergman kernel function and to compute it for the unit ball. Chapter III also includes a nice treatment of the Euler Beta function in $n$ variables.
Chapter IV focuses on that part of linear algebra concerned with positive definite Hermitian forms. I prove the spectral theorem for Hermitian linear operators in finite dimensions in Chapter IV. I defer the proof for compact Hermitian operators on a Hilbert space to Chapter V. Chapter IV includes a careful proof that a finite-dimensional operator is positive definite if and only if its leading principal minor determinants are positive numbers. The proof interprets the minor determinants in terms of sums of squares, and thus anticipates some discussion in Chapters VI and VII. The book has occasional applications. In Chapter IV, for example, we show that if \( \{x_j\} \) is a collection of distinct positive numbers, then the matrix whose \( j, k \)-th entry is the reciprocal of \( x_j + x_k \) must be positive definite.

Many classical inequalities (Hadamard, Minkowski, etc.) follow from the results in this chapter, and some appear as exercises. Chapter IV closes with two sections on elementary Fourier analysis, and includes several beautiful inequalities. Hilbert’s inequality, the Herglotz theorem, and one form of Wirtinger’s inequality appear here.

Chapter V includes a detailed discussion of compact operators on Hilbert space. In particular I prove a simple proposition that interprets compactness as an inequality, and I use this to prove some of the standard results about compact operators. The chapter also includes a section on integral operators, and it closes with an introductory glimpse at singular integral operators. For example, I give a concrete definition of a fractional derivative operator, and then indicate how pseudodifferential operators provide a nice approach to this notion.

Chapter VI considers a point of view I like to call nonlinear Hermitian linear algebra. Consider a real-valued polynomial of several complex variables. The coefficients of such a polynomial may be identified with a Hermitian matrix; we want to apply, in this nonlinear setting, the linear algebra developed in the book. One goal is to understand polynomials whose values are nonnegative. What is the relationship of this condition to the nonnegative definiteness of the matrix of coefficients? To study this question I introduce eight positivity conditions and discuss the implications among them. I provide many examples. Also appearing here is a somewhat unusual discussion of
plurisubharmonicity, the complex variable analogue of convexity. This includes a surprising result: for nonnegative bihomogeneous functions, plurisubharmonicity is equivalent to logarithmic plurisubharmonicity. I know of no real-variables analogue of this statement.

The mathematics of Chapter VII arises from the following complex variables analogue of Hilbert’s seventeenth problem. Suppose that a polynomial in several complex variables takes on positive values on a set. Must it agree with the quotient of squared norms of holomorphic polynomial mappings there? I have studied this question in recent research papers, both jointly with David Catlin and on my own. Theorem VII.1.1, a stabilization result, is one of the primary goals of the book. The proof here (due to Catlin and myself) applies compact and integral operators in a somewhat unexpected way. The theorem was proved earlier by Quillen using a different but intriguingly related method.

Theorem VII.1.1 gives a decisive answer to the question relating positivity of the values of a polynomial to positivity conditions on the matrix of coefficients. It leads to three additional results about writing nonnegative polynomials as quotients of squared norms. I also apply it to study proper holomorphic mappings between balls in different dimensions.

It is possible to reinterpret some of the results of Chapters VI and VII in the language of holomorphic line bundles. I had originally planned on closing the book with such a chapter. This material would introduce the reader to a modern point of view whose value in mathematics and mathematical physics continues to develop. After soliciting opinions from other mathematicians, I decided not to include this chapter. Doing so would open many doors but not adequately close them. I therefore end the book with a brief afterword, where I mention some generalizations and give references to the research literature.

The book has many exercises; they are numbered beginning with 1 in each chapter. All other items are numbered by stating the kind of item, the number of the chapter, the number of the section, and the item number within the section. For example, Theorem II.3.2 is the second item in Section 3 of Chapter II.
I acknowledge the contributions of many people. Most important are those of my wife Annette and our four small children. Their love has inspired me throughout.

From the mathematical point of view I owe a great debt to David Catlin, whose understanding of most of these ideas far exceeds my own. It was his idea to involve the Bergman kernel in the study of positivity conditions for polynomials. By pursuing this idea we were able to do more than I had originally expected.

I want to specifically acknowledge four mathematicians who read lengthy parts of various preliminary versions, provided comments, and went far beyond the call of duty. Dan Grayson and Alex Isaev read much of the material without expecting anything in return. Their superb comments played a major role in my revision. Harold Boas and Ken Ross read preliminary versions as members of the Committee on Carus Monographs, and each spotted a large number of potential improvements. I owe much to all four of these people. Of course I take full responsibility for the remaining flaws.

Many other mathematicians have provided useful comments on either the mathematics herein or on preliminary versions of the manuscript. They include Salah Baouendi, David Berg, Steve Bradlow, Mike Christ, Charlie Epstein, Fritz Haslinger, Xiaojun Huang, Bob Jer-rard, Robert Kaufman, J. J. Kohn, Steve Krantz, László Lempert, Jeff McNeal, Anoush Najarian, Bruce Reznick, Linda Rothschild, Alberto Scalari, Ragnar Sigurddson, Emil Straube, and Alex Tumanov. I wish to specifically thank Steve Krantz for suggesting that I consider writing a Carus monograph. I also want to thank Kang-Tae Kim for inviting me to KSCV4 (the 4th Korean Several Complex Variables meeting) in 1999. My lecture notes for that meeting helped me get started on this book.

I wish to acknowledge the NSF for research support, and IAS, MSRI, AIM, and ESI for profitable time spent there. Spending time at these institutes has been invaluable for me; this has developed my taste and aspirations within mathematics. I took a sabbatical from teaching in order to write this book, and partially supported my family with money I received from the 1999 Bergman prize. I therefore wish to
acknowledge the AMS for awarding me this prize. I wish to thank the MAA staff and especially Beverly Ruedi for their efforts in turning my computer files into a finished book. Finally, I acknowledge the contribution of my home institution, the University of Illinois, for providing me an excellent library and computer facilities to help me write this book.

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This terse appendix provides some basic information on the prerequisites for this book. It consists of definitions and statements of some concepts and results used, but not developed, in the book.

A.1 Algebra

Definition. (Group) A group is a mathematical system consisting of a set \( G \), a binary operation, \((g, h) \mapsto gh\), and a distinguished element 1, such that the following hold:

1) (1 is the identity element.) \(1g = g1 = g\) for all \(g \in G\).

2) (Existence of inverses). For all \(g \in G\), there is a (necessarily unique) \(g^{-1} \in G\) such that \(g^{-1}g = gg^{-1} = 1\).

3) (Associative law). For all \(f, g, h \in G\), we have \((fg)h = f(gh)\).

Mathematicians generally say things such as “Let \(G\) be a group.” This is an abbreviation for something more precise such as “Let \(G\) be a set, with a binary operation on \(G\) and an element 1 in \(G\) satisfying the three axioms in the Definition.” This type of abbreviated language applies to fields, vector spaces, and other mathematical systems. The brevity gained more than compensates for the precision lost. For example, with this language the letter \(G\) stands for both the group and the
underlying set of elements. One also says “\( G \) is a group under multiplication”; this clause has a clear meaning, yet it blurs the issue whether \( G \) means the set or the mathematical system.

A group is called Abelian (or commutative) if \( gh = hg \) for all \( g \) and \( h \) in \( G \). Often one writes \( g + h \) instead of \( gh \) when the group is Abelian. This applies especially in cases where there is another operation considered as multiplication.

**Definition. (Field)** A field is a mathematical system consisting of a set \( F \), two binary operations called addition and multiplication, and two distinguished elements 0 and 1, such that axioms 1), 2), and 3) are true. We write addition as \((x, y) \mapsto x + y\) and multiplication as \((x, y) \mapsto xy\).

1) \( F \) is an Abelian group under addition with identity element 0.  
2) \( F - \{0\} \) is an Abelian group under multiplication with identity element 1.  
3) For all \( x, y, z \in F \) the distributive law holds: 

\[(x + y)z = xz + yz.\]

We write the additive inverse of \( x \) as \(-x\), and the multiplicative inverse of a nonzero \( x \) as \( \frac{1}{x} \) or \( x^{-1} \).

**Definition. (Ordered field)** An ordered field is a mathematical system consisting of a field \( F \) and a subset \( P \) of \( F \) with the following properties. We call \( P \) the set of positive elements in \( F \).

1) \( 0 \) is not in \( P \).  
2) Suppose \( x \neq 0 \); then either \( x \) is in \( P \) or \(-x \) is in \( P \).  
3) \( 1 \) is in \( P \).  
4) If \( x, y \in P \), then \( x + y \) and \( xy \) are in \( P \).
Statement 2) does not allow the possibility that both $x$ and $-x$ are in $P$. In particular, if $x = -x$ for some $x$ in a field, then the field cannot be ordered.

There are examples of fields that admit more than one ordering; therefore the term ordered field includes the choice of the subset $P$.

Let $F$ be an ordered field. We write $x > y$ if and only if $x - y \in P$. We write $x \geq y$ if and only if either $x = y$ or $x > y$. We write $x < y$ if and only if $y > x$, and we write $x \leq y$ if and only if $y \geq x$. The axioms for an ordered field imply the usual rules for manipulating inequalities.

There are many examples of fields. The rational number system $\mathbb{Q}$ and the real number system $\mathbb{R}$ are ordered fields; in each case the positive set consists of positive numbers in the usual sense. Finite fields cannot be ordered. We illustrate this in a special case. Let $p$ be a prime number, and let $\mathbb{Z}_p$ denote the set $\{0, 1, \ldots, p-1\}$ of ordinary integers. We make $\mathbb{Z}_p$ into a field by performing arithmetic and multiplication modulo $p$. The numbers 0 and 1 are the identity elements. The additive inverse of 1 is $p - 1$. If the field were ordered, $p - 1$ would be positive as it is the sum of this many copies of 1. On the other hand, it would be negative as the additive inverse of the positive element 1. The assumption that an ordering exists thus leads to a contradiction.

**Definition. (Vector space)** A vector space $V$ over a field $F$ is a mathematical system consisting of an Abelian group $V$ and a function $F \times V \to V$, written $(c, v) \mapsto c \cdot v$ and called scalar multiplication, such that axioms 1) through 5) hold. Let $0$ denote the additive identity in $V$ and let 0 denote the additive identity in $F$. Let 1 denote the multiplicative identity in $F$. Then:

1) $0 \cdot v = 0$ for all $v \in V$.
2) $1 \cdot v = v$ for all $v \in V$.
3) $c \cdot (v + w) = c \cdot v + c \cdot w$ for all $v, w \in V$ and all $c \in F$.
4) $(c_1 + c_2) \cdot v = c_1 \cdot v + c_2 \cdot v$ for all $c_1, c_2 \in F$ and all $v \in V$.
5) $c_1 \cdot (c_2 \cdot v) = (c_1c_2) \cdot v$ for all $c_1, c_2 \in F$ and all $v \in V$. 

Elements of $V$ are called vectors, and elements of $F$ are called scalars. Generally (and in this book) one denotes $0$ and $\mathbf{0}$ by the same symbol.

A vector space is called \textit{finite-dimensional} if there is a finite subset \{\(v_1, \ldots, v_n\)\} of $V$ such that the following holds. For each $v \in V$ there are elements $c_j \in F$ such that

$$v = \sum_{j=1}^{n} c_j v_j.$$  

Such a sum is called a \textit{linear combination} of the $v_j$.

\section*{A.2 Analysis}

It is a bit artificial to distinguish between algebra and analysis. For example, one might say that fields belong to algebra while ordered fields belong to analysis. Certainly inequalities are a major feature of analysis. It is the completeness axiom for the real numbers that gets analysis going.

We recall the standard definition of completeness for the real number system. Let $F$ be an ordered field, and let $S$ be a nonempty subset of $F$. Then $S$ is bounded above if there is some $m \in F$ such that $x \leq m$ for all $x \in S$. Such an $m$ is an \textit{upper bound} for $S$.

\textbf{Definition.} An ordered field $F$ is complete if every nonempty subset $S \subset F$ that is bounded above has a least upper bound in $F$.

The least upper bound of $S$ is also called the \textit{supremum} of $S$; it need not be in $S$. The supremum of $S$ is the unique $m_0$ such that $m_0 \leq m$ for every upper bound $m$ of $S$.

Up to isomorphism, the real number system $\mathbb{R}$ is the unique complete ordered field. The rational number system $\mathbb{Q}$ is an ordered field, but is not complete. The complex number system $\mathbb{C}$ is a complete field, but cannot be ordered.
A normed linear space is a vector space where it makes sense to talk about the length of a vector. More precisely we have the following definition:

**Definition. (Normed linear space)** Let $V$ be a vector space over $F$, where either $F = \mathbb{R}$ or $F = \mathbb{C}$. We say that $V$ is a *normed linear space* if there is a real-valued function $|| \cdot ||$ on $V$, called the *norm*, such that:

1) $||v|| > 0$ for all nonzero $v$ in $V$.
2) $||cv|| = |c| \, ||v||$ for all $c \in F$ and for all $v \in V$.
3) $||v + w|| \leq ||v|| + ||w||$ for all $v$ and $w$ in $V$.

As a consequence of 2), we have $||0|| = 0$.

The simplest examples are of course $\mathbb{R}^n$ and $\mathbb{C}^n$ with the Euclidean norms as defined in the text. It is easy to prove, but never needed in this book, that all norms on a finite-dimensional vector space are equivalent, in the sense that they lead to the same topology. This means that they define the same collection of open sets. It suffices for us to recall the notion of open set in the metric space setting. Metric spaces have *distance functions* whose properties model those of normed linear spaces. When $V$ is a normed linear space, we define a function $d : V \times V \to \mathbb{R}$ by $d(u, v) = ||u - v||$. This function measures the distance between two points in $V$. More generally we have the following definition.

**Definition. (Metric space)** A *metric space* $(M, d)$ is a mathematical system consisting of a set $M$ and a function $d : M \times M \to \mathbb{R}$ (called the distance function) such that

1) $d(x, y) > 0$ if $x \neq y$.
2) $d(x, x) = 0$ for all $x \in M$.
3) $d(x, y) = d(y, x)$ for all $x$ and $y$ in $M$. 
4) \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \) in \( M \) (the triangle inequality).

Often one writes \( M \) instead of \((M, d)\), especially when the distance function is clear from context. Normed linear spaces provide good examples of metric spaces.

**Definitions.** The open ball \( B_r(x) \) of radius \( r \) about \( x \) is the subset of \( M \) consisting of all \( y \) for which \( d(x, y) < r \). The closed ball \( \overline{B}_r(x) \) of radius \( r \) about \( x \) is the subset of \( M \) consisting of all \( y \) for which \( d(x, y) \leq r \).

**Definitions.** A subset \( A \subset M \) is called open if, for each \( x \in A \), there is a positive number \( \epsilon \) (generally depending on \( x \)) so that \( B_\epsilon(x) \subset A \). A subset \( K \subset M \) is called closed if its complement is open.

It is elementary to verify that an open ball is open and a closed ball is closed.

**Definitions.** A sequence \( \{x_\nu\} \) in a metric space \((M, d)\) converges to \( x \) (or “has limit \( x \)”) if, for each \( \epsilon > 0 \), there is a positive integer \( N \) so that \( n \geq N \) implies \( d(x_n, x) < \epsilon \).

A sequence \( \{x_\nu\} \) in a metric space \((M, d)\) is a Cauchy sequence if, for each \( \epsilon > 0 \), there is a positive integer \( N \) so that \( n, m \geq N \) implies \( d(x_n, x_m) < \epsilon \).

Convergent sequences are always Cauchy sequences; the converse assertion holds in complete metric spaces.

**Definition.** A metric space \((M, d)\) is complete if every Cauchy sequence in \( M \) has a limit in \( M \).

**Definition.** A Banach space is a complete normed linear space.
The next two lemmas make the connection between closed sets and completeness.

**Lemma.** A subset $K$ of a metric space $(M, d)$ is closed if the following is true: whenever $\{x_\nu\}$ is a sequence in $K$ whose limit $x$ exists in $M$, then necessarily $x \in K$.

The next lemma holds in particular for closed subspaces of Banach and Hilbert spaces, and is used several times in the book.

**Lemma.** A closed subset $K$ of a complete metric space $(M, d)$ is itself a complete metric space (with the same distance function).

Many results from elementary real analysis fit into the general framework of metric spaces, and the usual epsilon-delta arguments from elementary calculus generally work in this setting. Limit and convergence arguments acquire a certain elegance. Early in the book there is some detail when these items arise, but especially later some of this standard material is used without reference. The reader might consult [Ah], [F], and [TBB] for more information.

We continue with a list of terms the reader should know, and then close the appendix with a brief discussion of some convergence results.

**Definitions the reader should know in the metric space setting:** compactness, completeness, connectedness, continuous function, dense set, equicontinuous family of functions, sequence, subsequence, uniform convergence, uniformly continuous function.

Let $f$ be a bounded real-valued function on a set $A$; the set of values $f(A)$ is then a non-empty subset of $\mathbb{R}$ that is bounded above. Therefore it has a supremum $m$. Several times in this book, and very often in pure and applied mathematics, one needs to know that $f$ achieves its supremum somewhere. Thus we seek an $x$ with $f(x) = m$. The next result gives us circumstances when this is possible.
Theorem. (Min-Max Theorem) Suppose that \((M, d)\) is a compact metric space, and \(f : M \to \mathbb{R}\) is a continuous real-valued function. Then \(f\) is uniformly continuous, \(f\) is bounded, and there are points in \(M\) at which \(f\) achieves its infimum and its supremum.

The next result makes use of a metric space of functions. Let \((M, d)\) be a compact metric space, and let \(C(M)\) denote the linear space of continuous complex-valued functions on \(M\), with norm given by

\[
\|f\| = \sup_{x \in M} |f(x)|.
\]

Then \(C(M)\) is a Banach space. We are interested in describing its compact subsets. To do so, we briefly discuss equicontinuity.

Let \(\mathcal{F}\) be a family of continuous functions on a metric space \((M, d)\). For each \(f \in \mathcal{F}\), and for \(x \in M\), the usual epsilon-delta definition provides the meaning for the phrase “\(f\) is continuous at \(x\).” In general there is no reason to expect, for a given \(\epsilon\), to be able to choose \(\delta\) independently of \(x\) or of \(f\). In case \((M, d)\) is compact, each \(f \in \mathcal{F}\) is uniformly continuous, so we can then choose \(\delta\) independently of \(x\) for a fixed \(f\). Equicontinuity is the analogous idea, where we want to choose \(\delta\) independently of \(f\).

Definitions. A collection \(\mathcal{F}\) of continuous complex-valued functions on a metric space \((M, d)\) is equicontinuous at \(x\) if, for each \(\epsilon > 0\), there is a \(\delta > 0\) so that

\[
d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon
\]

for all \(f \in \mathcal{F}\). We say that \(\mathcal{F}\) is equicontinuous if it is equicontinuous at each \(x \in M\). Finally \(\mathcal{F}\) is uniformly bounded if there is a constant \(C\) such that \(|f(x)| \leq C\) for all \(x \in M\) and for all \(f \in \mathcal{F}\).

Theorem A.2.1. (Arzelà-Ascoli) Suppose that \(M\) is a compact metric space and \(\mathcal{F}\) is a family of continuous complex-valued functions on \(M\).
Then $\mathcal{F}$ is a compact subset of the Banach space $C(M)$ if and only if the following three conditions hold:

1) $\mathcal{F}$ is equicontinuous.
2) $\mathcal{F}$ is uniformly bounded.
3) $\mathcal{F}$ is closed.
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JOHN P. D’ANGELO

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