J. William Helton
with the assistance of
Joseph A. Ball, Charles R. Johnson
and John N. Palmer

OPERATOR THEORY, ANALYTIC FUNCTIONS
MATRICES, AND ELECTRICAL ENGINEERING

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MATRICES, AND ELECTRICAL ENGINEERING

J. William Helton
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10 9 8 7 6 5 4 3 2 95 94 93 92 91 90
Contents

Preface vii

Part I. Engineering motivation 3
1. Engineering background
   What is the transfer function—how analytic functions arise in engineering-connection laws.

2. Engineering problems
   Engineering problems (vs. analysis); a control paradigm.

Part II. Analytic function theory 11
How much classical analysis can you get by staring at pairs of invariant subspaces? There is a unified way of obtaining the theories of classical interpolation, $H^\infty$ approximation, Corona = Bezout identities, commutant lifting, Wiener-Hopf factorization, integrable systems (Toda Lattice, KdV), matrix LU decompositions, interpolation, upper triangular approximation. The method gives excellent results when all functions induced are differentiable and in these cases frequently extends existing results in various ways, e.g., treats added symmetries, allows poles in $H^\infty$ functions.

3. Fractional maps and Grassmannians 21
   Basic projective lore. The correspondences between linear fractional actions on operators and linear actions on subspaces.

4. Representing shift invariant subspaces 35
5. Applications to factorization, interpolation, and approximation 43
6. Further applications 57
7. Matrix analogs and generalizations 73
   A general theory which contains both the matrix and analytic function case.
CONTENTS

Part III. Matrices
The distance of a matrix to a matrix subspace, the spectrum of a matrix with respect to an algebra of matrices, and other matrix stories.

8. Some matrix problems in engineering 89
   Engineering problems (vs. matrices).

9. Optimization, matrix inequalities, and matrix completions 95
   Matrix optimization and completion problems.

10. The spectrum of a matrix with respect to an algebra 109
    A matrix spectrum.

   Part IV. The general $H^\infty$ optimization problem

11. Nonlinear $H^\infty$ optimization 121
    Qualitative theory and a few words about quantitative methods.

Bibliography 125
Preface

These notes expand ten lectures given at a regional conference in Lincoln, Nebraska. The objective is to describe the idea behind a broad variety of topics in a brief volume. The conference assembled a wide variety of scientists: pure mathematicians on one hand and engineers on the other. Consequently one of the constraints on these notes was that they stake out a middle ground and make the extremes accessible to most conference participants. However, these extremes provided a distinctive flavor which is missing from these notes. In particular talks by other participants gave substantial practice in applying the results of Chapter 5 and Chapter 8 to control problems.

This volume splits into four parts which are nearly independent. An approximate description of what you must have read in order to read a particular chapter is:

1 2
Part I

3 4 5 6 7
Part II

8 most of 10
Part III

SKETCHY NO PROOFS (proofs use 5)
Part IV
Many readers might want to skip Part I entirely and go directly to Part II.

While the impetus to write these notes was a ten lecture series, they correspond more closely to the 30-lecture graduate topics course in which they were developed. I think that a reasonable topics course could be based on sections of these notes. A reasonable plan is as follows:

- Part I supplemented with engineering texts Dorf [Dor] or Ogata [O] which are very easy to read.
- In Part II do 3, 4, 5, 6A, 6D, 6E, 6F, a quick pass at 7, or do [BG] in detail. General supplemental references are [Hls], [D], [Du1-Du2], [G], [RR2], while specialized references are [S1-S2], [BH1-BH5, 8], [BG], [SW1-SW2], and [Y].
- Parts III and IV as they stand are a bit sketchy for classroom use.

A warning is that throughout these notes the terms chapter and homework exercise are used with tongue in cheek. Usually what we term chapter requires several chapters and the homework would be awfully rough on a beginner.

These notes were written by several people. Joe Ball and I wrote Chapter 6.A and Chapter 8. Charlie Johnson wrote Chapter 9. Chapter 6.F is based mostly on lectures and discussions with John Palmer and somewhat on discussions with Nolan Wallach. Also John Doyle checked Chapter 10 while Bruce Francis checked Chapters 1 and 2. My students Dave Schwartz and Orlando Merino checked most of the manuscript; they made valuable remarks. Neola Crimmins did a marvelous job of typing the original manuscript and implementing many revisions.
## Notation Guide

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{C}^n)</td>
<td>Complex (n)-space.</td>
<td>3</td>
</tr>
<tr>
<td>(L^2[0, \infty])</td>
<td>Square integrable complex-valued functions on ([0, \infty]).</td>
<td>4</td>
</tr>
<tr>
<td>(L^2[0, \infty, K])</td>
<td>(e^{Kt} \cdot L^2[0, \infty]) or ({f: e^{-Kt}f(t) \in L^2[0, \infty]}).</td>
<td>4</td>
</tr>
<tr>
<td>(\tilde{f}(s))</td>
<td>Laplace transform of (f(t)).</td>
<td>4</td>
</tr>
<tr>
<td>R.H.P.</td>
<td>Right half-plane.</td>
<td>5</td>
</tr>
<tr>
<td>(H^2(\text{R.H.P.}))</td>
<td>The set ({h \in L^2[-i\infty, i\infty]; h\ \text{is analytic in R.H.P. with} \int_{-\infty}^{\infty}</td>
<td>h(i\omega + a)^2 d\omega \leq M &lt; \infty \text{ for all } a &lt; 0}).</td>
</tr>
<tr>
<td>(\tilde{B})</td>
<td>For (B) an operator (\tilde{B}g \equiv (\tilde{B}g)) for all (g \in L^2[0, \infty]).</td>
<td>5</td>
</tr>
</tbody>
</table>

### Chapter 1

\[\Gamma(\omega, z)\] Positive-valued function of \(\omega \in \mathbb{R}\) and of \(z \in \mathbb{C}^n\). Later use \(\Gamma(e^{i\theta}, z)\). | 11 |

\(A^\infty(\text{R.H.P.})\) Functions in \(H^\infty(\text{R.H.P.})\) which are continuous on closed R.H.P. including infinity. | 11 |

\((\text{REALIZE})\) Design a circuit that corresponds to a function \(f\) in a set \(E\) of admissible functions. | 11 |

\(S_{\omega}(c)\) The set \(\{z \in \mathbb{C}^N: \Gamma(\omega, z) < c\}\). | 11 |

\(\Delta_{\bar{K}}\) The set \(\{f: |K(i\omega) - f(i\omega)| < R(i\omega)\}\). | 11 |

### Chapter 2

\(G_g(W)\) Linear fractional transformation. \[G_g(w) = (\alpha w + \beta)(\kappa w + \gamma)^{-1},\] where \(g = (\alpha \beta \gamma)\). | 22 |
GL(2, C) Invertible 2 × 2 matrices with complex entries.

S(w) Graph of w ∈ M_{m×n} = \{(w_z) : z ∈ C^n\}.

ΔY The disk in M_{m×n} given by \{w : w^*w + f^*w + w*f + d ≤ 0\} where Y is the selfadjoint matrix \((a, f)\).

[,]Y Sesquilinear form on C^2 defined by [u, v]_Y = [Y u, v].

K [, ] Krein space, a Hilbert space K with a nondegenerate sesquilinear form [, ].

K+, K− Hilbert orthogonal subspaces whose direct sum is K, so that [, ] = [, ]_Y, Y = (I 0) with respect to this decomposition.

M' Closed [, ]-orthogonal complement of the subspace M.

M + N Sum of M and N if they are disjoint, [, ]-orthogonal and if M + N is closed.

C(w_1, w_2, w_3, w_4) Cross ratio of the complex numbers C(w_1, w_2, w_3, w_4) = (w_1 − w_2)(w_2 − w_3)^{-1}(w_3 − w_4)(w_4 − w_1)^{-1}.

Chapter 4


L_∞^p, L_∞^p_{M×N}, H_∞^p, H_∞^p_{M×N} Vector and matrix-valued analogs of L_∞, H_∞ on the unit disk.

H_p^p_\mathbb{N} All functions in L_p^p_\mathbb{N} whose positive Fourier coefficients vanish.

T_A Toeplitz operator.

M⊥ Annihilator of M in L_q^q_\mathbb{N}.

Chapter 5

T^f The set \{f ∈ \mathbb{B}H^\infty : f(z_j) = w_j, j = 1, \ldots, p\}.

clos T^f The closure of the set T^f in L_∞.

Λ_{z,w} The “Pick” matrix \(\{1−w^*_z \bar{w}_j\}_{i,j=1}^P\).

κ(\epsilon^{iθ}) The Szegö reproducing kernel \(\frac{1}{2\pi(1−\epsilon^{iθ})} \).

C_κ The set \(\{(h_1, h_2) ∈ \mathbb{B}H_2^\infty : \kappa(\phi) ≜ a_1 h_1 + a_2 h_2\) is rational, has \(|\phi(\epsilon^{iθ})| ≡ 1\), with winding number about zero ≤ l\}. Here \(a_1, a_2 ∈ H^\infty\), \(κ > 0\), and \(l\) an integer are given.
Chapter 6

(AS) Given $\psi, Y \in L^\infty$, $a_1, a_2 \in H^\infty$, $l$ integer, $\kappa > 0$ find $L^l$.

$L^l$ \{ $(\alpha, \phi)$ : $\alpha \in B L^\infty$, $\phi$ inner of degree $l$ s.t. $\exists G_1, G_2 \in H^\infty$ with $\alpha = \psi G_1 + Y G_2$, $\phi \kappa = a_1 G_1 + a_2 G_2$ \}.

$T^l$ \{ $\alpha \in Y + \psi H^\infty$; $\| \alpha \| \leq 1$ \}.

$T(\tilde{T}, T)$ \{ $A \in L(H, \tilde{H})$ : $\tilde{T} A = AT$, $H, \tilde{H}$ Hilbert spaces; $T \in L(H)$, $\tilde{T} \in L(\tilde{H})$ contractions \}.

$\text{CID}^l(A)$ The set of all almost contractive intertwining dilations of $A = \{ A_{\infty} \in L(K, \tilde{K}) : \tilde{U} A_{\infty} U$. $PA = AP$ and $I - A_{\infty}^* A_{\infty}$ has at most $l$ negative eigenvalues \}.

Here $U, \tilde{U}$ denote the minimal isometric dilations of $T, \tilde{T}$, to $K, \tilde{K}$, $P : K \rightarrow H$, $\tilde{P} : \tilde{K} \rightarrow \tilde{H}$ are orthogonal projections.

$X(s), X_+(s), X_-(s)$ A matrix-valued function, and its upper triangular and strictly lower triangular parts.

Chapter 7

$FL^{(k:n)}$ Set of all sequences $\{ M_j \}_{j=1}^n$ of subspaces of $L_k^2$ such that $z M_1 \subseteq M_n$, and for each $j$, $M_{j+1} \subset M_j$ and each $M_j$ is full range and simply invariant (FRSI) with respect to multiplication by $z$.

$L^p$-forward-reference flag $M_i = F_i + e^{i\theta} H^p_k$, $i = 1, \ldots, n$, and $C^k = F_1 \supset \cdots \supset F_n \supset 0$ a conventional flag of $n$ subspaces of $C^k$.

Backward periodic flag A collection $\{ M_j^X : 1 \leq j \leq n \}$ of subspaces of $L_k^2$ such that $z M_j^X \supset M_n^X$ and, for each $j$, $M_{j+1}^X \supset M_j^X$ and $M_j^X$ is FRSI with respect to backward shift multiplication by $z^{-1}$.

$M$ $[M_1, M_2, \ldots, M_n]$, the set of $L_k^{2n}$-functions such that the $j$th column is an element of the subspace $M_j \subset L_k^2$. 
\( H_{0\Omega_L}^\infty \) \hspace{1em} The set of functions \( L \) in \( H_{n \times n}^\infty \) such that \( L(0) \in \Omega_L \), the algebra of lower triangular \( n \times n \) matrices.

\( H_{0\Omega_L}^\infty \)-invariant \hspace{1em} Subspaces \( \mathbf{M} \) of \( L_{k \times n}^2 \) such that \( F \in \mathbf{M} \), \( L \in H_{0\Omega_L}^\infty \Rightarrow FL \in \mathbf{M} \).

\( H_{0\Omega_L}^\infty \)-simply invariant \hspace{1em} Invariant subspace \( \mathbf{M} \) such that \( \bigcap_{L \in H_{0\Omega_L}^\infty} \{FL : F \in \mathbf{M} \} = \{0\} \).

\( H_{0\Omega_L}^\infty \)-full range \hspace{1em} Invariant subspace \( \mathbf{M} \) such that \( \bigcup_{L \in L_{n \times n}^\infty} \{FL : F \in \mathbf{M} \} \) is dense in \( L_{k \times n}^2 \).

Block invariant \hspace{1em} A subspace which is \( H_{0\Omega_L}^\infty \)-simply invariant and \( H_{0\Omega_L}^\infty \)-full range.

\( * \)-Block invariant \hspace{1em} Subspace which is simply invariant and full range with respect to \( H_{0\Omega_L}^\infty \).

\( \overline{H_{0\Omega_u}^\infty} \) \hspace{1em} \( \{F \in L_{k \times n}^\infty : F \in H_{n \times n}^\infty \text{ and } F(0) \text{ belong to } \Omega_u \text{ the algebra of } n \times n \text{ upper triangular matrices}\} \). \( \overline{F} \) = complex conjugate (pointwise and elementwise) of \( F \).

\( \nu \) \hspace{1em} \( \{\nu_1, \ldots, \nu_n\} \subset \mathbb{N}^n \).

\( N(\nu) \) \hspace{1em} \( \nu_1 + \cdots + \nu_n \).

\( \Omega(\nu) \) \hspace{1em} The set of all \( N(\nu) \times n \) matrices, written as \( n \times n \) block matrices \( F = [F_{ij}]_{1 \leq i, j \leq n} \), where the size of \((i, j)\) block \( F_{ij} \) is \( \nu_i \times 1 \).

\( \Omega_L(\nu) \) \hspace{1em} Set of all block matrices \( F = [F_{ij}]_{1 \leq i, j \leq n} \) which are block lower triangular (\( F_{ij} = 0 \) for \( i < j \)).

\( \Omega_u(\nu) \) \hspace{1em} Set of all block upper triangular matrices.

\( H_{0\Omega_L(\nu)}^\infty \) \hspace{1em} \( \{F \in H_{n(\nu) \times n}^2 : F(0) \in \Omega_L(\nu)\} \).

\( \mathbf{M}^{\perp_J} \) \hspace{1em} Orthogonal complement of \( \mathbf{M} \) in the \( J \)-inner product.

\( [F, G]_J \) \hspace{1em} \( \frac{1}{2\pi} \int_0^{2\pi} \text{tr}(G^*(e^{i\theta})JF(e^{i\theta})) \, d\theta \), defined on \( L_{k \times n}^2 \).

\( \Omega_L^m \) \hspace{1em} \( \{[F_{ij}] \in \Omega_L : F_{ij} = 0 \text{ for } i > j + m\} \).

Chapter 8

\( \eta \) \hspace{1em} The intersection \( S \cap \Delta_Y : \cap \Delta_Y : \cap \cdots \cap \Delta_Y \).

Chapter 9

\( A[\alpha] \) \hspace{1em} Principal submatrix of \( A \) contained in the rows and columns indicated by the index set \( \alpha, \beta \subseteq \{1, 2, \ldots, n\} \).
<table>
<thead>
<tr>
<th><strong>$A(\alpha)$</strong></th>
<th>Complementary principal submatrix of $A$ resulting from deletion of the rows and columns indicated by $\alpha$.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$p(A)$</strong></td>
<td>Value of the parameter at the matrix $A$.</td>
</tr>
<tr>
<td><strong>$\Gamma$</strong></td>
<td>Set of matrices which agree with respect to the information to be used.</td>
</tr>
<tr>
<td><strong>$H$-matrix</strong></td>
<td>Matrix with a diagonal multiple which is strictly diagonally dominant.</td>
</tr>
<tr>
<td><strong>Partial matrix</strong></td>
<td>Rectangular array in which some entries are specified and others are unspecified.</td>
</tr>
<tr>
<td><strong>Completion of a partial matrix</strong></td>
<td>A specification of the unspecified entries.</td>
</tr>
<tr>
<td><strong>Partial Hermitian</strong></td>
<td>The specified diagonal entries are real and if $a_{ji}$ is specified, so is $a_{ij}$, with $a_{ij} = \overline{a_{ji}}$.</td>
</tr>
<tr>
<td><strong>Partial positive definite</strong></td>
<td>Partial Hermitian matrix each of whose specified principal submatrices is positive definite.</td>
</tr>
<tr>
<td><strong>Contraction</strong></td>
<td>Rectangular matrix with each singular value $\leq 1$.</td>
</tr>
<tr>
<td><strong>Partial contraction</strong></td>
<td>Rectangular partial matrix with each specified submatrices a contraction.</td>
</tr>
<tr>
<td><strong>Path</strong></td>
<td>Sequence ${(i_1, i_2), {i_2, i_3}, \ldots, {i_{k-1}, i_k}}$ of concatenated edges (of an undirected graph).</td>
</tr>
<tr>
<td><strong>Circuit</strong></td>
<td>Path with $i_k = i_1$.</td>
</tr>
<tr>
<td><strong>Simple circuit</strong></td>
<td>$i_1, i_2, \ldots, i_{k-1}$ are all different, and $i_k = i_1$.</td>
</tr>
<tr>
<td><strong>Minimal simple circuit</strong></td>
<td>No proper subset of vertices is the set of vertices of a simple circuit.</td>
</tr>
<tr>
<td><strong>Chord (or a circuit)</strong></td>
<td>Edge which joins two vertices which are not adjacent in the circuit.</td>
</tr>
<tr>
<td><strong>Chordal graph</strong></td>
<td>Has no minimal simple circuits of 4 or more edges.</td>
</tr>
<tr>
<td><strong>Decomposable</strong></td>
<td>$n \times m$ partial matrix $A$ for which exists $Q_1$ ($n$-by-$n$) and $Q_2$ ($m$-by-$m$) such that $Q_1 A Q_2 = [B_{11} B_{12}; B_{21} B_{22}]$ where the off-diagonal blocks consist entirely of unspecified entries.</td>
</tr>
<tr>
<td><strong>$\sigma_Q(W)$</strong></td>
<td>The set ${ \xi \in Q : W - \xi \text{ is not invertible} }$, (spectrum of $W$ with respect to the set $Q$).</td>
</tr>
</tbody>
</table>
\( \mu_Q(W) \) The spectral radius of \( W \) w.r.t. the set \( Q \)
\[ \sup_{\xi \in Q} \inf_x \frac{\| \xi x \|}{\| x \|}. \]

Chapter 11

\((\text{OPT})\)

Find \( \inf_{f \in A_N} \sup_{\theta} \Gamma(e^{i\theta}, f(e^{i\theta})) \), where
\( \Gamma(e^{i\theta}, z) \) is a nonnegative-valued function of \( e^{i\theta} \) and \( z \in \mathbb{C}^N \).

\( \kappa(A) \)

The condition number
\[ \sup_{\tilde{h}} \{ \inf_\theta \left| \sum_j a_j h_j : \tilde{h} \in \overline{BA}_N \right| \text{ where } \sum_j a_j h_j \leq 0 \} \text{ where } A_N = \text{set of } N \text{-vector valued functions analytic and continuous on the disk} \]
\( \overline{BA}_N = \{ \tilde{h} \in A_N : \sum_j |h_j|^2 \leq 1 \} \), and
\[ a_j(e^{i\theta}) = (\partial \Gamma / \partial z)(e^{i\theta}, f_0(e^{i\theta})). \]

Throughout \( B \) is a prefix which stands for open unit ball, e.g., \( BL^2 = \{ f \in L^2 : \| f \|_{L^2} < 1 \} \). For the closed unit ball we use \( \overline{B} \). Also \( L^p_N, H^p_N \) sometimes denotes a function space on the circle and sometimes a function space on the imaginary axis. There should be little trouble in telling which usage is intended in any particular context. After Chapter 2 the circle is used almost exclusively.
Dedicated to
Joanne
Maxene and John Helton
Frances and Jim
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