Conference Board of the Mathematical Sciences

CBMS

Regional Conference Series in Mathematics

Number 120

Ergodic Theory and Fractal Geometry

Hillel Furstenberg

American Mathematical Society
with support from the
National Science Foundation

Ergodic Theory and Fractal Geometry

http://dx.doi.org/10.1090/cbms/120

Conference Board of the Mathematical Sciences

CBMS

Regional Conference Series in Mathematics

Number 120

Ergodic Theory and Fractal Geometry

Hillel Furstenberg

Published for the
Conference Board of the Mathematical Sciences
by the
American Mathematical Society
Providence, Rhode Island
with support from the
National Science Foundation

NSF/CBMS Regional Conference in the Mathematical Sciences on Ergodic Methods in the Theory of Fractals, held at Kent State University, June 18–23, 2011

2010 *Mathematics Subject Classification.* Primary 28A80, 37A30; Secondary 30D05, 37F45, 47A35.

For additional information and updates on this book, visit
www.ams.org/bookpages/cbms-120

Library of Congress Cataloging-in-Publication Data

Furstenberg, Harry.
Ergodic theory and fractal geometry / Hillel Furstenberg.
pages cm. — (Conference Board of the Mathematical Sciences Regional Conference series in mathematics ; number 120)
"Support from the National Science Foundation."
"NSF-CBMS Regional Conference in the Mathematical Sciences on Ergodic Methods in the Theory of Fractals, held at Kent State University, June 18–23, 2011."
Includes bibliographical references and index.
ISBN 978-1-4704-1034-6 (alk. paper)
1. Ergodic theory—Congresses. 2. Fractals—Congresses. I. Conference Board of the Mathematical Sciences. II. National Science Foundation (U.S.) III. Title.

QA313.F87 2014
515′.48—dc23

2014010556

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to `reprint-permission@ams.org`.

© 2014 by the American Mathematical Society. All rights reserved.
The American Mathematical Society retains all rights
except those granted to the United States Government.
Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 19 18 17 16 15 14

Dedicated to the memory of Benoit Mandelbrot

Contents

Preface

Dynamics in all its variations is the study of change. In the usual physical context, change takes place within time. The objects of geometry are static and if there is any change, it is "in the eye of the beholder". In fractal geometry this point takes on meaning, particularly in the form of changing degree of magnification and "zooming in" on an object. This suggests developing dynamical concepts appropriate to this framework.

In these notes, based on a series of lectures delivered at Kent State University in 2011, we show that ergodic theoretic concepts can be applied to the process of changing magnification to give insight to phenomena peculiar to fractals. An important step is showing how fractal dimension can be interpreted in terms of ergodic averages in an appropriate measure preserving system. The familiar phenomenon of self similarity appears as the analogue of periodicity in classical dynamics. We don't pursue the full implications of recurrence in the geometric context, but some examples of the related Ramsey type questions are considered.

The theory developed here and the major ideas originated in the papers [**F**] and [**F**′]. It will develop that there is a close connection between dimension theory and rates of growth of trees. This is exploited in [**FW**] where analogs of Szemerédi's theorem are demonstrated in the context of trees.

I am indebted to Dmitry Ryabogin and Fedor Nazarov for transcribing the lectures as well as for working out many details that were not provided in the lectures as I presented them.

Hillel Furstenberg, January, 2014
Jerusalem, Israel

Bibliography

[B] A. Beardon, *Iteration of Rational Functions*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1991.

[F] H. Furstenberg, *Intersections of Cantor sets and transversality of semigroups*, in Problems in Analysis, ed. R. Gunning, Symposium in honor of S. Bochner 1969, Princeton University Press, Princeton, N.J. (1970), 41–59.

[F′] H. Furstenberg, *Ergodic fractal measures and dimension conservation*, ETDS **28** (2008), no.2, 405–422.

[Fe] H. Federer, *Geometric Measure Theory*, Springer, New York, 1969.

[FW] H. Furstenberg and B. Weiss, *Markov processes and Ramsey theory for trees*, Combinatorics, Probability and Computing **12** (2003), 547–563.

[KF] A. Kolmogorov and S. Fomin, *Introductory Real Analysis*, translated and edited by Richard A. Silverman, corrected reprinting, Dover Publications, Inc, New York, 1975; ISBN 0-486-61226-0.

[K] U. Krengel, *Ergodic Theorems*, Studies in Mathematics, de Gruyter, Berlin, 1985.

[M] P.T. Maker, *The ergodic theorem for a sequence of functions*, Duke Mathematical Journal **6** (1940), 27–30.

[Mat] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.

[Mi] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematics Studies 160 (2006), Princeton University Press, Princeton, N.J., viii+304.

Index

Fractal geometry represents a radical departure from classical geometry, which focuses on smooth objects that "straighten out" under magnification. Fractals, which take their name from the shape of fractured objects, can be characterized as retaining their lack of smoothness under magnification. The properties of fractals come to light under repeated magnification, which we refer to informally as "zooming in". This zooming-in process has its parallels in dynamics, and the varying "scenery" corresponds to the evolution of dynamical variables.

The present monograph focuses on applications of one branch of dynamics—ergodic theory—to the geometry of fractals. Much attention is given to the all-important notion of fractal dimension, which is shown to be intimately related to the study of ergodic averages. It has been long known that dynamical systems serve as a rich source of fractal examples. The primary goal in this monograph is to demonstrate how the minute structure of fractals is unfolded when seen in the light of related dynamics.

ISBN 978-1-4704-1034-6

9 781470 410346

CBMS/120

For additional information and updates on this book, visit

www.ams.org/bookpages/cbms-120

AMS *on the* Web
www.ams.org