Conference Board of the Mathematical Sciences

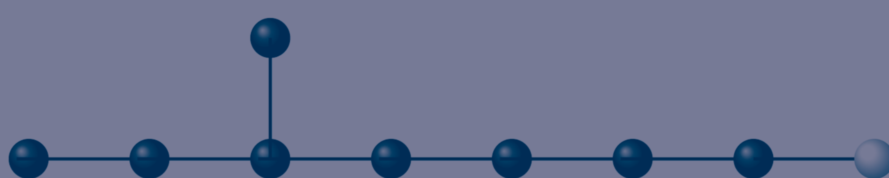
CBMS

Regional Conference Series in Mathematics

Number 131

Discrete Painlevé Equations

Nalini Joshi



AMERICAN MATHEMATICAL SOCIETY

with support from the

NATIONAL SCIENCE FOUNDATION

10.1090/cbms/131

Conference Board of the Mathematical Sciences

CBMS

Regional Conference Series in Mathematics

Number 131

Discrete Painlevé Equations

Nalini Joshi

Published for the
Conference Board of the Mathematical Sciences
by the

AMERICAN MATHEMATICAL SOCIETY

with support from the

NSF-CBMS Regional Conference in the Mathematical Sciences
on Discrete Painlevé Equations, held at The University of Texas
Rio Grande Valley, Edinburg, Texas, May 16–20, 2016.

Partially supported by NSF Grant DMS number 1543860. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

2010 *Mathematics Subject Classification.* Primary 33E17, 39A45, 14E07, 14E15, 32S45.

For additional information and updates on this book, visit
www.ams.org/bookpages/cbms-131

Library of Congress Cataloging-in-Publication Data

Names: Joshi, Nalini, author. | Conference Board of the Mathematical Sciences. | National Science Foundation (U.S.)
Title: Discrete Painleve equations / Nalini Joshi. Other titles: Painleve equations
Description: Providence, Rhode Island : Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, [2019] | Series: CBMS regional conference series in mathematics ; number 131 | "Support from the National Science Foundation." | Includes bibliographical references and index.
Identifiers: LCCN 2018058757 | ISBN 9781470450380 (alk. paper)
Subjects: LCSH: Painlevé equations. | Differential equations, Nonlinear. | AMS: Special functions (33-XX deals with the properties of functions as functions) – Other special functions – Painlevé-type functions. msc | Difference and functional equations – Difference equations – Equations in the complex domain. msc | Algebraic geometry – Birational geometry – Birational automorphisms, Cremona group and generalizations. msc | Algebraic geometry – Birational geometry – Global theory and resolution of singularities. msc | Several complex variables and analytic spaces – Singularities – Modifications; resolution of singularities. msc
Classification: LCC QA372 .J67 2019 | DDC 515/.352–dc23
LC record available at https://lccn.loc.gov/2018058757

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `https://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 24 23 22 21 20 19

Contents

Preface

The discrete Painlevé equations are nonlinear difference equations that arise as compatibility conditions of linear systems. The deceptive simplicity of this statement hides deep layers of mathematical properties, which are outlined in these lectures.

These properties are discrete versions of the ones that characterise integrable partial differential equations, such as the Korteweg-de Vries equation, and integrable ordinary differential equations, such as the Painlevé equations. These integrable PDEs and ODEs all arise as compatibility conditions of associated linear problems, which are called Lax pairs. The discovery of solitons [**ZK65**] and the inverse scattering transform method [**GGKM67**, **AKN74**] for solving integrable PDEs provides the beginnings of this theory.

The Painlevé equations were discovered more than a century ago [**Cla18**, **Pai06**], but modern interest was sparked by the surprising discovery that they arise as symmetry reductions of integrable PDEs [**ARS78**, **ARS80a**, **ARS80b**]. They appear widely in applications, ranging from fluid dynamics, plasmas, optics and general relativity to random matrix theory [**TW94**]. The first identification of a difference equation as a discrete Painlevé equation came from the theory of orthogonal polynomials [**Sho39**, **FIK91**]. Now more than twenty classes of such integrable equations are known.

The original naming of discrete Painlevé equations came from their continuum limits, but these limits are perhaps their least interesting property. They have a separate rich spectrum of properties, independent of continuum limits, that make them worthy of attention. They share a deep geometric property characterising their initial value (or phase) spaces. They arise when we walk from one tile to another on a lattice defined by reflections associated with an affine Coxeter or Weyl group. They are dynamical systems with zero algebraic entropy [**BV99**]. Their general solutions provide new higher transcendental functions. Like the differential case, there is reason to expect that these new functions may arise universally as models in many contexts.

Discrete Painlevé equations are second-order nonlinear, nonautonomous difference equations governing functions w of a discrete variable n, say. Nonautonomous means they have coefficients that are explicit functions of n. There are three different types of equations, according to whether the coefficients are linear, exponential or elliptic functions of n. These respective cases are labelled as *additive*, *multiplicative*, or, *elliptic* difference equations in the literature, denoted with prefixes *d*-, *q*-, or *ell*- in front of the equation's name.

In the autonomous limit, their solutions parametrise elliptic curves, with the discrete motion on each curve provided by addition theorems. In the general case, the solutions are still associated with curves, but instead of moving along one curve, the solutions move from curve to curve in a way that is explicitly described through algebro-geometric operations.

In limiting cases of parameters, explicit solutions of discrete Painlevé equations are known to include rational functions of their coefficients and special functions, such as basic and generalized hypergeometric functions. Like the latter functions, all the solutions satisfy recurrence relations and transformations, which help to characterise their properties.

This book collects the material I presented as principal lecturer at a Conference Board of Mathematical Sciences and National Science Foundation conference in Texas in 2016. Because the audience varied widely in their background knowledge, the lectures necessarily started with elementary introductory material. The present book aims to develop the theory at a higher level, but no attempt is made here to state detailed, general theorems or to provide complete proofs. Instead, the book relies on providing essential points of many arguments through explicit examples that I hope will be useful for applied mathematicians.

The book is oriented towards a reader with a graduate level of knowledge of mathematics, covering complex analysis and differential equations theory to projective geometry. But because it ranges from asymptotics and methods of applied mathematics to reflection groups, foliations and similar abstract theory, appendices are provided to cover further background material.

Many colleagues and students have contributed to the creation and writing of this book. I would like to thank Kenichi Maruno who first proposed the idea that I should be principal speaker at a CBMS-NSF conference and Andras Balogh and Baofeng Feng who not only organized the conference and my lectures, but also made sure everyone was transported and fed. I would also like to thank Mark Ablowitz, Huda Alrashdi, Percy Deift, Basil Grammaticos, Giorgio Gubbiotti, Jarmo Hietarinta, Philip Howes, Kenji Kajiwara, Elynor Liu, Christopher Lustri, Steven Luu, Nobutaka Nakazono, Matt Nolan, Frank Nijhoff, Masatoshi Noumi, Milena Radnović, Eric Rains, Alfred Ramani, Pieter Roffelsen, Hidetaka Sakai, Shonal Singh, Yang Shi, Yingying Sun, Dinh Tran, Yasuhiko Yamada, Claude Viallet, and Da-jun Zhang for asking perceptive questions over many years, which shaped my view of this field, and for providing feedback on the manuscript. Special thanks goes to Hugh Garden for his insightful advice on drawing figures with Ti*k*Z and to the School of Information Technology and Mathematical Sciences at the University of South Australia who provided me with a beautiful, tranquil space to write.

Bibliography

[Abh76] Shreeram S. Abhyankar, *Historical ramblings in algebraic geometry and related algebra*, The American Mathematical Monthly **83** (1976), no. 6, 409–448, DOI 10.2307/2318338. MR0401754

[AC91] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, vol. 149, Cambridge University Press, Cambridge, 1991. MR1149378

[AC04] Maria Alberich-Carramiñana, *Geometry of the plane Cremona maps*, Lecture Notes in Mathematics, vol. 1769, Springer-Verlag, Berlin, 2002. MR1874328

[Ada28] C. Raymond Adams, *On the linear ordinary q-difference equation*, Annals of Mathematics **30** (1928/29), no. 1-4, 195–205, DOI 10.2307/1968274. MR1502876

[AHH00] M. J. Ablowitz, R. Halburd, and B. Herbst, *On the extension of the Painlevé property to difference equations*, Nonlinearity **13** (2000), no. 3, 889–905, DOI 10.1088/0951-7715/13/3/321. MR1759006

[AKN74] Mark J. Ablowitz, David J. Kaup, Alan C. Newell, and Harvey Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Applied Mathematics **53** (1974), no. 4, 249–315. MR0450815

[And18] G.E. Andrews. *q-Hypergeometric and Related Functions*, chapter 17. In DLMF [**DLMF18**], 2018. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[AR18] R. Askey and R. Roy. *Gamma Function*, chapter 5. In DLMF [**DLMF18**], 2018. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[ARS78] M. J. Ablowitz, A. Ramani, and H. Segur, *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Lettere Al Nuovo Cimento **23** (1978), no. 9, 333–338. MR513779

[ARS80a] M. J. Ablowitz, A. Ramani, and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type. I*, Journal of Mathematical Physics **21** (1980), no. 4, 715–721, DOI 10.1063/1.524491. MR565716

[ARS80b] M. J. Ablowitz, A. Ramani, and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type. II*, Journal of Mathematical Physics **21** (1980), no. 5, 1006–1015, DOI 10.1063/1.524548. MR574872

[AS70] *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, Edited by Milton Abramowitz and Irene A. Stegun, Dover Publications, Inc., New York, 1966. MR0208797

[AS77] Mark J. Ablowitz and Harvey Segur, *Exact linearization of a Painlevé transcendent*, Physical Review Letters **38** (1977), no. 20, 1103–1106, DOI 10.1103/PhysRevLett.38.1103. MR0442981

[AS81] Mark J. Ablowitz and Harvey Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, vol. 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981. MR642018

[AHJN] James Atkinson, Phil Howes, Nalini Joshi, and Nobutaka Nakazono, *Geometry of an elliptic difference equation related to Q4*, Journal of the London Mathematical Society **93** (2016), no. 3, 763–784, DOI 10.1112/jlms/jdw020. MR3509963

[Bak33] H. F. Baker, *Principles of geometry. Volume 2. Plane geometry*, Cambridge Library Collection, Cambridge University Press, Cambridge, 2010. Reprint of the 1922 original. MR2857757

[Bir11] George D. Birkhoff, *General theory of linear difference equations*, Transactions of the American Mathematical Society **12** (1911), no. 2, 243–284, DOI 10.2307/1988577. MR1500888

[Bir13] G. D. Birkhoff. *The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations*. Proceedings of the American Academy of Arts and Sciences, 49(9): 521–568, 1913.

[BO78] Carl M. Bender and Steven A. Orszag, *Advanced mathematical methods for scientists and engineers*, McGraw-Hill Book Co., New York, 1978. International Series in Pure and Applied Mathematics. MR538168

[Bor04] Alexei Borodin, *Isomonodromy transformations of linear systems of difference equations*, Annals of Mathematics **160** (2004), no. 3, 1141–1182, DOI 10.4007/annals.2004.160.1141. MR2144976

[Bou07] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie*, Hermann, Paris, 1972. Actualités Scientifiques et Industrielles, No. 1349. MR0573068

[Bur11] W. Burnside. *Theory of Groups of Finite Order*. Cambridge University Press, 1911.

[BV99] M. P. Bellon and C.-M. Viallet, *Algebraic entropy*, Communications in Mathematical Physics **204** (1999), no. 2, 425–437, DOI 10.1007/s002200050652. MR1704282

[Cai43] Jeanne Le Caine, *The linear q-difference equation of the second order*, American Journal of Mathematics **65** (1943), 585–600, DOI 10.2307/2371867. MR0008889

[Car11] R. D. Carmichael, *Linear difference equations and their analytic solutions*, Transactions of the American Mathematical Society **12** (1911), no. 1, 99–134, DOI 10.2307/1988737. MR1500883

[Car12] R. D. Carmichael, *The general theory of linear q-difference equations*, American Journal of Mathematics **34** (1912), no. 2, 147–168, DOI 10.2307/2369887. MR1506145

[CJ99] Clio Cresswell and Nalini Joshi, *The discrete first, second and thirty-fourth Painlevé hierarchies*, Journal of Physics A: Mathematical and General **32** (1999), no. 4, 655–669, DOI 10.1088/0305-4470/32/4/009. MR1671841

[CKA98] S. J. Chapman, J. R. King, and K. L. Adams, *Exponential asymptotics and Stokes lines in nonlinear ordinary differential equations*, Proceedings of the Royal Society of London A: Mathematical, Physical, and Engineering Sciences **454** (1998), no. 1978, 2733–2755, DOI 10.1098/rspa.1998.0278. MR1650775

[Cla18] P. A. Clarkson. *Painlevé Transcendents*, chapter 32. In DLMF [**DLMF18**], 2018. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[Cox71] H. S. M. Coxeter, *Frieze patterns*, Acta Arithmetica **18** (1971), 297–310, DOI 10.4064/aa-18-1-297-310. MR0286771

[CS93] Christopher M. Cosgrove and George Scoufis, *Painlevé classification of a class of differential equations of the second order and second degree*, Studies in Applied Mathematics **88** (1993), no. 1, 25–87, DOI 10.1002/sapm199388125. MR1194166

[Dei99] P. A. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics, vol. 3, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. MR1677884

[DJ11] J. J. Duistermaat and N. Joshi, *Okamoto's space for the first Painlevé equation in Boutroux coordinates*, Archive for Rational Mechanics and Analysis **202** (2011), no. 3, 707–785, DOI 10.1007/s00205-011-0437-8. MR2854669

[DLMF18] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.20 of 2018-09-15, 2018. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[Dui10] Johannes J. Duistermaat, *Discrete Integrable Systems*: *QRT maps and elliptic surfaces*, Springer Monographs in Mathematics, Springer, New York, 2010. MR2683025

[Dun18] T.M. Dunster. *Legendre and Related Functions*, chapter 14. In DLMF [**DLMF18**], 2018. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[dV37] Patrick du Val, *On the Kantor group of a set of points in a plane*, Proceedings of the London Mathematical Society **42** (1936), no. 1, 18–51, DOI 10.1112/plms/s2-42.1.18. MR1577027

[FA81] A. S. Fokas and M. J. Ablowitz, *Linearization of the Korteweg-de Vries and Painlevé* II *equations*, Physical Review Letters **47** (1981), no. 16, 1096–1100, DOI 10.1103/PhysRevLett.47.1096. MR630505

[FIK91] A. S. Fokas, A. R. It· s, and A. V. Kitaev, *Discrete Painlevé equations and their appearance in quantum gravity*, Communications in Mathematical Physics **142** (1991), no. 2, 313–344. MR1137067

[Fri53] K. O. Friedrichs. *Special topics in analysis.* Lecture Notes. New York University, 1953.

[Fuc05] R. Fuchs. *Sur quelques équations différentielles linéaires du second ordre.* Comptes Rendus de l'Academie des Sciences Paris, 141:555–558, 1905.

[Ful13] U. Fulton, *Teoriya peresecheniĭ* (Russian), "Mir", Moscow, 1989. Translated from the English and with a foreword by V. I. Danilov. MR1024784

[Ǵ76] R. Gerard, *Geometric theory of differential equations in the complex domain*, Complex analysis and its applications (Lectures, Internat. Sem., Trieste, 1975), Internat. Atomic Energy Agency, Vienna, 1976, pp. 269–308. MR0488082

[Ǵ83] Raymond Gérard, *La géométrie des transcendantes de P. Painlevé* (French), Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, pp. 323–352. MR728428

[Gal13] H. Galbrun, *Sur la représentation des solutions d'une équation linéaire aux différences finies pour les grandes valeurs de la variable* (French), Acta Mathematica **36** (1913), no. 1, 1–68, DOI 10.1007/BF02422377. MR1555083

[Gam10] B. Gambier, *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est a points critiques fixes* (French), Acta Mathematica **33** (1910), no. 1, 1–55, DOI 10.1007/BF02393211. MR1555055

[GGKM67] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura. Method for solving the Korteweg-deVries equation. *Physical Review Letters*, 19:1095–1097, Nov 1967.

[GGKM74] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura, *Korteweg-deVries equation and generalization. VI. Methods for exact solution*, Communications in Pure and Applied Mathematics **27** (1974), 97–133, DOI 10.1002/cpa.3160270108. MR0336122

[GH78] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics. MR507725

[GM90] David J. Gross and Alexander A. Migdal, *Nonperturbative two-dimensional quantum gravity*, Physical Review Letters **64** (1990), no. 2, 127–130, DOI 10.1103/PhysRevLett.64.127. MR1031944

[Gri89] Phillip A. Griffiths, *Introduction to Algebraic Curves*, Translations of Mathematical Monographs, vol. 76, American Mathematical Society, Providence, RI, 1989. Translated from the Chinese by Kuniko Weltin. MR1013999

[GRP91] B. Grammaticos, A. Ramani, and V. Papageorgiou, *Do integrable mappings have the Painlevé property?*, Physical Review Letters **67** (1991), no. 14, 1825–1828, DOI 10.1103/PhysRevLett.67.1825. MR1125950

[GS72] Raymond Gérard and Antoinette Sec, *Feuilletages de Painlevé* (French), Bulletin de la Société Mathématique de France **100** (1972), 47–72. MR0306552

[Har77] Robin Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157

[Hau03] Herwig Hauser, *The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)*, Bulletin of the American Mathematical Society **40** (2003), no. 3, 323–403, DOI 10.1090/S0273-0979-03-00982-0. MR1978567

[HJN16] J. Hietarinta, N. Joshi, and F. W. Nijhoff, *Discrete Systems and Integrability*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2016. MR3587455

[HJS64] W. A. Harris Jr. and Y. Sibuya, *Note on linear difference equations*, Bulletin of the American Mathematical Society **70** (1964), 123–127, DOI 10.1090/S0002-9904-1964-11047-8. MR0157140

[HL99] Aimo Hinkkanen and Ilpo Laine, *Solutions of the first and second Painlevé equations are meromorphic*, Journal d'Analyse Mathématique **79** (1999), 345–377, DOI 10.1007/BF02788247. MR1749318

[HS99] Po-Fang Hsieh and Yasutaka Sibuya, *Basic theory of ordinary differential equations*, Universitext, Springer-Verlag, New York, 1999. MR1697415

[Hum92] James E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460

[HV07] Jarmo Hietarinta and Claude Viallet, *Searching for integrable lattice maps using factorization*, Journal of Physics A: Mathematical and Theoretical **40** (2007), no. 42, 12629–12643, DOI 10.1088/1751-8113/40/42/S09. MR2392894

[Imm06] Geertrui K. Immink, *Asymptotics of analytic difference equations*, Lecture Notes in Mathematics, vol. 1085, Springer-Verlag, Berlin, 1984. MR765699

[Inc56] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1944. MR0010757

[JBH92] Nalini Joshi, Damien Burtonclay, and Rodney G. Halburd, *Nonlinear nonautonomous discrete dynamical systems from a general discrete isomonodromy problem*, Letters in Mathematical Physics **26** (1992), no. 2, 123–131, DOI 10.1007/BF00398809. MR1193633

[JK94] Nalini Joshi and Martin D. Kruskal, *A direct proof that solutions of the six Painlevé equations have no movable singularities except poles*, Studies in Applied Mathematics **93** (1994), no. 3, 187–207, DOI 10.1002/sapm1994933187. MR1298423

[JL15] N. Joshi and C. J. Lustri, *Stokes phenomena in discrete Painlevé I*, Proceedings of the Royal Society of London A: Mathematical, Physical, and Engineering Sciences **471** (2015), no. 2177, 20140874, 22, DOI 10.1098/rspa.2014.0874. MR3348378

[JLL17] N. Joshi, C. J. Lustri, and S. Luu, *Stokes phenomena in discrete Painlevé II*, Proceedings of the Royal Society of London A: Mathematical, Physical, and Engineering Sciences **473** (2017), no. 2198, 20160539, 20, DOI 10.1098/rspa.2016.0539. MR3621486

[JLL19] N. Joshi, C.J. Lustri, and S. Luu, *Nonlinear q-Stokes phenomena for q-Painlevé I*, Journal of Physics A: Mathematical and Theoretical **52** (2019), no. 6, DOI 10.1088/1751-8121/aaf77c.

[JM81a] Michio Jimbo and Tetsuji Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*, Physica D: Nonlinear Phenomena **2** (1981), no. 3, 407–448, DOI 10.1016/0167-2789(81)90021-X. MR625446

[JM81b] Michio Jimbo and Tetsuji Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III*, Physica D: Nonlinear Phenomena **4** (1981/82), no. 1, 26–46, DOI 10.1016/0167-2789(81)90003-8. MR636469

[JN16] Nalini Joshi and Nobutaka Nakazono, *Lax pairs of discrete Painlevé equations: $(A_2 + A_1)^{(1)}$ case*, Proceedings of the Royal Society of London A: Mathematical, Physical, and Engineering Sciences **472** (2016), no. 2196, 20160696, 14, DOI 10.1098/rspa.2016.0696. MR3597921

[JN17] Nalini Joshi and Nobutaka Nakazono, *Elliptic Painlevé equations from next-nearest-neighbor translations on the $E_8^{(1)}$ lattice*, Journal of Physics A: Mathematical and Theoretical **50** (2017), no. 30, 305205, 17, DOI 10.1088/1751-8121/aa7915. MR3673467

[JNS16] N. Joshi, N. Nakazono, and Y. Shi. Reflection groups and discrete integrable systems. *Journal of Integrable Systems*, 1(1):1–37, 2016.

[Jos97] Nalini Joshi, *A local asymptotic analysis of the discrete first Painlevé equation as the discrete independent variable approaches infinity*, Methods and Applications of Analysis **4** (1997), no. 2, 124–133, DOI 10.4310/MAA.1997.v4.n2.a2. Dedicated to Martin David Kruskal. MR1603106

[Jos14] Nalini Joshi, *Quicksilver solutions of a q-difference first Painlevé equation*, Studies in Applied Mathematics **134** (2015), no. 2, 233–251, DOI 10.1111/sapm.12066. MR3313454

[JS96] Michio Jimbo and Hidetaka Sakai, *A q-analog of the sixth Painlevé equation*, Letters in Mathematical Physics **38** (1996), no. 2, 145–154, DOI 10.1007/BF00398316. MR1403067

[Kac94] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219

[K63] M. D. Kruskal. Asymptotology. in *Mathematical Models in Physical Sciences*, Dobrot, S., ed. Prentice-Hall, Englewood Cliffs, New Jersey (1963) 17–48.

[KC92] Martin D. Kruskal and Peter A. Clarkson, *The Painlevé-Kowalevski and poly-Painlevé tests for integrability*, Studies in Applied Mathematics **86** (1992), no. 2, 87–165, DOI 10.1002/sapm199286287. MR1140912

[KC01] J. R. King and S. J. Chapman, *Asymptotics beyond all orders and Stokes lines in nonlinear differential-difference equations*, European Journal of Applied Mathematics **12** (2001), no. 4, 433–463, DOI 10.1017/S095679250100434X. MR1852310

[KJH97] M. D. Kruskal, N. Joshi, and R. Halburd, *Analytic and asymptotic methods for nonlinear singularity analysis: a review and extensions of tests for the Painlevé property*, Integrability of nonlinear systems (Pondicherry, 1996), Lecture Notes in Phys., vol. 495, Springer, Berlin, 1997, pp. 171–205, DOI 10.1007/BFb0113696. MR1636294

[KMNOY] Kenji Kajiwara, Tetsu Masuda, Masatoshi Noumi, Yasuhiro Ohta, and Yasuhiko Yamada, ${}_{10}E_9$ *solution to the elliptic Painlevé equation*, Journal of Physics A: Mathematical and General **36** (2003), no. 17, L263–L272, DOI 10.1088/0305-4470/36/17/102. MR1984002

[KNT11] Kenji Kajiwara, Nobutaka Nakazono, and Teruhisa Tsuda, *Projective reduction of the discrete Painlevé system of type* $(A_2 + A_1)^{(1)}$, International Mathematics Research Notices. IMRN **4** (2011), 930–966. MR2773334

[KNY01] Kenji Kajiwara, Masatoshi Noumi, and Yasuhiko Yamada, *A study on the fourth q-Painlevé equation*, Journal of Physics A: Mathematical and General **34** (2001), no. 41, 8563–8581, DOI 10.1088/0305-4470/34/41/312. MR1876614

[KNY17] Kenji Kajiwara, Masatoshi Noumi, and Yasuhiko Yamada, *Geometric aspects of Painlevé equations*, Journal of Physics A: Mathematical and General **50** (2017), no. 7, 073001, 164, DOI 10.1088/1751-8121/50/7/073001. MR3609039

[KTGR00] M. D. Kruskal, K. M. Tamizhmani, B. Grammaticos, and A. Ramani, *Asymmetric discrete Painlevé equations*, Regular and Chaotic Dynamics **5** (2000), no. 3, 273–280, DOI 10.1070/rd2000v005n03ABEH000149. MR1789477

[Lag85] E. Laguerre. Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels. *Journal de Mathématiques Pures et Appliquées*, 1:135–166, 1885.

[Lax68] Peter D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968), 467–490, DOI 10.1002/cpa.3160210503. MR0235310

[Lax76] Peter D. Lax, *Almost periodic solutions of the KdV equation*, SIAM Review **18** (1976), no. 3, 351–375, DOI 10.1137/1018074. MR0404889

[Lef63] Solomon Lefschetz, *Differential equations: Geometric theory*, Second edition. Pure and Applied Mathematics, Vol. VI, Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963. MR0153903

[Lyn42] R.C Lyness. *Cycles.* The Mathematical Gazette, 26:62, 1942. Note 1581.

[Lyn45] R.C Lyness. *Cycles.* The Mathematical Gazette, 29:231–233, 1945. Note 1847.

[Mal83] B. Malgrange. *Déformations isomonodromiques des singularités régulières.* Les rencontres physiciens-mathématiciens de Strasbourg-RCP25 31:1–26, 1983.

[McM71] E.M. McMillan. *A problem in the stability of periodic systems*, In E.E Brittin and H. Odabasi, editors, *Topics in Modern Physics, a Tribute to E. V. Condon*, pages 219–244. Colorado Assoc. Univ. Press, Boulder, 1971.

[Mik09] A. V. Mikhailov (ed.), *Integrability*, Lecture Notes in Physics, vol. 767, Springer-Verlag, Berlin, 2009. MR2868499

[Mil00] L.M. Milne-Thomson, *The Calculus of Finite Differences*, Chelsea Publishing Company, 2000.

[Mil70] J. Milnor. *Foliations and foliated vector bundles,* mimeographed notes. Institute for Advanced Study, Princeton, 1970.

[JMU81] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and* τ*-function*, Physica D: Nonlinear Phenomena **2** (1981), no. 2, 306–352, DOI 10.1016/0167-2789(81)90013-0. MR630674

[MS59] H. P. Mulholland and C. A. B. Smith, *An inequality arising in genetical theory*, American Mathematical Monthly **66** (1959), 673–683, DOI 10.2307/2309342. MR0110721

[MS16] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76. MR0440554

[Mur04] Mikio Murata, *New expressions for discrete Painlevé equations*, Funkcialaj Ekvacioj Serio Internacia **47** (2004), no. 2, 291–305, DOI 10.1619/fesi.47.291. MR2108677

[Mur09] Mikio Murata, *Lax forms of the q-Painlevé equations*, Journal of Physics A: Mathematical and Theoretical **42** (2009), no. 11, 115201, 17, DOI 10.1088/1751-8113/42/11/115201. MR2485835

[NS75] A. M. Nīkolaĭčuk and I. M. Spītkovs′kiĭ, *Factorization of Hermitian matrix-valued functions, and its applications to boundary value problems* (Russian), Ukrain. Mat. Ž. **27** (1975), no. 6, 767–779, 861. MR0409839

[Nis88] Keiji Nishioka, *A note on the transcendency of Painlevé's first transcendent*, Nagoya Mathematical Journal **109** (1988), 63–67, DOI 10.1017/S0027763000002762. MR931951

[Nis10] Seiji Nishioka, *Transcendence of solutions of q-Painlevé equation of type* $A_7^{(1)}$, Aequationes Math. **79** (2010), no. 1-2, 1–12, DOI 10.1007/s00010-010-0007-4. MR2640274

[Nør16] N. E. Nörlund, *Sur les équations linéaires aux differences finies à coefficients rationnels* (French), Acta Mathematica **40** (1916), no. 1, 191–249, DOI 10.1007/BF02418545. MR1555137

[Nou04] Masatoshi Noumi, *Painlevé Equations Through Symmetry*, Translations of Mathematical Monographs, vol. 223, American Mathematical Society, Providence, RI, 2004. Translated from the 2000 Japanese original by the author. MR2044201

[NY99] Masatoshi Noumi and Yasuhiko Yamada, *Symmetries in the fourth Painlevé equation and Okamoto polynomials*, Nagoya Mathematical Journal **153** (1999), 53–86, DOI 10.1017/S0027763000006899. MR1684551

[ORG01] Y. Ohta, A. Ramani, and B. Grammaticos, *An affine Weyl group approach to the eight-parameter discrete Painlevé equation*, Journal of Physics A: Mathematical and General **34** (2001), no. 48, 10523–10532, DOI 10.1088/0305-4470/34/48/316. Symmetries and integrability of difference equations (Tokyo, 2000). MR1877472

[Oka79] Kazuo Okamoto, *Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé* (French), Japanese Journal of Mathematics. New series **5** (1979), no. 1, 1–79. MR614694

[Oka86I] Kazuo Okamoto, *Studies on the Painlevé equations. IV. Third Painlevé equation* $P_{\rm III}$, Funkcialaj Ekvacioj. Serio Internacia **30** (1987), no. 2-3, 305–332. MR927186

[Oka86III] Kazuo Okamoto, *Studies on the Painlevé equations. III. Second and fourth Painlevé equations,* $P_{\rm II}$ *and* $P_{\rm IV}$, Mathematische Annalen **275** (1986), no. 2, 221–255, DOI 10.1007/BF01458459. MR854008

[Olv74] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Computer Science and Applied Mathematics. MR0435697

[OR17] Christopher M. Ormerod and Eric Rains, *A symmetric difference-differential Lax pair for Painlevé VI*, Journal of Integrable Systems **2** (2017), no. 1, xyx003, 20, DOI 10.1093/integr/xyx003. MR3691966

[Pai02] P. Painlevé, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme* (French), Acta Mathematics **25** (1902), no. 1, 1–85, DOI 10.1007/BF02419020. MR1554937

[Pai06] P. Painlevé. *Sur les équations différentielles du second ordre á points critiques fixes.* Comptes Rendus de l'Academie des Sciences Paris, 143:1111–1117, 1906.

[PNGR92] V. G. Papageorgiou, F. W. Nijhoff, B. Grammaticos, and A. Ramani, *Isomonodromic deformation problems for discrete analogues of Painlevé equations*, Physics Letters A **164** (1992), no. 1, 57–64, DOI 10.1016/0375-9601(92)90905-2. MR1162062

[Pic89] E. Picard. *Mémoire sur la théorie des fonctions algébriques de deux variables.* Journal de Mathématiques pures et appliquées, 5:135–320, 1889.

[PS81] Roger Penrose and Cedric A. B. Smith, *A quadratic mapping with invariant cubic curve*, Mathematical Proceedings of the Cambridge Philosophical Society **89** (1981), no. 1, 89–105, DOI 10.1017/S0305004100057972. MR591975

[PS90] V. Periwal and D. Shevitz. *Unitary-matrix models as exactly solvable string theories.* Physical review letters, 64(12):1326, 1990.

[QRT88] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson, *Integrable mappings and soliton equations*, Physics Letters. A **126** (1988), no. 7, 419–421, DOI 10.1016/0375-9601(88)90803-1. MR924318

[QRT89] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson, *Integrable mappings and soliton equations. II*, Physica D. Nonlinear Phenomena **34** (1989), no. 1-2, 183–192, DOI 10.1016/0167-2789(89)90233-9. MR982386

[Rai16] E. Rains. Generalized Hitchin systems on rational surfaces. *arXiv preprint arXiv:1307/4033v2*, 2016.

[RCG09] A. Ramani, A. S. Carstea, and B. Grammaticos, *On the non-autonomous form of the Q_4 mapping and its relation to elliptic Painlevé equations*, Journal of Physics A: Mathematical and Theoretical **42** (2009), no. 32, 322003, 8, DOI 10.1088/1751-8113/42/32/322003. MR2525848

[RCGO02] A. Ramani, A. S. Carstea, B. Grammaticos, and Y. Ohta, *On the autonomous limit of discrete Painlevé equations*, Physica A: Statistical Mechanics and its Applications **305** (2002), no. 3-4, 437–444, DOI 10.1016/S0378-4371(01)00619-7. MR1928121

[RG96] A. Ramani and B. Grammaticos, *Discrete Painlevé equations: coalescences, limits and degeneracies*, Physica A: Statistical Mechanics and its Applications **228** (1996), no. 1-4, 160–171, DOI 10.1016/0378-4371(95)00439-4. MR1399286

[RGH91] A. Ramani, B. Grammaticos, and J. Hietarinta, *Discrete versions of the Painlevé equations*, Physical Review Letters **67** (1991), no. 14, 1829–1832, DOI 10.1103/PhysRevLett.67.1829. MR1125951

[RSZ13] Jean-Pierre Ramis, Jacques Sauloy, and Changgui Zhang, *Local analytic classification of q-difference equations* (English, with French summary), Astérisque **355** (2013), vi+151. MR3185985

[RW18] W.P. Reinhardt and P.L. Walker. *Theta Functions*, chapter 20. In DLMF [**DLMF18**], 2018. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[Sak01] Hidetaka Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Communications in Mathematical Physics **220** (2001), no. 1, 165–229, DOI 10.1007/s002200100446. MR1882403

[Sak07] Hidetaka Sakai, *Problem: discrete Painlevé equations and their Lax forms*, Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies, RIMS Kôkyûroku Bessatsu, B2, Research Institute for Mathematical Sciences (RIMS), Kyoto, 2007, pp. 195–208. MR2310030

[Sha94] Igor R. Shafarevich, *Basic Algebraic Geometry. 1*, 2nd ed., Springer-Verlag, Berlin, 1994. Varieties in projective space; Translated from the 1988 Russian edition and with notes by Miles Reid. MR1328833

[Sho39] J. Shohat, *A differential equation for orthogonal polynomials*, Duke Mathematical Journal **5** (1939), no. 2, 401–417, DOI 10.1215/S0012-7094-39-00534-X. MR1546133

[SM59] P.A.G. Scheuer and S.P.H. Mandel. *An inequality in population genetics.* Heredity, 13:519–524, 1959.

[Tak03] Tomoyuki Takenawa, *Weyl group symmetry of type $D_5^{(1)}$ in the q-Painlevé V equation*, Funkcial. Ekvac. **46** (2003), no. 1, 173–186, DOI 10.1619/fesi.46.173. MR1996297

[Trj33] W. J. Trjitzinsky, *Analytic theory of linear q-difference equations*, Acta Mathematics **61** (1933), no. 1, 1–38, DOI 10.1007/BF02547785. MR1555369

[TW94] Craig A. Tracy and Harold Widom, *Level-spacing distributions and the Airy kernel*, Communications in Mathematical Physics **159** (1994), no. 1, 151–174. MR1257246

[Ume90] Hiroshi Umemura, *Second proof of the irreducibility of the first differential equation of Painlevé*, Nagoya Mathematical Journal **117** (1990), 125–171, DOI 10.1017/S0027763000001835. MR1044939

[UW97] Hiroshi Umemura and Humihiko Watanabe, *Solutions of the second and fourth Painlevé equations. I*, Nagoya Mathematical Journal **148** (1997), 151–198, DOI 10.1017/S0027763000006486. MR1492945

[UW98] Hiroshi Umemura and Humihiko Watanabe, *Solutions of the third Painlevé equation. I*, Nagoya Mathematical Journal **151** (1998), 1–24, DOI 10.1017/S0027763000025149. MR1650348

[Was02] Wolfgang Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Dover Publications, Inc., New York, 1987. Reprint of the 1976 edition. MR919406

[WW02] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions; Reprint of the fourth (1927) edition. MR1424469

[ZK65] N. J. Zabusky and M. D. Kruskal. *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*. Physical Review Letters, 15:240–243, 1965.

[ZS74] V. E. Zaharov and A. B. Šabat, *A plan for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I* (Russian), Funkcional. Anal. i Priložen. **8** (1974), no. 3, 43–53. MR0481668

Index

Discrete Painlevé equations are nonlinear difference equations, which arise from translations on crystallographic lattices. The deceptive simplicity of this statement hides immensely rich mathematical properties, connecting dynamical systems, algebraic geometry, Coxeter groups, topology, special functions theory, and mathematical physics.

This book necessarily starts with introductory material to give the reader an accessible entry point to this vast subject matter. It is based on lectures that the author presented as principal lecturer at a Conference Board of Mathematical Sciences and National Science Foundation conference in Texas in 2016. Instead of technical theorems or complete proofs, the book relies on providing essential points of many arguments through explicit examples, with the hope that they will be useful for applied mathematicians and physicists.

ISBN 978-1-4704-5038-0

9 781470 450380

CBMS/131

For additional information and updates on this book, visit

www.ams.org/bookpages/cbms-131

AMS
www.ams.org