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Number 135

## Fitting Smooth Functions to Data

Charles Fefferman  
Arie Israel

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To Julie and Rachel.



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## Preface

Fix positive integers  $m, n$ , and let  $f$  be a real-valued function on an (arbitrary) given compact set  $E$  in  $\mathbb{R}^n$ . How can we tell whether  $f$  extends to a function  $F$  in  $C^m(\mathbb{R}^n)$ ?

H. Whitney started the study of this problem in 1934. He solved the one-dimensional case and proved the classic Whitney extension theorem, solving completely an easier version of the problem in  $n$  dimensions.

There is a finite, computational version of Whitney's problem. We start with a real-valued function  $f$  defined on a finite set  $E$  in  $\mathbb{R}^n$  and ask how to compute a function  $F$  in  $C^m(\mathbb{R}^n)$  that agrees with  $f$  on  $E$  and has  $C^m$  norm nearly as small as possible. We hope to compute such an  $F$  using minimal computer time and memory.

As in the original Whitney problem, much of the interest and challenge of the finite version arises when we ask for algorithms that always work, regardless of the geometry of the set  $E$ . If we assume that  $E$  looks anything like a lattice in  $\mathbb{R}^n$ , our problems become much easier.

It is natural to look also for functions  $F$  that agree only approximately with a given  $f$ , and to consider other function space norms in addition to the  $C^m$  norm.

Over the last 15 years, I've been fascinated by the above questions and their variants. This book explains what I've found, mostly in joint work with several collaborators (see below). The results include the solution to Whitney's problem, an efficient algorithm for the finite version, and analogues for Hölder and Sobolev spaces in place of  $C^m$ .

Our goal here is not to provide complete proofs or complete descriptions of algorithms, but rather to explain many of the basic ideas simply and painlessly. I hope readers will enjoy the book, and perhaps become interested in some of the significant unsolved problems.

This book is organized as follows. We first present an overview stating our main results in Chapter 1. We continue in Chapter 2 with a proof of the classical Whitney extension theorem. We discuss the  $C^m$  interpolation problem for finite data in Chapter 3, and the Whitney extension problem for  $C^m$  in Chapter 4. In Chapter 5 we describe our results for the Whitney extension and interpolation problems for Sobolev spaces. Chapter 6 presents our results on extension, interpolation, and selection problems for vector-valued functions. We close by listing a few basic unsolved problems in Chapter 7.

The book is based on lectures presented at a CBMS regional workshop held at the University of Texas at Austin in the summer of 2019. My coauthor Arie Israel transcribed and extended my lectures to produce the manuscript. I thank Arie for his efforts, which included finding and correcting several mathematical errors. I take responsibility for any that remain.

I'm grateful for the generous support that helped me to obtain and disseminate the results presented here. In particular, the Conference Board of the Mathematical Sciences and the University of Texas made possible the Austin workshop; and the American Institute of Mathematics, the Banff International Research Station, the College of William and Mary, the Fields Institute, and the Technion hosted and supported previous workshops on Whitney problems. The NSF, ONR, AFOSR, and BSF supported my work over many years.<sup>1</sup>

It has been a joy to collaborate with Arie Israel, Bo'az Klartag, Garving (Kevin) Luli, and Pavel Shvartsman. I thank them, as well as all the participants in the Whitney workshops mentioned above, for their interest and their ideas. I am grateful to P. Shvartsman for his careful reading and useful comments on a preliminary version of this manuscript. Finally, I'm grateful to E. Bierstone, Yu. Brudnyi, G. Glaeser, P. Milman, W. Pawłucki, P. Shvartsman, and of course, H. Whitney; their work on Whitney's problem brought to light ideas on which this book builds. Now let's get started.

C. Fefferman

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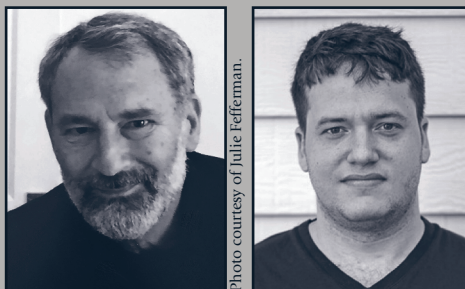
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