Knots and Links

Dale Rolfsen

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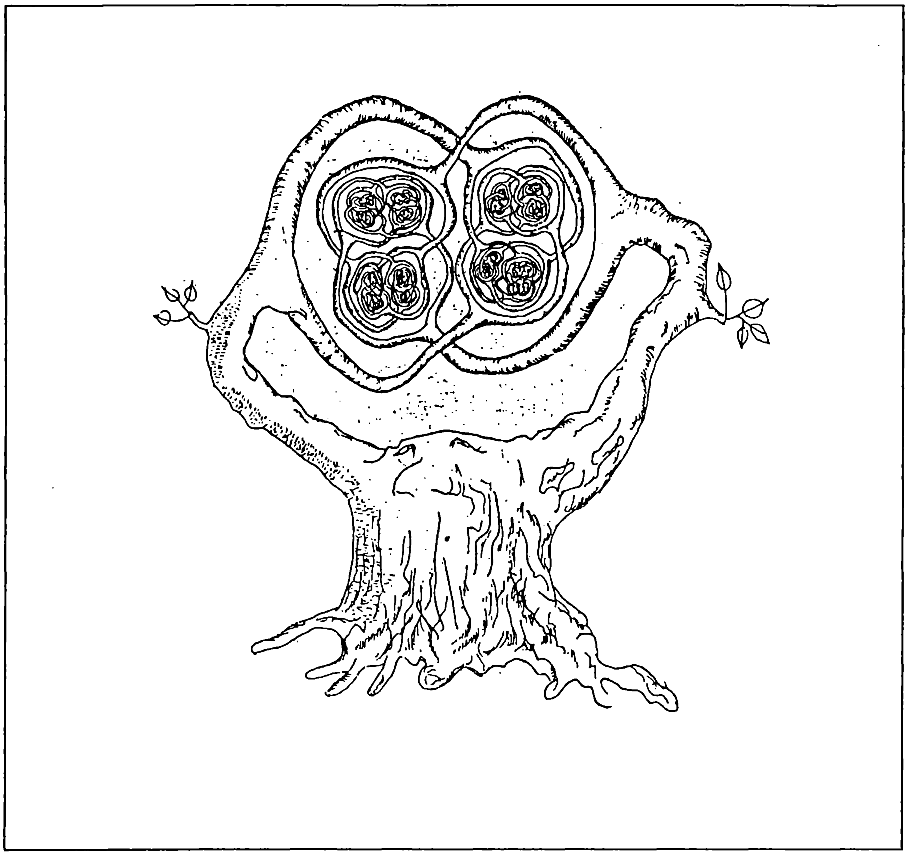

Yggdrasil. The tree of Norse mythology whose branches lead to heaven. Realized as Alexander's horned sphere in this etching by Bill Meyers.

Previous Page: This knot (7_4 in the table) is one of the eight glorious emblems of Tibetan Buddhism. Just as a knot does not exist without reference to its embedding in space, this emblem is a reminder of the interdependence of all things in the phenomenal world.

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To Amy, Catherine and Gloria

Preface to the AMS Chelsea edition

This book was written as a textbook for graduate students or advanced undergraduates, or for the nonexpert who wants to learn about the mathematical theory of knots. A very basic understanding of algebraic topology is assumed (and outlined in Appendix A).

Since the first edition appeared in 1976, knot theory has been transformed from a rather specialized branch of topology to a very popular, vibrant field of mathematics. The impetus for this change was largely the work of Vaughan Jones, who discovered a new polynomial invariant of knots through his work in operator algebras. This led to astonishing connections between knot theory and physics, and such diverse disciplines as algebraic geometry, Lie theory, statistical mechanics and quantum theory.

Friends have encouraged me to revise *Knots and Links* to include an account of these exciting developments. I decided not to do this for several reasons. First of all, a number of good books on knot theory have appeared since *Knots and Links*, which cover these later developments. Secondly, the present book is already a fairly large tome, and it would be doubled in size if I were to do justice to advances in the field since publication. This didn't seem like a good idea. Finally, I believe this book remains valuable as an introduction to the exciting fields of knot theory and low dimensional topology. For similar reasons, the ''forthcoming entirely new book'' mentioned in the preface to the second printing will very likely never materialize.

Knots and Links would not have existed in the first place, had it not been for Mike Spivak, owner and founder of *Publish or Perish Press*. Spivak has been a faithful friend and constant source of encouragement. It was he who suggested I use my class notes as the basis for a book -- this book was the result. It was also his suggestion that, when the last printing of the book by *Publish or Perish* ran out recently, I seek another publisher. I am extremely pleased that, with the new AMS Chelsea Classics edition, this book will remain available in a high-quality format and at a reasonable price. I'd also like to thank Edward Dunne and Sergei Gelfand and the staff at AMS Books for facilitating this edition.

Finally, I would like to extend my gratitude to a number of friends and colleagues who have pointed out errors in the previous editions. Nathan Dunfield verified all the Alexander polynomials of the knots and links in the tables, and found exactly four errors, which are corrected in this edition (9^2_{29}, 9^2_{55}, 9^2_{57} and 9^2_{59}). Other corrections for this new edition are a matrix entry on page 220, correction of Lemma 8E18, p. 222, and an exercise at the bottom of page 353. These last two were pointed out by Steve Boyer. Thanks also to Jim Bailey, Steve Bleiler, Jim Hoste, Peter Landweber, Olivier Collin and others whose help I may have forgotten. No doubt there are still errors, which I would be glad to hear about. Any future corrections will be posted on the AMS Books website. The url is given on the copyright page.

Dale Rolfsen

(rolfsen@math.ubc.ca)

Vancouver, June, 2003

PREFACE TO THE SECOND PRINTING

This new printing is essentially the same as the original edition, except that I have corrected the errors that I know about. Several colleagues and students have been very helpful in pointing out these errors, and I wish to thank them for their help. Special thanks to Professors Jim Hoste and Peter Landweber for finding lots of them and sending me detailed lists. One of the most embarrassing errors is the duplication in the knot table: 10_{161} and 10_{162} are really the same knot, as K. Perko has pointed out. Also, in the table, the drawing of 10_{144} was wrong.

I didn't make any attempt to update the book with new material. I took the advice of a kind friend who told me not to tamper with a "classic." A lot has happened in knot theory in the decade and a half since this book was written. I will do my best to report that in a forthcoming entirely new book. However, some notable developments really ought to be mentioned. The old conjecture that knots are determined by their complements was recently solved in the affirmative by C. Gordon and J. Leucke. Likewise, we now know the Smith Conjecture to be true, although the Poincaré and Property P conjectures still stand. Equally exciting is the "polynomial fever" rampant for the past five years, inspired by V. Jones' discovery of a new polynomial so powerful that it could distinguish the two trefoils. This breakthrough led to the discovery of plenty of new polynomials, giving us a large new collection of very sharp tools and adding fundamentally to our understanding of that wonder of our natural world: knots.

The best thing that has happened to knot theory, however, is that many more scientists are now interested in it -- not just topologists -- and contributing in their unique ways. Jones led the way by introducing operator algebras and representation theory to the subject. Since then deep contributions have been made by algebraic and differential geometry, and by mathematical physics. Surprising connections with statistical mechanics and quantum field theory are just now being explored and promise to make the end of the 20th century a real golden age for knot theory. Knot theory is not only utilizing ideas from other disciplines, but is beginning to return the favor. Besides stimulating new directions of research in mathematics and physics, ideas of knot theory are being used effectively in such fields as stereochemistry and molecular biology. So knot theory can begin to call itself applied mathematics!

Since the appearance of "Knots and Links," several excellent books on the subject of mathematical knot theory have appeared. Most notable are "Knots," by Burde and Zieschang, and "On knots," by Kauffman. These are highly recommended. Each has an emphasis different from the present work, and the three can be regarded as mutually complementary.

Finally, my sincere thanks go to my publisher, Mike Spivak, for agreeing to put out this new printing, for his patience in making the corrections, and for his realization that it was hopeless to expect my promised new book on knots in the very near future.

Dale Rolfsen

Vancouver, Canada

February 9, 1990

PREFACE

This book began as a course of lectures that I gave at the University of British Columbia in 1973-74. It was a graduate course officially called "Topics in geometric topology." That would probably be a more accurate title for this book than the one it has. My bias in writing it has been to treat knots and links, not as the subject of a theory unto itself, but rather as (1) a source of examples through which various techniques of topology and algebra can be illustrated and (2) a point of view which has real and interesting applications to other branches of topology. Accordingly, this book consists mainly of examples.

The students in that course were graduate level and all had some background in point-set topology and a little algebraic topology. But I think an intelligent undergraduate mathematics student, who is willing to learn algebraic topology as he goes along, should be able to handle the ideas here. As part of my course, the students lectured to each other from Rourke and Sanderson's book [1972] on piecewise-linear topology. So I've used some PL techniques without much explanation, but not to excess.

If you scan through the pages you'll find that there are lots of exercises. Some are routine and some are difficult. My philosophy in teaching the course was to have the students prove things for themselves as much as possible, so the exercises are central to the ideas developed in these notes. Do as many as you can.

I would like to express my thanks to the people who helped me to prepare this manuscript during rather nomadic times for me. They are: Cathy Agnew (Vancouver), Yit-Sin Choo (Vancouver), Cynthia Coddington (Heriot Bay, B. C.), Joanne Congo (Vancouver), Sandra Flint (Cambridge), Judy Gilbertson (Laramie, Wyoming), Carol Samson (Vancouver) and Maria del Carmen Sanchez del Valle (Mexico City). Special thanks are due to Jim Bailey, who took notes in the course on which this book is based, compiled the table which forms appendix C, and helped in many other ways. Also to his friend Ali Roth who drew the knots and links so beautifully. David Gillman gave an excellent series of three lectures on Dehn's Lemma, and I'm grateful to him for writing up the notes for inclusion here as appendix B.

Many friends and mathematicians have given me encouragement and advice, both mathematical and psychological. Among them are Andrew Casson, Francisco Gonzalez-Acuna, Cameron Gordon, Cherry Kearton, Robion Kirby, Raymond Lickorish and Joe Martin, whose own lecture notes were very helpful to me. Finally I want to thank Mary-Ellen Rudin for her advice, which I should have followed sooner: "Don't try to get everything in that book."

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