Comparison Theorems in Riemannian Geometry

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Preface to the AMS Chelsea Printing

In the period since the first edition of this book appeared (1976), Riemannian geometry has experienced explosive growth. This circumstance was ultimately the determining factor in our decision not to try to bring the book up to date. We came to realize that if such an attempt were made, the book would never see the light of day. So our book remains essentially as it was. We hope that it will continue to play a useful role by setting out some (still relevant) fundamentals of the subject.

There do exist a number of valuable surveys and expositions of many of the developments that have taken place in the intervening years. Some of these are indicated in a relatively small collection of additional references that have been added to the bibliography. The surveys of Berger, in particular, provide a sweeping overview and include very extensive bibliographies.

We have also added a highly selective (and quite inadequate) set of references to original sources, reflecting what we see as some of the most significant developments over the past 30 years. We apologize in advance to anyone whose work has been unjustly slighted.

We are indebted to Satyaki Dutta for having retyped the manuscript and for a substantial amount of proofreading. We would also like to express our thanks to all those colleagues who, in the 20 years since the first edition went out of print, have urged us to consider having it republished.

Jeff Cheeger and David G. Ebin, 2007.

Preface

In this book we study complete Riemannian manifolds by developing techniques for comparing the geometry of a general manifold M with that of a simply connected model space of constant curvature M_H. A typical conclusion is that M retains particular geometrical properties of the model space under the assumption that its sectional curvature K_M, is bounded between suitable constants. Once this has been established, it is usually possible to conclude that M retains topological properties of M_H as well.

The distinction between strict and weak bounds on K_M is important, since this may reflect the difference between the geometry of say the sphere and that of Euclidean space. However, it is often the case that a conclusion which becomes false when one relaxes the condition of strict inequality to weak inequality can be shown to fail only under very special circumstances. Results of this nature, which are known as rigidity theorems, generally require a delicate global argument. Here are some examples which will be treated in more detail in Chapter 8.

Topological Theorem. If M is a complete manifold such that $K_M \geq \delta > 0$, then M has finite fundamental group.

Geometrical Antecedent. If M is a complete manifold such that $K_M \geq \delta > 0$, then the diameter of its universal covering space $\widetilde{M}$ is $\leq \Pi/\sqrt{\delta}$. In particular, $\widetilde{M}$ is compact.

Even if we assume M to be compact, the preceding statements are false if only $K_M \geq 0$. However, we can show the following.

Rigid Topological Theorem. Let M be a compact manifold such that $K_M \geq 0$. Then there is an exact sequence

$$0 \to \Phi \to \Pi_1(M) \to B \to 0$$

where is a finite group and B is a crystallographic group on $\mathbb{R}_k$ for some $k \leq \dim M$, and therefore satisfies an exact sequence

$$0 \to \mathbb{Z}_k \to B \to \Psi \to 0$$

where Ψ is a finite group.

Rigid Geometrical Antecedent. Let M be a compact manifold such that $K_M \geq 0$. Then $\widetilde{M}$ splits isometrically as $\overline{M} \times \mathbb{R}_k$ (same k as above), where $\overline{M}$ is compact and $\mathbb{R}_k$ has its standard flat metric. Thus, if $K_M \geq 0$, $\widetilde{M}$ may not be compact, but it is at worst the isometric product of a compact manifold and Euclidean space. The infinite part of $\Pi_1(M)$ comes precisely from the Euclidean factor.

The reader of this book should have a basic knowledge of differential geometry and algebraic topology, at least the equivalent of a one term course is each. Our purpose is to provide him with a fairly direct route to some interesting geometrical theorems, without his becoming bogged down in a detailed study of connections and tensors. In keeping with this approach, we have limited ourselves primarily to those techniques which arise as outgrowths of the second variation formula and to some extent of Morse theory.

In Chapter 1 we have included a rapid treatment of the more elementary material on which the later chapters are based. Of course, we do not recommend that the less knowledgeable reader regard this as a comprehensive introduction to Riemannian geometry. Likewise, Chapters 3 and 4 are provided in part for the convenience of the reader. In Chapter 3 which deals with homogeneous spaces, we also summarize without proof the relevant material on Lie groups. In Chapter 4 the main theorems of Morse theory are stated, again mostly without proofs. An exception, however, is Lemma 4.11, which is perhaps less standard than the other material. For the unproven results in both chapters, excellent references are readily available. Our main geometrical tools, the Rauch Comparison Theorems and the more global Toponogov Theorem, are discussed in Chapters 1 and 2 respectively. Chapter 5 deals with closed geodesics and the injectivity radius of the exponential map. Chapters 6-9 form the core of our study. Chapter 6 contains the Sphere Theorem – M simply connected and $1 \geq K_M > 1/4$ implies M homeomorphic to a sphere – as well as Berger's rigidity theorem which covers the case $1 \geq K_M \geq 1/4$. The last three chapters deal with material of recent origin. Chapter 7 is primarily concerned with the differentiable version of the Sphere Theorem. Chapter 8 takes up the structure theory of complete noncompact manifolds of nonnegative curvature, while Chapter 9 gives some results on the fundamental group of compact manifolds of nonpositive curvature.

It is a pleasure to thank Carole Alberghine and Lois Cheeger for their patient work, typing the original manuscript.

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