

# COMPARISON THEOREMS IN RIEMANNIAN GEOMETRY

JEFF CHEEGER  
DAVID G. EBIN

AMS CHELSEA PUBLISHING  
American Mathematical Society • Providence, Rhode Island



**COMPARISON THEOREMS  
IN RIEMANNIAN GEOMETRY**



Photo courtesy of Ann Billingsley

David G. Ebin and Jeff Cheeger

# COMPARISON THEOREMS IN RIEMANNIAN GEOMETRY

JEFF CHEEGER  
DAVID G. EBIN

AMS CHELSEA PUBLISHING  
American Mathematical Society • Providence, Rhode Island



2000 *Mathematics Subject Classification*. Primary 53C20; Secondary 58E10.

---

For additional information and updates on this book, visit  
[www.ams.org/bookpages/chel-365](http://www.ams.org/bookpages/chel-365)

---

**Library of Congress Cataloging-in-Publication Data**

Cheeger, Jeff.

Comparison theorems in riemannian geometry / Jeff Cheeger, David G. Ebin.

p. cm. — (AMS Chelsea Publishing)

Originally published: Amsterdam : North-Holland Pub. Co. ; New York : American Elsevier Pub. Co., 1975, in series: North-Holland mathematical library ; v. 9.

Includes bibliographical references and index.

ISBN 978-0-8218-4417-5 (alk. paper)

1. Geometry, Riemannian. 2. Riemannian manifolds. I. Ebin, D. G. II. American Mathematical Society. III. Title.

QA649.C47 2008

516.3'73—dc22

2007052113

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 1975 held by the American Mathematical Society. All rights reserved.

Reprinted by the American Mathematical Society, 2008

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 13 12 11 10 09 08

## Contents

Preface to the AMS Chelsea Printing	vii
Preface	ix
Chapter 1. Basic Concepts and Results	1
1. Notation and preliminaries	1
2. First variation of arc length	3
3. Exponential map and normal coordinates	6
4. The Hopf-Rinow Theorem	9
5. The curvature tensor and Jacobi fields	11
6. Conjugate points	14
7. Second variation of arc length	16
8. Submanifolds and the second fundamental form	18
9. Basic index lemmas	20
10. Ricci curvature and Myers' and Bonnet's Theorems	23
11. Rauch Comparison Theorem	24
12. The Cartan-Hadamard Theorem	30
13. The Cartan-Ambrose-Hicks Theorem	31
14. Spaces of constant curvature	34
Chapter 2. Toponogov's Theorem	35
Chapter 3. Homogeneous Spaces	47
Chapter 4. Morse Theory	69
Chapter 5. Closed Geodesics and the Cut Locus	81
Chapter 6. The Sphere Theorem and its Generalizations	93
Chapter 7. The Differentiable Sphere Theorem	105
Chapter 8. Complete Manifolds of Nonnegative Curvature	119
Chapter 9. Compact Manifolds of Nonpositive Curvature	137
Bibliography	149
Additional Bibliography	155
Index	157



## Preface to the AMS Chelsea Printing

In the period since the first edition of this book appeared (1976), Riemannian geometry has experienced explosive growth. This circumstance was ultimately the determining factor in our decision not to try to bring the book up to date. We came to realize that if such an attempt were made, the book would never see the light of day. So our book remains essentially as it was. We hope that it will continue to play a useful role by setting out some (still relevant) fundamentals of the subject.

There do exist a number of valuable surveys and expositions of many of the developments that have taken place in the intervening years. Some of these are indicated in a relatively small collection of additional references that have been added to the bibliography. The surveys of Berger, in particular, provide a sweeping overview and include very extensive bibliographies.

We have also added a highly selective (and quite inadequate) set of references to original sources, reflecting what we see as some of the most significant developments over the past 30 years. We apologize in advance to anyone whose work has been unjustly slighted.

We are indebted to Satyaki Dutta for having retyped the manuscript and for a substantial amount of proofreading. We would also like to express our thanks to all those colleagues who, in the 20 years since the first edition went out of print, have urged us to consider having it republished.

Jeff Cheeger and David G. Ebin, 2007.





## Preface

In this book we study complete Riemannian manifolds by developing techniques for comparing the geometry of a general manifold  $M$  with that of a simply connected model space of constant curvature  $M_H$ . A typical conclusion is that  $M$  retains particular geometrical properties of the model space under the assumption that its sectional curvature  $K_M$ , is bounded between suitable constants. Once this has been established, it is usually possible to conclude that  $M$  retains topological properties of  $M_H$  as well.

The distinction between strict and weak bounds on  $K_M$  is important, since this may reflect the difference between the geometry of say the sphere and that of Euclidean space. However, it is often the case that a conclusion which becomes false when one relaxes the condition of strict inequality to weak inequality can be shown to fail only under very special circumstances. Results of this nature, which are known as rigidity theorems, generally require a delicate global argument. Here are some examples which will be treated in more detail in Chapter 8.

**Topological Theorem.** If  $M$  is a complete manifold such that  $K_M \geq \delta > 0$ , then  $M$  has finite fundamental group.

**Geometrical Antecedent.** If  $M$  is a complete manifold such that  $K_M \geq \delta > 0$ , then the diameter of its universal covering space  $\widetilde{M}$  is  $\leq \Pi/\sqrt{\delta}$ . In particular,  $\widetilde{M}$  is compact.

Even if we assume  $M$  to be compact, the preceding statements are false if only  $K_M \geq 0$ . However, we can show the following.

**Rigid Topological Theorem.** Let  $M$  be a compact manifold such that  $K_M \geq 0$ . Then there is an exact sequence

$$0 \rightarrow \Phi \rightarrow \Pi_1(M) \rightarrow B \rightarrow 0$$

where  $\Phi$  is a finite group and  $B$  is a crystallographic group on  $\mathbb{R}_k$  for some  $k \leq \dim M$ , and therefore satisfies an exact sequence

$$0 \rightarrow \mathbb{Z}_k \rightarrow B \rightarrow \Psi \rightarrow 0$$

where  $\Psi$  is a finite group.

**Rigid Geometrical Antecedent.** Let  $M$  be a compact manifold such that  $K_M \geq 0$ . Then  $\widetilde{M}$  splits isometrically as  $\overline{M} \times \mathbb{R}_k$  (same  $k$  as above), where  $\overline{M}$  is compact and  $\mathbb{R}_k$  has its standard flat metric. Thus, if  $K_M \geq 0$ ,  $\widetilde{M}$  may not be compact, but it is at worst the isometric product of a compact

manifold and Euclidean space. The infinite part of  $\Pi_1(M)$  comes precisely from the Euclidean factor.

The reader of this book should have a basic knowledge of differential geometry and algebraic topology, at least the equivalent of a one term course in each. Our purpose is to provide him with a fairly direct route to some interesting geometrical theorems, without his becoming bogged down in a detailed study of connections and tensors. In keeping with this approach, we have limited ourselves primarily to those techniques which arise as outgrowths of the second variation formula and to some extent of Morse theory.

In Chapter 1 we have included a rapid treatment of the more elementary material on which the later chapters are based. Of course, we do not recommend that the less knowledgeable reader regard this as a comprehensive introduction to Riemannian geometry. Likewise, Chapters 3 and 4 are provided in part for the convenience of the reader. In Chapter 3 which deals with homogeneous spaces, we also summarize without proof the relevant material on Lie groups. In Chapter 4 the main theorems of Morse theory are stated, again mostly without proofs. An exception, however, is Lemma 4.11, which is perhaps less standard than the other material. For the unproven results in both chapters, excellent references are readily available. Our main geometrical tools, the Rauch Comparison Theorems and the more global Toponogov Theorem, are discussed in Chapters 1 and 2 respectively. Chapter 5 deals with closed geodesics and the injectivity radius of the exponential map. Chapters 6-9 form the core of our study. Chapter 6 contains the Sphere Theorem –  $M$  simply connected and  $1 \geq K_M > 1/4$  implies  $M$  homeomorphic to a sphere – as well as Berger's rigidity theorem which covers the case  $1 \geq K_M \geq 1/4$ . The last three chapters deal with material of recent origin. Chapter 7 is primarily concerned with the differentiable version of the Sphere Theorem. Chapter 8 takes up the structure theory of complete noncompact manifolds of nonnegative curvature, while Chapter 9 gives some results on the fundamental group of compact manifolds of nonpositive curvature.

It is a pleasure to thank Carole Alberghine and Lois Cheeger for their patient work, typing the original manuscript.

## Bibliography

M. BERGER

- [1959] Variétés Riemanniennes à courbure positive, *Bull. Soc. Math. France* **87**.
- [1960] Les variétés Riemanniennes  $\frac{1}{4}$ -pincées, *Ann. Scuola Norm. Sup. Pisa (III)* **14**(2).
- [1960] Sur quelques variétés Riemanniennes suffisamment pincées, *Bull. Soc. Math. France* **88**.
- [1961] Les variétés Riemanniennes homogènes normales simplement connexes à courbure strictement positive, *Ann. Scuola Norm. Sup. Pisa (III)* **15**(3).
- [1961] Sur les variétés à courbure positive de diamètre minimum, *Comment. Math. Helv.* **35**(1).
- [1962] Les variétés Riemanniennes dont la courbure satisfait certaines conditions, *Proc. Intern. Congr. Mathematicians*.
- [1962] An extension of Rauch's metric comparison theorem and some applications, *Illinois J. Math.* **6**(4).
- [1962] On the characteristic of positively pinched Riemannian manifolds, *Proc. Natl. Acad. Sci.* **48**(11).

R. BISHOP and R. CRITTENDEN

- [1964] *Geometry of Manifolds* (Academic Press, New York).

S. BOCHNER and K. YANO

- [1953] Curvature and Betti Numbers, *Ann. Math. Stud. (Princeton Univ. Press, Princeton, N.J.)*.

A. BOREL

- [1963] Compact Clifford-Klein forms of symmetric spaces, *Topology* **2**(2).

Yu.D. BURAGO and V.A. TOPONOGOV

- [1973] On three-dimensional Riemannian manifolds whose curvature is bounded from above, *Mat. Zametki* **13** (in Russian).

J. CHEEGER

- [1967] Comparison and finiteness theorems for Riemannian manifolds, Thesis Princeton Univ., Princeton, N.J.
- [1969] Pinching theorems for a certain class of Riemannian manifolds, *Am. J. Math.* **91**(3).
- [1970] Finiteness theorems for Riemannian manifolds, *Am. J. Math.* **92**(1).
- [1973] Some examples of manifolds of nonnegative curvature, *J. Differential Geometry* **8**(9).

J. CHEEGER and D. GROMOLL

- [1968] The structure of complete manifolds of nonnegative curvature, *Bull. Ann. Math. Soc.* **74**(6).
- [1971] The splitting theorem for manifolds of nonnegative Ricci curvature, *J. Differential Geometry* **6**(1).
- [1972] On the structure of complete manifolds of nonnegative curvature, *Ann. Math.* **96**(3), 413-443.
- [1972] The injectivity radius of  $\frac{1}{4}$ -pinched Riemannian manifold, Reprint.

S. S. CHERN

- [1944] A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, *Ann. Math.* **45**(2).

C. CHEVALLEY

- [1946] Theory of Lie Groups (Princeton Univ. Press, Princeton, N.J.).

S. COHN-VOSSEN

- [1935] Kürzeste Wege Totalkrümmung auf Flächen, *Comp. Math.* **2**, 69-133.
- [1936] Totalkrümmung und geodätische Linien auf einfachzusammenhängenden offenen vollständigen Flächenstücken, *Rec. Math. Moscou* **43**, 139-163.

H. S. M. COXETER

- [1957] Non-Euclidean Geometry (Univ. of Toronto Press, Toronto).

R. CRITTENDEN

- [1962] Minimum and conjugate points in symmetric spaces, *Can. J. Math.* **14**, 320-328.

P. EBERLEIN

- [1971] Manifolds admitting no metric of constant negative curvature, *J. Differential Geometry* **5**(1).

D. EBIN and J. MARSDEN

- [1970] Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. Math.* **92**(1).

M. H. FREEDMAN

- [1982] The topology of four-dimensional manifolds, *J. Differential Geometry* **173**, 357-453.

D. GROMOLL

- [1966] Differenzierbare Strukturen und Metriken positiver Krümmung auf Spären, *Math. Ann.* **164**.

D. GROMOLL and W. MEYER

- [1969] On complete open manifolds of positive curvature, *Ann. Math.* **90**(1).
- [1974] An exotic sphere with non-negative sectional curvature, *Ann. Math.* **100**(1).

D. GROMOLL and J. WOLF

- [1971] Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature, *Bull. Am. Math. Soc.* **77**(4).

S. HELGASON

- [1962] Differential Geometry and Symmetric Spaces (Academic Press, New York).

N. HICKS

- [1965] Notes on Differential Geometry (Van, Nostrand, New York, 1965).

N. J. HITCHIN

- [1972] The space of harmonic spinors, Ph.D. Thesis, University of Oxford.

W. KLINGENBERG

- [1959] Contributions to Riemannian geometry in the large, *Ann. Math* **69**.  
[1961] Über Riemannische Mannigfaltigkeiten mit positiver Krümmung, *Comment. Math. Helv.* **35**.  
[1963] Manifolds with restricted conjugate locus, *Ann. Math.* **75**(3).  
[1969] Differentialgeometrie im Grossen, *Ber. Math. Forschungsinst. Oberwolfach*.

W. KLINGENBERG, D. GROMOLL and W. MEYER

- [1968] Riemannische Geometrie im Grossen (Springer, Berlin).

S. KOBAYASHI and K. NOMIZU

- [1963] Foundations of Differential Geometry, Vol. 1 (Interscience, New York).  
[1969] Foundations of Differential Geometry, Vol. 2 (Interscience, New York).

H. B. LAWSON and S. T. YAU

- [1972] On compact manifolds of nonpositive curvature, *J. Differential Geometry* **7**.

J. MILNOR

- [1963] Morse Theory, Ann. Math. Stud. (Princeton Univ. Press, Princeton, N.J.).  
[1968] A note on curvature and the fundamental group, *J. Differential Geometry* **2**.

M. MORSE

- [1934] The Calculus of Variations in the Large (Am. Math. Soc., Providence, R.I.; reprinted 1966).

S. MEYERS and N. STEENROD

- [1939] The group of isometries of a Riemannian manifold, *Ann. Math.* **40**.

K. NOMIZU

- [1956] Lie Groups and Differential Geometry (Math. Soc. Japan, Tokyo).

B. O'NEILL

- [1966] The fundamental equations of a submersion, *Michigan Math. J.* **13**.

R. S. PALAIS

- [1969] The Morse Lemma for Banach spaces, *Bull. Am. Math. Soc.* **75**(5).

G. PERELMAN

- [2002] The Entropy Formula for the Ricci Flow and its Geometric Applications, <http://arXiv.org/abs/math.DG/0211159>

- [2003] Ricci Flow with Surgery on Three-Manifolds,  
<http://arxiv.org/abs/math.DG/0303109>
- [2003] Finite Extinction Time for the Solutions to the Ricci Flow on Certain  
 Three-Manifolds, <http://arxiv.org/abs/math.DG/0307245>
- M. M. POSTNIKOV  
 [1965] The Variation Theory of Geodesics (Nauka, Moscow; in Russian).
- H. E. RAUCH  
 [1951] A contribution to Riemannian geometry in the large, *Ann. Math.* **54**.
- E. RUH  
 [1971] Curvature and differentiable structures on spheres, *Bull. Am. Math. Soc.* **77**(1).
- J. T. SCHWARTZ  
 [1964] Nonlinear functional analysis, Notes, New York Univ., New York.
- J. P. SERRE  
 [1951] Homologie singulière des espaces fibrés, *Ann. Math.* **54**.
- Y. SHIKATA  
 [1967] On the differentiable pinching problem, *Osaka J. Math.* **4**(2).
- S. SMALE  
 [1956] Generalized Poincaré conjecture in dimensions greater than 4, *Ann. Math.* **64**.
- E. SPANIER  
 [1966] Algebraic Topology (McGraw-Hill, New York).
- S. STERNBERG  
 [1964] Lectures of Differential Geometry (Prentice-Hall, Englewood Cliffs, N.J.).
- M. SUGIMOTO, K. SHIOHAMA and H. KARCHER  
 [1971] On the differentiable pinching problem, *Math. Ann.* **195**.
- V. A. TOPONOGOV  
 [1959] Riemann spaces with the curvature bounded below, *Uspehi Mat. Nauk* **14** (in Russian).  
 [1964] Spaces with straight lines, *Am. Math. Soc. Transl.* **37**.
- Y. TSUKAMOTO  
 [1961] On Riemannian manifolds with positive curvature, *Mem. Fac. Sci. Kyushu Univ. (A)* **15**(2).  
 [1963] A proof of Berger's theorem, *Mem. Fac. Sci. Kyushu Univ. (A)* **17**(2).  
 [1966] On certain Riemannian manifolds of positive curvature, *Tohoku Math. J. (2)* **18**(1).
- N. WALLACH  
 [1972] Three new examples of compact manifolds admitting Riemannian structures of positive curvature, *Bull. Am. Math. Soc.* **78**.  
 [1972] Homogeneous Riemannian manifolds of strictly positive curvature, *Ann. Math.* **75**.

## F. WARNER

- [1966] Extension of the Rauch comparison theorem to submanifolds, *Trans. Am. Math. Soc.*

## A. WEINSTEIN

- [1967] On the homotopy type of positively pinched manifolds. *Arch. Math.* **18**.  
[1969] Symplectic structures on Banach manifolds, *Bull. Am. Math. Soc.* **75**(5).

## J. H. C. WHITEHEAD

- [1932] Convex regions in the geometry of paths, *Quart. J. Math. Oxford* **3**.

## J. WOLF

- [1966] Spaces of Constant Curvature (McGraw-Hill, New York).





## Additional Bibliography

- [1] Marcel Berger. Systoles et applications selon Gromov. *Astérisque*, (216):Exp. No. 771, 5, 279–310, 1993. Séminaire Bourbaki, Vol. 1992/93.
- [2] Marcel Berger. *Riemannian geometry during the second half of the twentieth century*, volume 17 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2000. Reprint of the 1998 original.
- [3] Marcel Berger. *A panoramic view of Riemannian geometry*. Springer-Verlag, Berlin, 2003.
- [4] Yu. Burago, M. Gromov, and G. Perel'man. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.
- [5] Peter Buser and Hermann Karcher. *Gromov's almost flat manifolds*, volume 81 of *Astérisque*. Société Mathématique de France, Paris, 1981.
- [6] Leonard S. Charlap. *Bieberbach groups and flat manifolds*. Universitext. Springer-Verlag, New York, 1986.
- [7] J. Cheeger and K. Grove. *Metric and Comparison Geometry*. Surveys in Differential Geometry; XI. International Press, Summerville, 2007.
- [8] Jeff Cheeger. Critical points of distance functions and applications to geometry. In *Geometric topology: recent developments (Montecatini Terme, 1990)*, volume 1504 of *Lecture Notes in Math.*, pages 1–38. Springer, Berlin, 1991.
- [9] Jeff Cheeger. *Degeneration of Riemannian metrics under Ricci curvature bounds*. Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa, 2001.
- [10] Jeff Cheeger. Degeneration of Einstein metrics and metrics with special holonomy. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, Surv. Differ. Geom., VIII, pages 29–73. Int. Press, Somerville, MA, 2003.
- [11] Jeff Cheeger, Kenji Fukaya, and Mikhael Gromov. Nilpotent structures and invariant metrics on collapsed manifolds. *J. Amer. Math. Soc.*, 5(2):327–372, 1992.
- [12] Sylvestre Gallot. Volumes, courbure de Ricci et convergence des variétés (d'après T. H. Colding et Cheeger-Colding). *Astérisque*, (252):Exp. No. 835, 3, 7–32, 1998. Séminaire Bourbaki. Vol. 1997/98.
- [13] M. Gromov. Synthetic geometry in Riemannian manifolds. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 415–419, Helsinki, 1980. Acad. Sci. Fennica.
- [14] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
- [15] M. Gromov. Sign and geometric meaning of curvature. *Rend. Sem. Mat. Fis. Milano*, 61:9–123 (1994), 1991.
- [16] M. Gromov and P. Pansu. Rigidity of lattices: an introduction. In *Geometric topology: recent developments (Montecatini Terme, 1990)*, volume 1504 of *Lecture Notes in Math.*, pages 39–137. Springer, Berlin, 1991.
- [17] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.
- [18] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, english edition, 2007.

- Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [19] Karsten Grove. Critical point theory for distance functions. In *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 357–385. Amer. Math. Soc., Providence, RI, 1993.
  - [20] Karsten Grove. *Riemannian geometry: a metric entrance*, volume 65 of *Lecture Notes Series (Aarhus)*. University of Aarhus, Department of Mathematics, Aarhus, 1999.
  - [21] Karsten Grove. Finiteness theorems in Riemannian geometry. In *Explorations in complex and Riemannian geometry*, volume 332 of *Contemp. Math.*, pages 101–120. Amer. Math. Soc., Providence, RI, 2003.
  - [22] Karsten Grove. Aspects of comparison geometry. In *Perspectives in Riemannian geometry*, volume 40 of *CRM Proc. Lecture Notes*, pages 157–181. Amer. Math. Soc., Providence, RI, 2006.
  - [23] P. Pansu. Effondrement des variétés riemanniennes, d’après J. Cheeger et M. Gromov. *Astérisque*, (121-122):63–82, 1985. Seminar Bourbaki, Vol. 1983/84.
  - [24] Peter Petersen. Convergence theorems in Riemannian geometry. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 167–202. Cambridge Univ. Press, Cambridge, 1997.
  - [25] Peter Petersen. Rigidity and compactness of Einstein metrics. In *Surveys in differential geometry: essays on Einstein manifolds*, *Surv. Differ. Geom.*, VI, pages 221–234. Int. Press, Boston, MA, 1999.
  - [26] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

## Index

- $\delta$ -pinched, 105, 111
- $\hat{A}$  genus, 117
- $K(\pi, 1)$  space, 137
- L'Hôpital's rule, 27
- action
  - effective, 51
  - proper discontinuous, 132, 138–140, 142, 144
  - transitive, 51
  - uniform, 132, 138
- adjoint, 54
- adjoint representation, 50
- affine connection, 1
- antipodal, 110
- antipodal point, 14, 81, 94, 96
- arc length, 4
- Ascoli's Theorem, 87
- Berger, 93, 97
- Berger Example, 88
- bi-invariant, 51
- Bianchi identity
  - first, 11
- Bieberbach, 132
- Bieberbach group, 132, 144
- bilinear form
  - non-degenerate, 69
  - symmetric, 69
- Bochner, 93
- Bochner and Yano, 148
- Bonnet's Theorem, 23
- Calabi, 105
- Cartan-Ambrose-Hicks Theorem, 31, 32, 51, 118
- Cartan-Hadamard Theorem, 30
- Cayley Plane, 62, 102
- characteristic form, 85
- Cheeger, 83, 97, 118, 119, 131
- Chern, 117
- Cohn-Vossen, 129
- compact group, 59
- compactness, 10
- concave, 140, 141
- conjugate, 14
- conjugate point, 14, 19, 20
  - order, 14
- connection
  - torsion-free, 2
- constant curvature, 35
  - spaces of, 34
- convex, 123
- convex hull, 145
- covariant derivative, 1, 3
- covering space, 48
  - universal, 24, 146
- critical point, 69
  - degenerate, 72, 74, 77
  - index of, 69
  - nondegenerate, 69, 102
- critical value, 69, 71
- crystallographic group, 132, 137
- curvature
  - constant, 16
  - non-positive, 31
  - Ricci, 23
  - scalar
    - positive, 105, 117
  - sign, 24
- curvature tensor, 11
  - covariant derivative, 118
- cut locus, 81
  - in the tangent space, 82
- cut point, 81, 82
- CW-complex, 69, 72, 77
- cylinder, 42
- de Rham Decomposition Theorem, 138, 143, 147

- deformation retract, 73, 75
- deformation retraction, 76
- dexp, 18
- diameter, 23
- diffeomorphic, 105
- diffeomorphism, 49, 51, 109
  - local, 6, 19
- Differentiable Sphere Theorem, 105
- displacement function, 137
- distance, 3
- dual space, 70
- Duistermaat, 70
  
- embedding, 52
- energy function, 75
- Euclidean metric, 3, 13
- Euler Characteristic, 85
- exact homotopy sequence, 89
- exponential map, 6, 51
  
- fibration, 51
- finite kernel, 137
- first variation formula, 4
- First Index Lemma, 23
- focal point, 19, 20, 38, 79
- form
  - negative definite, 65
- Frobenius' Theorem, 47
- fundamental group, 24
  
- Gauss Lemma, 7, 87
- genus, 137
- geodesic, 5
  - broken, 75, 76
  - closed, 81, 83
    - smooth, 83
  - minimal, 10, 98, 101
  - non-broken, 76
  - normal, 5
- geodesic triangle, 35
- geodesic segment, 90, 110
- geodesic triangle, 37
- gotally geodesic, 66
- gradient, 7
- Grassmann manifold, 14
- Gromoll, 97, 105, 119, 131, 137
- group
  - abelian, 59
  - solvable, 137
  - torsion free, 145
- group action
  - isometric, 138
  
- half-space, 119–121
  - open, 126
  - supporting, 126
- Hausdorff space, 132
- Helgason, 102
- Hessian, 69
- hinge, 35, 37, 141
  - right, 100
  - right thin, 38
  - small, 37, 38
  - thin, 40
  - thin acute, 39
  - thin obtuse, 38
  - thin right, 38, 39
- Hitchin, 105
- Hodge Theory, 93, 117
- homeomorphic, 97
- homeomorphism, 96
- homogeneous space, 47
  - normal, 58
- homology sphere, 93
- homomorphism, 48, 49
- homotopy class
  - free, 86
  - nontrivial, 89
- homotopy equivalence, 69, 71, 119, 130
- homotopy sequence, 130
- Homotopy Sphere Theorem, 102
- homotopy type, 72, 77
- Hopf, 117
- Hopf conjecture, 117
- Hopf-Rinow Theorem, 9
- horizontal lift, 58
- hypersurface, 124
  
- ideal, 48
  - simple, 64
- imbedded, 99
- implicit function theorem, 6, 70, 71
- index
  - lower semicontinuous, 74
- index form, 17, 20, 76, 77
- Index Lemma, 79
- Index Theorem, 117
- induced bundle, 3
- induced connection, 3
- injectivity radius, 82, 93
- integral manifold
  - maximal connected, 48
- Intermediate-Value Theorem, 8, 95, 96
- invariant, 51
- inverse function theorem, 81
- involutive distribution, 48
- isometric, 93, 97

- isometric splitting, 138
- isometry, 2, 24, 51, 53
  - global, 131
  - local, 24, 32, 102, 131
- of metric spaces, 24
- orientation preserving, 85
- semi-simple, 137, 138, 142–144
- isotopic, 105
- isotropy, 53
- isotropy group, 51
  
- Jacobi equation, 12, 102
- Jacobi field, 11, 12, 15, 18, 20, 78, 79, 97, 101, 111, 113, 114
  - broken, 76
  - space of, 102
- Jacobi field equation, 111
- Jacobi identity, 11, 47, 51
  
- Karcher, 105
- Killing field, 148
- Killing form, 64, 88
- Klingenberg, 83, 85, 87, 93
- Kobayashi, 138
  
- law of cosines, 128
- Lawson, 137
- left invariant, 47
- length, 3
- lens spaces, 118
- Lichnerowicz, 105, 117
- Lie algebra, 47
  - finite dimensional, 48
  - homomorphism of, 48
  - infinite dimension, 47
  - semi-simple, 48
- Lie derivative, 47
- Lie group, 47
  - compact, 65
- Lie subgroup, 49, 51
- line, 131
- linear transformation
  - space of, 70
- lune, 99
  
- manifold, 1
- manifold of nonpositive curvature, 137
- maximal rank, 70
- metric
  - bi-invariant, 54, 55, 66, 88
  - G-invariant, 52
- metric space, 3
  - complete, 9
- Meyer, 119
- Meyer's Theorem, 85
- Milnor, 69, 119, 135, 137
- minimal geodesic, 124, 127
- minimum
  - weak, 100
- Morse, 69
- Morse Index Theorem, 20, 77, 78, 88
- Morse lemma, 70
- Morse Theory, 69, 102, 119
  - fundamental theorem, 77
- Myers' Theorem, 23
  
- Nomizu, 138
- normal bundle, 18, 119
- normal coordinate ball, 10, 31, 90
- normal coordinates, 6, 7
  
- O'Neill formula, 55
- orthogonal projection, 18
  
- paraboloid, 129
- paraboloid of revolution, 121
- parallel distribution, 147
- parallel field, 2, 38, 143
- parallel translate, 2
- piecewise smooth, 4
- Poincaré conjecture, 103
  - generalized, 103
- Pontryagin number, 85
- Postnikov, 79
- Preismann, 137, 145
- projective space, 102
  - complex, 62, 63
  - quaternionic, 62
- quadrilateral, 147
  - geodesic, 141, 143
  
- Rauch, 93
- Rauch I, 84
- Rauch Comparason Theorem, 20, 24, 93
- Rauch I, 25, 35, 38, 77, 84, 112, 115, 128, 142
- Rauch II, 25, 26, 38, 101, 116, 127
- Rauch Theorem
  - first, 25
  - second, 25, 29
- ray, 43, 119, 121
- rays
  - family of, 11
- real analytic, 47, 51
- regularity theorem, 123
- Ricci curvature, 59, 85
- Riemannian connection, 1, 3, 57

- Riemannian manifold
  - complete, 9
- Riemannian metric, 1, 19
- right invariant, 48
- rigidity phenomenon, 134
- rigidity theorem, 93, 137
- Ruh, 105
- Sard's Theorem, 77, 88
- Schwarz inequality, 75
- second fundamental form, 18
- second variation formula, 17
- second variation of arc length, 16
- sectional curvature, 13
  - non-negative, 55
  - non-positive, 30
- semi-simple, 64
- Shiohama, 105
- simply connected, 48, 61
- Singer, 105, 117
- singular value, 14, 19
- smooth variation, 4
- soul, 119
- space forms, 34
- Spanier, 130
- sphere
  - exotic, 105
  - topological, 103
- Sphere Theorem, 93, 99, 105
- spherical trigonometry, 110
- strongly convex, 89, 123
- subalgebra, 48
- subgroup
  - abelian, 137
  - closed, 51
  - finite normal, 133
  - free abelian, 132
  - infinite cyclic, 137
  - normal, 49
  - normal solvable, 145
  - one-parameter, 49
  - solvable of finite index, 145
- submanifold, 18
  - geodesic, 19
- submersion, 55
  - Riemannian, 55, 67
- subspaces
  - horizontal, 56
  - vertical, 56
- Sugimoto, 105, 108
- symmetric
  - affine, 61
  - globally, 61
  - locally, 61
- symmetric bilinear form, 77
- symmetric space, 61, 88, 97
  - dual, 65
  - positive curvature, 93
  - simply connected, 62
- symplectic group, 68
- Synge, 85
- tangent bundle, 1
- tangent cone, 125
- tangent space, 1
- tangent vector, 1
- Taylor expansion, 11
- Toponogov, 96, 131
- Toponogov's Theorem, 35, 85, 94, 100, 114, 116, 120, 127, 129, 141
- torsion, 2
- torsion subgroup, 143
- totally convex, 119, 127
- totally convex set, 119, 123, 138
  - proper, 119
- totally geodesic, 20, 67, 99, 119, 124
- totally geodesic rectangle, 128
- totally geodesic submanifold, 123, 141
- trace, 23
- transvection, 66
- triangle
  - small, 37–39
  - thin, 40–42
  - thin right, 41
- Tsukamoto, 95
- twisted sphere, 113
- twisted sphere, 105
- two-dimensional manifold, 137
- two-parameter variation, 16
- unit normal, 19
- unit tangent vector, 143
- unitary group, 63
  - special, 63
- variation field, 12
- vector field
  - right invariant, 48
- vector field, 1, 4
  - $\pi$ -related, 56
  - along a map, 3
  - left invariant, 48
  - piecewise smooth, 18
- vector group, 59
- vector space
  - finite-dimensional, 70

Weinstein, 43, 118  
Whitehead, 89  
Whitehead Theorem, 130  
Wolf, 137  
  
Yano, 93  
Yau, 137  
  
Zorn's Lemma, 142, 147











ISBN 978-0-8218-4417-5



9 780821 844175

**CHEL/365.H**

