Generalized Functions, Volume 3

Theory of Differential Equations

I. M. Gel´fand
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Translator's Note

According to the wish of Professor Gel'fand, this translation has been compared with the 1964 German translation,[1] and all improvements and omissions contained in the latter were taken over here. Certain minor corrections were made without being mentioned and a few notes were added (which are identified as translator's notes), especially in the last chapter, which is closely related to Chapter I in Volume 4 of this series.

No serious attempt has been made to coordinate the terminology with that used in previously published volumes, partly because the present translator does not entirely agree with it (e.g. the use of conjugate space for what is called here dual space, or function of bounded support, for what is more frequently called function of compact support). On the other hand, there are no radical departures from the notation and terminology of the authors—in particular, no attempt has been made to "modernize" it.[2]

The theory of partial differential equations, and of generalized eigenfunction expansions has made tremendous progress in the past few years. Not being a specialist in these fields the translator has made no attempt to update the literature on the subject (except for a few obvious references).

It was the express wish of Professor Gel'fand to refer the reader to the "excellent book of Hörmander" for some of the more recent developments.[3] This book is indeed the most valuable contribution to the literature on partial differential equations and should be read by any serious student of the subject.

Finally, I would like to thank Professor Gel'fand for supplying me with a copy of the German edition of this book and other literature which was useful in the translation.

October, 1967 MEINHARD E. MAYER

[1] "Verallgemeinerte Funktionen (Distributionen). Volume III—Einige Fragen zur Theorie der Differentialgleichungen." VEB-Deutscher Verlag der Wissenschaften, Berlin, 1964.

[2] As was done in the recently published French translation, *Math. Rev.* 1080 (1966), rev. Nr. 6001.

[3] L. Hörmander, "Linear Partial Differential Operators." Springer-Verlag, Berlin-Heidelberg-Göttingen and Academic Press, New York, 1963.

Preface to the Russian Edition

In the present volume, the third in the series "Generalized Functions," the apparatus of generalized functions is applied to the investigation of the following problems of the theory of partial differential equations: the problems of determining uniqueness and correctness classes for solutions of the Cauchy problem for systems with constant (or only time-dependent) coefficients and the problem of eigenfunction expansions for self-adjoint differential operators.

In subsequent volumes, the authors intend to discuss boundary value problems for elliptic equations and the Cauchy problem for equations with variable coefficients and for quasilinear equations, as well as problems related to complex extensions of all independent variables.

The authors use this occasion to thank the participants of the Seminar on Generalized Functions and Partial Differential Equations at Moscow State University, where various sections of this volume were repeatedly discussed. In particular, they are grateful to V. M. Borok, A. G. Kostyuchenko, Ya. I. Zhitomirskii and G. N. Zolotarev. The authors would also like to thank I. I. Shulishova for setting up detailed indexes for the first three volumes and to M. S. Agranovich, who has carefully edited the whole text and whose criticism has contributed considerable improvements.

Moscow, 1958

I. M. GEL'FAND
G. E. SHILOV

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NOTES AND REFERENCES

Chapter I

Spaces of type W have been introduced and analyzed by B. L. Gurevich [25]. These spaces not only represent generalizations of the spaces K_p, Z^p and $Z_p{}^p$ which had been introduced previously by I. M. Gel'fand and G. E. Shilov [22], but also lead to improvements in the definitions and proofs.

A slightly more general class of spaces, together with their duality theory is due to L. Hörmander [30] (see also Additional References [1]).

B. Ya. Levin's theorem on the existence of an entire function with given generalized growth indicatrix can be found in Ref. [39], Chapter 2.

Chapter II

Sections 1–5. Holmgren's method is treated in [28]. The first general theorems on uniqueness classes for Cauchy problems for equations of the evolution type,[1] with constant or t-dependent coefficients are due to I. G. Petrovskiĭ [45]. He has shown that the class of all functions which are continuous and bounded for $-\infty < x_j < \infty$ is a uniqueness class for each of these systems. This theorem has been extended by V. E. Lyantse [41] and L. Schwartz [50] to the class of all functions of power-law growth, Schwartz making use of the theory of distributions (generalized functions over S). The construction of the uniqueness classes consisting of exponentially increasing functions for arbitrary Petrovskiĭ-correct systems is due to the authors: in Ref. [22] it was derived by making use of the Fourier transforms of exponentially increasing functions and in Ref. [23] by means of the operator method, considering operators $\exp(tp(d/dx))$ in the spaces $S_\alpha{}^\beta$. The concept of reduced order of a system was first introduced in [22], in terms of the order of the resolvent matrix $e^{tP(s)}$.

[1] That is, systems of equations of the form

$$\frac{\partial u(x, t)}{\partial t} = \sum_{|q| \leqslant m} a_q(x, t)\, D_x{}^q u(x, t) + b(x, t).$$

The characterization of uniqueness classes given in this book is more complete than in Ref. [22] and [23] (the exponent $p_0' - \epsilon$ is replaced by p_0'). This improvement is due independently to K. I. Babenko [2], B. L. Gurevich [25] and S. D. Éĭdelman [14]. G. N. Zolotarev [56] is the author of the theorem that the functions of exponential order $\leqslant p_0' + \epsilon$, $\epsilon > 0$ arbitrary, do not form a uniqueness class for any system of the evolution type (for $p_0' > 1$).

The following authors have contributed to the problem of determining the uniqueness classes for the Cauchy problem: E. Holmgren [29] and A. N. Tikhonov (Tychonoff) [52] for the heat equation and related equations, O. A. Ladyzhenskaya [38] for general parabolic equations and S. D. Éĭdelman [12] for general parabolic systems. Necessary and sufficient conditions for functions satisfying an inequality of the type $|f(x)| \leqslant Ce^{\Phi(x)}$ to form a uniqueness class for Cauchy problems have been established by S. Täcklind [51] for the heat equation and by G. N. Zolotarev [55] for Petrovskiĭ-parabolic systems.

The original definition of a hyperbolic system and the existence of solutions for sufficiently smooth initial functions is due to I. G. Petrovskiĭ [45]. A slightly more general definition, permitting to formulate the converse theorem (i.e., that any system with solutions for sufficiently smooth initial data is hyperbolic) has been given by L. Gårding [16] (cf. in this connection, Section 3, Chapter III). Similar theorems have been subsequently proved by L. Schwartz [50] within the framework of distribution theory. Another approach to uniqueness theorems, on the basis of Laplace transforms in t (which seems to be applicable only to the case of one space variable) is due to E. Hille (cf. Ref. [27] and further references given there).

Section 6. The reduction of the problem of determining the growth of the function $e^{tP(s)}$ in the complex *s-plane* to the investigation of the growth of the real parts of the characteristic roots of the matrix $P(s)$, by making use of the inequality (6) is due to G. E. Shilov [48]. The estimation of the coefficients of the Newton interpolation formula by means of complex integration has been taken from the book [24] of A. O. Gel'fond. The formula for the computation of the reduced order is due to V. M. Borok [6]. She has shown in the same paper that every system with integral reduced order p_0 can be reduced to a system of the form $\partial u/\partial t = P_0(i\,\partial/\partial x)u$, where the differential operator P_0 is of order not higher than p_0 (for hyperbolic systems: 1). If the reduced order is rational $p_0 = p/q$, the system can be reduced to the form $\partial^n u/\partial t^n = P_0(i\,\partial/\partial x)u$ with $n \leqslant q$ and the order of P_0 not larger than p.

Section 7. The main result of this section is due to G. E. Shilov [49].

It is given here in an improved version ($h < 1$ is replaced by $h < (2p_0)'$), which was obtained by I. I. Shulishova in her Diploma Thesis (Master's Thesis).

The note at the end originated with Yu. A. Dement'ev.

The results of Appendix 1 are due to B. L. Gurevich.

The results contained in Appendix 2 have not been published so far and are due to A. G. Kostyuchenko and G. E. Shilov.

A. G. Kostyuchenko [34] has also obtained the results presented in Appendix 3. These results are given here in a more complete form. The theorem of Éĭdelman on the solution of the Cauchy problem for systems with elliptic operators is contained in Ref. [13].

Chapter III

The first general theorems on uniqueness classes for the Cauchy problem for systems of evolution type with constant or t-dependent coefficients have been found by I. G. Petrovskiĭ [45]. He has shown that "condition A" (cf. p. 107; this condition is our condition of Petrovskiĭ-correctness) is necessary and sufficient in order that the class of functions which are bounded together with their derivatives up to a certain order for $-\infty < x_j < \infty$, form a uniqueness class for the Cauchy problem for systems of the form $\partial u/\partial t = P(i\,/\partial x)u$. The following authors have indicated correctness classes consisting of functions of exponential type: Holmgren [29] and Tikhonov [52] for the heat equation, Täcklind [51] for the equation $\partial u/\partial t = \partial^{2p}u/\partial x^{2p}$, Ladyzhenskaya [38] for general parabolic equations, and Éĭdelman [12] for Petrovskiĭ-parabolic systems. The general construction of correctness classes for arbitrary systems of evolution type have been carried out by Shilov [48]. The presentation in this volume is the first systematic exposition of the subject.

Section 2. Petrovskiĭ-parabolic systems were first introduced in Ref. [45]. For the general definition cf. the paper by Shilov [48].

Characteristics for systems with one space variable have been computed by V. M. Borok [7]. Petrovskiĭ-parabolic systems with coefficients depending on the space variables have been investigated by Éĭdelman [13].

Since the fundamental solution of a parabolic system is an infinitely differentiable function of x, each solution belonging to a uniqueness class has the same property, although the initial function need only be locally integrable. V. M. Borok [5] has shown that only parabolic systems have this property.

Section 3. Systems which are Petrovskiĭ-hyperbolic were investigated in Ref. [45] and by Gårding [16].

Section 4. What we call Petrovskiĭ-correct systems, Petrovskiĭ himself called systems with "condition A." S. A. Galpern [15] has introduced a class of systems which occupy within the class of Petrovskiĭ-correct system the same place, as the Petrovskiĭ-hyperbolic equations occupy among the general hyperbolic systems. Systems of positive genus p_0 have been introduced by the authors [22] under the name "regular systems." Subsequently Kostyuchenko and Shilov [35] have proved the theorem that each such system has a solution within the class of functions of order $\exp \epsilon|x|^{p_0'}$ ($\epsilon > 0$ arbitrary). This proof is the basis of the general existence theorems established in this section. In Ref. [14] Éĭdelman has shown that certain systems in physics and mechanics belong to the class of regular systems (e.g., the equation describing sound propagation in a viscous gas, given in this section). Formulas for the computation of characteristics for systems with one space variable have been given by V. M. Borok [7]. Theorem 4 is due to Kostyuchenko and Shilov and has not previously been published.

For systems which we called "conditionally correct," V. E. Lyantse [41] has indicated a uniqueness class consisting of infinitely differentiable functions with exponential growth.

F. John [31] has arrived at the class of conditionally correct systems by different considerations. He has described those systems which admit at least one solution with a nonvanishing initial function of compact support.

The results in subsection 4.3 (correctness in the class of analytic functions) are due to Kostyuchenko [48]. This is the first detailed account of the subject. L. Ehrenpreis [11] has also investigated the solvability of systems which have entire functions as initial data.

Chapter IV

The history of the problem of eigenfunction expansions has been sketched in the preamble of this chapter. The reduction of quadratic integral forms to canonical form and the proof of the completeness of the eigenfunctions of a regular Sturm-Liouville problem (Steklov's theorem) are due to D. Hilbert [26]. The completeness of the eigenfunctions for compact operators was first proved by F. Riesz [47]. The spectral resolution of unbounded self-adjoint operators in a Hilbert space is due to J. von Neumann [43]. The problem of extension of symmetric operators, in particular, semibounded operators (theorems of von Neumann, Friedrichs, Riesz) is treated in [36] (cf. also the repeatedly quoted book by Riesz and Nagy). Eigenfunction expansions for an ordinary differential operator on a semiaxis are originally due to H. Weyl [54] (cf. also the more recent investigations of E. C. Titchmarsh [52] and B. M. Levitan [40]). For differential operators of nth order this problem has been investigated by M. G. Kreĭn [37] and K. Kodaira [32]. A. Ya. Povzner [46] has treated the problem of eigenfunction expansions for operators of the form $-\Delta u + qu$, defined in the whole space. F. I. Mautner [42] has proved an expansion theorem for general self-adjoint operators for which the resolvent is an integral operator with kernel of Carleman type. F. Browder [87] and L. Gårding [17,18] have shown that all elliptic differential operators belong to this class, consequently the expansion theorem is valid for any elliptic operator.

Section 2. Differentiation of functionals of strongly bounded variation has been considered independently first by I. M. Gel'fand [19] and then by N. Dunford [10] and B. J. Pettis [44].

Sections 3–7. The results of these sections are due to Gel'fand and Kostyuchenko [21]. F. Browder [9] has extended the fundamental theorem to the case of maximal symmetric operators (cf. in this connection the paper [33] of Kostyuchenko, where the theorems on the structure of generalized eigenfunctions have been carried over to this case). After Ref. [21] appeared, Yu. M. Berezanskiĭ [3] proposed a different way of obtaining the eigenfunction expansion in the space $L_2(R_n)$. In a communication at the 3rd Soviet-Union Mathematical Congress (June, 1956), Gårding reported that he has established a theorem on generalized eigenfunction expansions for selfadjoint differential operators in $L_2(R_n)$. Subsequently Berezanskiĭ showed that the primitive functions of the generalized eigenfunctions (in R_n) do not increase faster than $|x|^{(5/2)n+1+\epsilon}$. Kostyuchenko [33] has improved this result, replacing $|x|^{(5/2)n+1+\epsilon}$ by $x^{n/2}$.

In another paper [4], Berezanskiĭ has extended to eigenfunction expansions Bochner's theorem on the representation of functions of positive type.

The Sobolev problem has been investigated by R. A. Alexandryan [1].

The results of Section 6 are due to Kostyuchenko [33], those in Section 7 are due to Gel'fand and Kostyuchenko [21]. The theorem of Gel'fand and Fomin on dynamical systems on manifolds of constant negative curvature can be found in Ref. [20].

Translator's Note

No systematic attempt has been made to include the literature which has appeared since the original manuscript was completed (1957). A few newer books which are pertinent have been included as "Additional References." For further references the reader is directed to survey articles and the Mathematical Reviews for the past nine years.

As was already remarked, the last chapter is related to Chapter I in volume 4 (cf. the notes and references to that Chapter, especially Section 4, for additional bibliography).

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