

Philip M. Anselone and John W. Lee

## The Heart of Calculus

## Explorations and Applications

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# Explorations and Applications 

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## Preface

This book is intended primarily as enrichment material for courses in first and second year calculus. It also has material suitable for use in differential equations and introductory real analysis. We endeavor to offer an engaging combination of topics at a challenging, yet accessible level. The book targets talented and well-motivated students and can be used in a variety of settings, such as honors courses, undergraduate seminars, independent study, capstone courses taking a fresh look at calculus, and summer enrichment programs.

Our primary goal in writing the book is to impart a deeper understanding of and facility with the mathematical reasoning that lies at the heart of calculus and to convey something of its beauty and depth.

There are sixteen chapters in the book, divided about equally between pure and applied mathematics. The first three chapters are on fundamentals of differential calculus. The last three are on the monumental discoveries of Newton and Kepler on celestial motion and gravitation. The intervening chapters present significant topics in pure and applied mathematics chosen for their intrinsic interest, historical influence, and continuing importance. The book develops topics from novel or unifying perspectives. Thus, we hope and expect that it will be a valuable resource for graduate teaching assistants as they develop their academic and pedagogical skills and be informative to seasoned veterans who appreciate fresh perspectives.

The chapters of the book are written to be largely independent of each other. There are occasional references to results from earlier chapters but adjustments appropriate to the situation at hand are easily made.

The book had its origin in a 1-2 hour per week undergraduate honors seminar at Oregon State University. The students in the seminar were enrolled in the department's regular 4-credit science and engineering calculus sequence. Seminar students took turns presenting various calculus topics or applications to their colleagues for critical discussion. Those not presenting material turned in polished written reports. This format motivated the oral or written projects that accompany each chapter of the book. By all accounts, the seminar was very successful.

Students in the honors seminar were mostly mathematics, physics, and engineering majors. The topics in the seminar and in the book reflect that.

The genesis of the book suggests the students for whom it is written: well-motivated, talented students who desire to understand both calculus and its applications at a deeper level.

Beginning calculus courses assume implicitly or explicitly that the rational numbers are dense in the reals and that the real number system is complete. They also use the max-min theorem, the intermediate value theorem, and the existence of least upper bounds and greatest
lower bounds without proof. We abide by these norms. These underpinnings of calculus are displayed just before Chapter 1 for ease of reference and to reinforce their importance.

The enrichment materials assume some prior exposure to calculus. The students should have completed the better part of a standard "plug and chug" introduction to calculus. However, suitably delivered parts of the first three chapters have been used in the first term of a standard calculus sequence.

Logical reasoning skills at the level of proof in Euclidean geometry are all that is needed for productive use of these materials. Nevertheless, a high level of rigor is maintained. The book is logically complete in the sense that proofs of theorems and other conclusions are given at least in summary form with easier steps left to the reader. With few exceptions, proofs depend only on tools from calculus and earlier mathematics. Analytical arguments are carefully structured to avoid epsilons and deltas. This keeps the presentation friendly and accessible to students at several levels of mathematical maturity.

Virtually all the chapters are natural complements to a first course on real analysis. Students can see for themselves how the results they are learning in real analysis arose out of the need and desire to set calculus on a firm foundation.

Mathematical modeling is an integral part of many chapters. The mathematical and physical assumptions on which a model stands are clearly stated, expected attributes of a model are confirmed mathematically, and its predictions are carefully examined. The enrichment topics for these chapters are natural supplements for a course in differential equations or a modeling course.

The first three chapters of the book, on fundamentals of differential calculus, play a unique role. On the one hand, they present some fundamental topics in novel and more unified ways than in standard calculus textbooks. On the other hand, key reasoning skills are developed and the refinements and techniques introduced play a role in developing the theory in later chapters, especially the applied chapters. We encourage students primarily interested in the applications chapters not to overlook the first three chapters.

The reader may wonder why we refer to integral tables or use techniques of integration in the age of symbolic integrators. The reason is simple. They are still useful. In fact, one of the most effective uses of a change of variable in an integral is to transform the integral to an alternative form that is more informative for theoretical purposes or more suitable for numerical purposes. A case in point is the transformation of an elliptic integral in Chapter 13.

The next to last section of each chapter is Problems and Remarks. The problems are invitations to readers to complete arguments merely sketched in the main text, to extend particular results in the chapter, or just to establish interesting related results. The remarks vary from chapter to chapter. They typically contain historical background and comments that extend or enrich the main results of the chapter.

Every chapter ends with a section titled Further Reading and Projects. There is a short list of references that are germane to the chapter. Each project is presented in an outline form that should help a student prepare a complete oral or written presentation. The projects vary quite a lot. Some may approach the principal results of the chapter from a different perspective. Others may lead the student in a fresh but related direction.

A student could read this book on his or her own. However, we believe the mathematical experience will be much better if supported by a faculty mentor or, even better, takes place in a seminar setting.

## Chapter Highlights

An overview of the sixteen chapters follows.

Chapter 1 Critical Points and Graphing Critical points of functions defined on intervals play central roles in calculus, especially in the solution of max-min problems. Less well known is the fact that, for virtually all functions defined on intervals that come up in calculus, the intervals where a function $f$ increases or decreases are determined by the behavior of $f$ at its critical points and at the endpoints of these intervals. Similarly, the intervals where $f$ is concave up or down are determined by the behavior of $f^{\prime}$ at critical points of $f^{\prime}$ and at endpoints of intervals of its domain. Moreover, the critical points determine the sign patterns of $f^{\prime}$ and $f^{\prime \prime}$ by inspection, without the need for any work with inequalities. This is the easiest way to determine the sign patterns, especially for transcendental functions. A useful by-product of this point of view is a refinement of standard curve sketching techniques, which we call the critical point graphing method. The refinement is simpler to use, requires less work, and is especially advantageous when applied to transcendental functions.

Chapter 2 Inverse Functions The treatment of inverse functions in calculus can be considerably simplified and unified compared to the now standard approach to these functions. The standard approach defines inverse functions in the general context of one-to-one maps of abstract sets. The important inverse functions of calculus are then treated case-by-case in a way that often implicitly assumes continuity of the inverse function as well as differentiability, when derivative formulas are established.

We present a unified approach to the important inverse functions of calculus. It shows that inverse functions inherit, free of charge, monotonicity, continuity, and differentiability of the primary function.

The key to the approach is to recognize that virtually all one-to-one functions (those functions that have inverses) that arise naturally in calculus are continuous and have interval domains. We prove by elementary means the geometrically compelling result that all continuous one-to-one functions on interval domains are monotonic (increasing or decreasing). Therefore, the one-to-one functions of interest in calculus are monotonic, continuous, and have interval domains. It turns out that restricting the discussion of inverse functions to such functions is a major simplification. It means that many significant conclusions about the relationship between a function and its inverse have obvious graphical interpretations that lead to natural proofs and the unified treatment we give.

Chapter 3 Exponential and Logarithmic Functions Exponential functions $a^{x}$ for $a>1$ and for $0<a<1$ play an essential role throughout calculus as do their inverse functions, the logarithmic functions. Although it is natural to introduce them in beginning differential calculus, a fully satisfactory approach that gets these functions in the game early and establishes their differentiability in a mathematically satisfying way has proved challenging. We present an approach that is an attractive alternative to deferring essential features of exponentials and logarithms to the integral calculus.

We present a direct proof that exponential functions are differentiable. It is based entirely on tools from differential calculus or earlier mathematics. The proof is motivated by familiar geometric properties of graphs of exponential functions and involves squeezing arguments
similar to those used to show that the trigonometric functions are differentiable. This makes it possible to give a comprehensive treatment of the derivative properties of exponential and logarithmic functions within the scope of differential calculus.

Chapter 4 Linear Approximation and Newton's Method From its inception, calculus provided effective and systematic means of approximation. Initially the methods were used to accurately evaluate exponential, logarithmic, trigonometric, and root functions as well as to determine roots of equations. The first steps in these developments were linear approximation (tangent line approximation) and Newton's (root-finding) method. To be useful, approximation schemes must include effective means of estimating the error and must provide a strategy that enables any desired degree of precision to be achieved. This chapter addresses these issues for linear approximation and Newton's method. The analysis is motivated by convincing geometric reasoning and made precise by arguments based on monotonicity and concavity. In particular, the treatment of Newton's method establishes easily identified initial guesses for which the Newton approximations will converge monotonically to a particular root for virtually any function of a single variable likely to come up in applications.

Chapter 5 Taylor Polynomial Approximation Taylor polynomial approximation is the natural extension of linear approximation. In linear approximation, the value of a function and its derivative are matched at a base point. Higher order Taylor polynomials simply extend this idea by matching successively higher order derivatives at the base point. Our treatment of the error in Taylor polynomial approximation is novel and builds on the treatment in Chapter 4 of the error in linear approximation. Geometric ideas based on monotonicity remain front and center. The key to error estimates in linear approximation is that the linear approximation is squeezed between two natural quadratic approximations. This approach extends directly to higher order Taylor polynomials. We show that all the standard error results for Taylor polynomial approximation follow from corresponding squeezing results. All the results come from repeated application of an evident fact: If $f^{\prime} \geq 0$ on an interval $I$, then $f$ is nondecreasing on $I$.

Chapter 6 Global Extreme Values In principle, the standard critical point method for solving max-min problems for functions of one variable extends to functions of two or more variables. However, in practice it is typically more challenging to carry out the procedure. The primary challenges come from the fact that the boundary of the domain of interest often contains infinitely many points and from the difficulty of showing that extreme values exist, especially for domains that are not closed or are unbounded.

In this chapter, we present a two-step max-min method that is easier to carry out, when applicable. The method is a special instance of a general principle that taking advantage of certain symmetries in a problem can greatly simplify its solution. Here the symmetry involves the expression for the function as it relates to its domain. The domain is represented as a family of lines or curves, usually tailored to the form of the objective function $f$ or the shape of its domain $D$. Extreme values are calculated first for $f$ on the lines or curves and then for $f$ on $D$. Thus, a two-dimensional problem is reduced to two one-dimensional problems. In this approach, the existence and values of extrema are established at the same time.

Chapter 7 Angular Velocity and Curvature The relationship between angular velocity and curvature is quite informative, but it is usually not mentioned in standard treatments of calculus.

The connection is informative both from a geometric perspective and from the point of view of connections with rotational motion. We present such connections. Curvature is typically introduced as an angular rate of change for plane curves and as the rate of change of the magnitude of the unit tangent vector for space curves. The approaches are not reconciled. We present a unified approach to curvature for plane and space curves based on the rate of change of a natural angle. The standard formulas for the curvature of plane and space curves are obtained along the way by simple geometric arguments.

Chapter $8 \pi$ and $e$ are Irrational Intriguing properties of numbers of special significance to mathematics, such as $\pi$ and $e$, have attracted mathematicians throughout the ages. In this chapter we use lines of reasoning introduced by the distinguished American number theorist Ivan Niven to show that $\pi$ and $e$ are irrational, and give extensions of his results. Niven's arguments use only ideas of beginning calculus, particularly repeated applications of integrations by parts. They greatly simplify related ideas used by Hermite to prove that $e$ is transcendental.

Chapter 9 Hanging Cables This is the first of eight chapters devoted to applications of mathematics to the world around us. The study of hanging cables (whether transmission lines or clotheslines) is a natural starting point. Everyday experience and simple experiments give a general idea of what to expect, but more insight can be gained by developing a mathematical model.

What is the shape of a hanging cable? That question came up early and the answer given by Galileo, a parabola, proved to be incorrect. How is tension in a cable related to its sag? A qualitative answer is clear. What is the precise relationship? How do cable properties and economic considerations interact? A mathematical model yields quantitative answers to such questions. The model is constructed by determining the forces acting on a segment of the cable. Tensions along the cable act tangentially and the force of gravity acts vertically. Since the cable is not in motion (it is not a cable on the Tacoma Narrows bridge!) the net force must be zero. This leads to a differential equation that can be solved by two integrations. Questions such as those posed above are discussed.

Chapter 10 The Buffon Needle Problem Several path-breaking developments in mathematics and its applications had their origins in seemingly frivolous problems. The Buffon needle problem is an example: Toss a needle on a hardwood floor. What is the probability that the needle touches one of the parallel lines between the wooden strips? An experimental determination of the probability was made in 1850 by Rudolf Wolf who tossed a needle over his shoulder 5,000 times and counted the number of crossings. The experiment can be regarded as a precursor of modern Monte Carlo methods that have far reaching applications, including the absorption of neutrons in a nuclear reactor. The original Buffon needle problem leads to a surprising connection to $\pi$. It also leads to a geometric view of probabilities in terms of areas. Laplace considered a natural extension of Buffon's problem for which the solution uses geometric probability and multiple integrals.

Chapter 11 Optimal Location Optimal location problems typically occur in large-scale industrial and public works projects where a site for a factory, power plant, or whatever needs to be determined to minimize certain costs. In this chapter we consider the problem of the optimal location of a power plant being built to serve three factories. Cost factors, such as the construction
of power lines and electrical leakage along them, are modeled as proportional to the distance of each factory from the site of the power plant. Analysis of the mathematical model that results involves both simple geometric arguments and critical point analysis. A unique optimal location is shown to exist. When standard critical point analysis is used to find the optimal location what emerges is an elegant geometric condition that characterizes the location but gives no explicit determination for it. Numerical methods are needed to approximate the optimal location. Methods based on successive approximations are discussed and numerical examples are given.

Chapter 12 Energy Work is a staple of one-variable integral calculus but energy is almost always deferred until multivariable calculus. This is unfortunate because the two topics are closely related and several topics that can enrich integral calculus are lost in the process.

This chapter is concerned with the energy of an object moving under the action of a external force, particularly the force of gravity. After a brief introduction to kinetic and potential energy, the principle of conservation of energy is derived by elementary means. It is illustrated first by the familiar example of a ball thrown up into the air in the presence of the presumed constant force of gravity. The kinetic energy of the ball is transformed into potential energy on the upward flight and back into kinetic energy on the downward flight. Next, a projectile launched vertically from the Earth is acted on by the variable force of gravity given by Newton's law of gravitation. Comparisons of kinetic and potential energy yield the escape velocity that is the minimum launch velocity for which the projectile flies off into space, never to return to the Earth.

Chapter 13 Springs and Pendulums An understanding of oscillatory motion is central to the development of accurate clocks, the design of machinery, and the properties of atoms. For these reasons as well as for its historical significance, we discuss simple harmonic motion and pendulum motion. The presentation of simple harmonic motion serves as a gentle introduction to the use of linear differential equations to study motion problems. Uniqueness of the solution to an associated initial value problem is treated in an $a d$ hoc manner and then related to conservation of energy. The pendulum serves as an introduction to motion governed by a nonlinear differential equation. A first integral of the second order pendulum equation is found, essentially by conservation of energy methods. The subsequent discussion focuses on the period of the pendulum and leads to an elliptic integral for the period. We approximate the period both by a small angle approximation and by the binomial series. Computable error bounds based on the geometric series are given.

Chapter 14 Kepler's Laws of Planetary Motion After decades of observations and calculations, the eminent German astronomer Johannes Kepler discovered the three laws of planetary motion that bear his name. A half century later, Isaac Newton built on Kepler's laws to discover the law of universal gravitation and to extend Kepler's laws beyond the planets to other heavenly bodies and even to artificial satellites, as Newton himself realized and explicitly mentioned.

These great discoveries are presented in Chapters 15 and 16. Chapter 14 sets the stage with a statement of Kepler's laws and a discussion of some implications and equivalent forms of the laws, expressed in rectangular and polar coordinates, and in vector forms. In particular, the connection between Kepler's second law and central forces makes a natural appearance here.

Chapter 15 Newton's Law of Universal Gravitation This chapter presents what is essentially Newton's original reasoning that Kepler's three laws of planetary motion imply that the gravitational force of the sun on its planets is a central, attractive, inverse square force and that there is a gravitational constant for the solar system that is determined by the sun and is independent of the planets. Newton observed that the moons of Jupiter satisfy Kepler's laws and, hence, that the gravitation force of Jupiter on its moons also is a central, attractive, inverse square force. He used data about the Earth and its moon and Galileo's measurements of the gravitational acceleration near the surface of the Earth to confirm that Galileo's measurements and the inverse square law were consistent. Newton combined the experimental evidence with his third law to deduce his now famous law of universal gravitation.

Chapter 16 From Newton To Kepler and Beyond This final chapter of the book closes the circle of ideas explored in Chapters 14 and 15 . Once again we follow Newton's reasoning, recast in modern notation. We derive Kepler's laws of planetary motion from Newton's law of universal gravitation. The derivation explains Kepler's original laws and extends their applicability to other heavenly bodies and artificial satellites. Now trajectories of heavenly bodies are not restricted to ellipses but can be any conic section. Which conic section depends on the velocity and position of the body at any instant of time and can be expressed in terms of its total kinetic and gravitational energy.

## Acknowledgements

This book owes a great debt to our students, especially the students in the honors seminars mentioned earlier. Their enthusiasm and desire for a deeper understanding of calculus were great motivators for us. Likewise, the book benefited enormously from discussions with our colleagues about how and what to teach in calculus, most especially how to give students an appreciation for the intrinsic beauty, depth, and scope of calculus.

We extend our sincere thanks and appreciation to the anonymous reviewers who read the original manuscript and to the MAA editors, Jerry Bryce, Susan Staples, and Steve Kennedy, who coordinated the review process. The reviewers' and editors' indepth analysis of the manuscript and the advice they gave us resulted in a significantly better final product. Finally, a special thank you is due to Beverly Joy Ruedi, MAA Electronic Production and Publishing Manager. She was invariably helpful and quick to respond with advice and solutions whenever we needed help coordinating our electronic documents with MAA publication requirements.

The preface closes on a sad note. Phil Anselone died while the original manuscript submitted to the MAA was under review. He lived a long and productive life of almost 88 years. Phil was a fine mathematician, good friend, and colleague. He and I enjoyed working together and hiking together for more than 40 years. Phil enjoyed collaboration and it was my good fortune to be one of his co-authors. Phil had a special spot in his heart for clear, concise, and accurate mathematical writing. He was convinced that the material in this book should be communicated to the mathematical community, that there was a place for it, and that the best place for it would be among MAA publications. Phil would have been delighted to have learned that the MAA agreed! I miss Phil and I missed his presence, enthusiasm, and mathematical insights as I made the final revisions to the manuscript.

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This book contains enrichment material for courses in first and second year calculus, differential equations, modeling, and introductory real analysis. It targets talented students who seek a deeper understanding of calculus and its applications. The book can be used in honors courses, undergraduate seminars, independent study, capstone courses taking a fresh look at calculus, and summer enrichment programs.

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