American Mathematical Society
Colloquium Publications
Volume 25

Lattice Theory

Garrett Birkhoff

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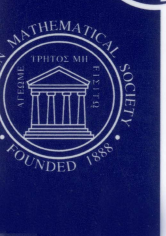

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http://dx.doi.org/10.1090/coll/025

American Mathematical Society

Colloquium Publications
Volume 25

Lattice Theory

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American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.* Primary 06–02.

Library of Congress Card Number 66-23707
International Standard Book Number 0-8218-1025-1
International Standard Serial Number 0065-9258

Second (Revised) Edition, 1948
Third Edition, seventh printing with corrections 1993.
Third Edition, eighth printing with corrections 1995.
Printed in the United States of America.

12 11 16 15 14 13 12 11

PREFACE TO THE THIRD EDITION

The purpose of this edition is threefold: to make the deeper ideas of lattice theory accessible to mathematicians generally, to portray its structure, and to indicate some of its most interesting applications. As in previous editions, an attempt is made to include current developments, including various unpublished ideas of my own; however, unlike previous editions, this edition contains only a very incomplete bibliography.

I am summarizing elsewhere† my ideas about the role played by lattice theory in mathematics generally. I shall therefore discuss below mainly its logical structure, which I have attempted to reflect in my table of contents.

The beauty of lattice theory derives in part from the extreme simplicity of its basic concepts: (partial) ordering, least upper and greatest lower bounds. In this respect, it closely resembles group theory. These ideas are developed in Chapters I–V below, where it is shown that their apparent simplicity conceals many subtle variations including for example, the properties of modularity, semimodularity, pseudo-complements and orthocomplements.

At this level, lattice-theoretic concepts pervade the whole of modern *algebra*, though many textbooks on algebra fail to make this apparent. Thus lattices and groups provide two of the most basic tools of " universal algebra ", and in particular the structure of algebraic systems is usually most clearly revealed through the analysis of appropriate lattices. Chapters VI and VII try to develop these remarks, and to include enough technical applications to the theory of groups and loops with operators to make them convincing.

A different aspect of lattice theory concerns the foundations of set theory (including general topology) and real analysis. Here the use of various (partial) orderings to justify transfinite inductions and other limiting processes involves some of the most sophisticated constructions of all mathematics, some of which are even questionable! Chapters VIII–XII describe these processes from a lattice-theoretic standpoint.

Finally, many of the deepest and most interesting applications of lattice theory concern (partially) ordered mathematical structures having also a binary addition or multiplication: lattice-ordered groups, monoids, vector spaces, rings, and fields (like the real field). Chapters XIII–XVII describe the properties of such systems,

† G. Birkhoff, *What can lattices do for you?*, an article in *Trends in Lattice Theory*, James C. Abbot, ed., Van Nostrand, Princeton, N.J., 1967.

and also those of positive linear operators on partially ordered vector spaces. The theory of such systems, indeed, constitutes the most rapidly developing part of lattice theory at the present time.

The labor of writing this book has been enormous, even though I have made no attempt at completeness. I wish to express my deep appreciation to those many colleagues and students who have criticized parts of my manuscript in various stages of preparation. In particular, I owe a very real debt to the following: Kirby Baker, Orrin Frink, George Grätzer, C. Grandjot, Alfred Hales, Paul Halmos, Samuel H. Holland, M. F. Janowitz, Roger Lyndon, Donald MacLaren, Richard S. Pierce, George Raney, Arlan Ramsay, Gian-Carlo Rota, Walter Taylor, and Alan G. Waterman.

My thanks are also due to the National Science Foundation for partial support of research in this area and of the preparation of a preliminary edition of notes, and to the Argonne National Laboratory and the Rand Corporation for support of research into aspects of lattice theory of interest to members of their staffs.

Finally, I wish to thank Laura Schlesinger and Lorraine Doherty for their skillful typing of the entire manuscript.

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Since the third edition **[LT3]** of *Lattice Theory* was published in 1967, containing two chapters on the role of lattices in universal algebra, there has been enormous progress in both areas. The journal *Algebra Universalis*, founded in 1971 by George Grätzer, has maintained a high quality of contributions to them ever since. Since 1984, a second journal *Order*, founded by Ivan Rival shortly after the publication of the volume **[Riv]** which he edited, has provided a second, more applications oriented vehicle of publication with combinatorial emphasis.

In addition to these journals, many books have been published explaining new developments concerned with the theories of ordered sets and lattices, and their many applications. About fifteen of the most important of these books are listed below. They supplement the many footnotes of **[LT3]**, which were intended to document major developments prior to around 1965 in 400 pages.

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E. J. Wilczyński, Projective differential geometry, 1910

1 **H. S. White,** Linear systems of curves on algebraic surfaces

F. S. Woods, Forms on noneuclidean space

E. B. Van Vleck, Selected topics in the theory of divergent series and of continued fractions, 1905

Since its original publication in 1940, this book has been revised and modernized several times, most notably in 1948 (second edition) and in 1967 (third edition). The material is organized into four main parts: general notions and concepts of lattice theory (Chapters I–V), universal algebra (Chapters VI–VII), applications of lattice theory to various areas of mathematics (Chapters VIII–XII), and mathematical structures that can be developed using lattices (Chapters XIII–XVII). At the end of the book there is a list of 166 unsolved problems in lattice theory, many of which still remain open.

It is excellent reading, and ... the best place to start when one wishes to explore some portion of lattice theory or to appreciate the general flavor of the field.

—Bulletin of the AMS

ISBN 978-0-8218-1025-5

COLL/25