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Gap and Density Theorems

Norman Levinson



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Dedicated to

NORBERT WIENER

*by one of the many men whose
careers he launched*

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PREFACE

The present volume is in many respects a companion volume to the earlier Colloquium Publication of Paley and Wiener. The same tools are used in both. (See Appendix.) Certain topics such as the closure of trigonometric sequences and some problems of Pólya are treated in both. The technique of the Fourier transform in the complex domain developed in Paley and Wiener has, in the present volume, been further extended in the solution of such problems as the general unrestricted gap Tauberian theorem.

The contents break up into four parts. Large portions of the first two parts have appeared in various journals during the past four years. The last two parts consist almost entirely of new material. The first part consists of Chapters I, II, III, and IV; the second part of Chapters V, VI, VII; the third part of Chapters VIII, and IX; and the fourth part of Chapters X, XI, and XII.

The present volume covers the various topics considered in some detail, most results being "best possible." Nevertheless for many of the topics treated there are several directions for further work in refining or developing these topics.

I am very much indebted to Professor J. D. Tamarkin for the painstaking manner in which he has examined and criticized this book.

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June 24, 1939

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A typical gap theorem of the type discussed in the book deals with a set of exponential functions $\{e^{i\lambda_n x}\}$ on an interval of the real line and explores the conditions under which this set generates the entire L_2 space on this interval. A typical gap theorem deals with functions f on the real line such that many Fourier coefficients of f vanish.

The main goal of this book is to investigate relations between density and gap theorems and to study various cases where these theorems hold. The author also shows that density- and gap-type theorems are related to various properties of zeros of analytic functions in one variable.

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