COLLOQUIUM PUBLICATIONS

American Mathematical Society
Colloquium Publications
Volume 33

Differential Algebra

Joseph Fels Ritt

http://dx.doi.org/10.1090/coll/033

American Mathematical Society
Colloquium Publications
Volume 33

Differential Algebra

Joseph Fels Ritt

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 12-02; Secondary 12H05.

Library of Congress Cataloging-in-Publication Data

Ritt, Joseph Fels, 1893–1951.
Differential algebra.
p. cm. — (American Mathematical Society Colloquium publications, ISSN 0065-9258 ; v. 33)
New York, American Mathematical Society, 1950.
Includes bibliography.
ISBN 978-0-8218-4638-4 (alk. paper)
1. Differential equations. I. Title. II. Colloquium publications (American Mathematical Society) ; v. 33.

QA1 .A3225 vol. 33 50008228

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 18 17 16 15 14 13

PREFACE

In 1932, the author published *Differential equations from the algebraic standpoint,*[1] a book dealing with differential polynomials and algebraic differential manifolds. In the sixteen years which have passed, the work of a number of mathematicians has given fresh substance and new color to the subject. The complete edition of the book having been exhausted, it has seemed proper to prepare a new exposition.

The title *Differential algebra* was suggested by Dr. Kolchin. The body of algebra deals with the operations of addition and multiplication. We are concerned here with three operations—addition, multiplication and differentiation.

If I am not mistaken, the general nature of the subject here treated is now well enough known among mathematicians to permit me to dispense with a detailed introduction, such as was given in A. D. E. My principal task is to show how much the present book owes to my associates. I am referring to H. W. Raudenbush, W. C. Strodt, E. R. Kolchin, Howard Levi, Eli Gourin and Richard M. Cohn.

Cohn's constructive proof of the theorem of zeros will be found in Chapter V. The theorem on embedded manifolds due to Gourin is contained in Chapter II. Chapter VI contains a discussion of Strodt's work on sequences of manifolds.

In Chapters I, III and IX, there are presented portions of Levi's work on ideals of differential polynomials and on the low power theorem. Of Kolchin's investigation of exponents of differential ideals, I have been able to give only a bare idea. Other work of Kolchin, for instance, proofs for the abstract case of results previously established for the analytic case, is given in Chapter II. His work on the Picard–Vessiot theory, which employs the methods of differential algebra, has just appeared in the Annals of Mathematics,[2] and may be permitted to speak for itself.

The contributions of Raudenbush can only be described as fundamental. The basis theorem of Chapter I was, in the analytic case, implicitly contained in A. D. E. It exists there in two parts; the first, the theorem on the completeness of infinite systems; the second, the theorem of zeros. Only casually had I noticed that the two theorems amounted to a basis theorem. I was acquainted with the fact that the theorem on the decomposition of manifolds amounted, in virtue of the theorem of zeros, to a theory of perfect and prime ideals of differential polynomials. In the summer of 1933, I suggested to Raudenbush the problem of constructing a theory of perfect ideals which would be valid in the abstract case. This he accomplished, and, in the course of his work, he brought the basis theorem to its present complete and abstract form. In the proof of

[1] These Colloquium publications, vol. 14. Called below A. D. E.

[2] Kolchin, 14. (See Bibliography, p. 180.)

the basis theorem, the procedure of taking powers is due to Raudenbush. The chains, characteristic sets and methods of reduction existed in the older theorem of completeness.

Raudenbush introduced generic zeros of prime ideals. Here he adapted a method of van der Waerden, which can be traced back to König. Raudenbush gave the first example of a system of differential polynomials with a weak basis. Systems with no strong bases were later produced by Kolchin.

The problems which this book treats are very concrete problems. They deal with situations of the classical theory of differential equations. Seldom would much be lost, as far as the results are concerned, if one limited oneself to the material of classical analysis. The abstract method which we generally employ has, however, a definite utility. It serves to separate algebraic methods from analytic methods. On the whole, it contributes to simplicity, although at times an abstract treatment is less natural than an analytical one. The form in which the results of differential algebra are being presented has thus been deeply influenced by the teachings of Emmy Noether, a prime mover of our period, who, in continuing Julius König's development of Kronecker's ideas, brought mathematicians to know algebra as it was never known before.

In this connection, I should like to say something concerning basis theorems. The basis theorem of Chapter I will be seen to play, in the present theory, the role held by Hilbert's theorem in the theories of polynomial ideals and of algebraic manifolds. When I began to work on algebraic differential equations, early in 1930, van der Waerden's excellent *Moderne Algebra* had not yet appeared. However, Emmy Noether's work of the twenties was available, and there was nothing to prevent one from learning in her papers the value of basis theorems in decomposition problems. Actually, I became acquainted with the basis theorem principle in the writings of Jules Drach[3] on logical integration, writings which date back to 1898. How a basis theorem is employed by him will now be described.

There are two distinct methods for characterizing an irreducible algebraic equation. On the one hand, an equation $f(x) = 0$ is irreducible if $f(x)$ cannot be factored. On the other, there is irreducibility if every equation which is satisfied by a single solution of $f(x) = 0$ is satisfied by all such solutions. The first formulation of irreducibility leads to the notion of irreducible algebraic manifold and to that of irreducible algebraic differential manifold. The second leads to the concept of irreducible system of algebraic differential equations which was employed by Koenigsberger and by Drach. A system of such equations, ordinary or partial, is irreducible if every differential equation which admits a single solution of the system admits all solutions. Drach undertakes to show that, given a system of partial differential equations, the repeated adjunction of new equations will eventually produce an irreducible system. For this he invokes a theorem of Tresse,[4] which states that, in every infinite system

[3] Drach, 4, pp. 292–296.

[4] Acta Mathematica, vol. 18 (1894), p. 4.

of partial differential equations, there is a finite subsystem from which the infinite system can be derived by differentiations and eliminations. A study of Tresse's paper will quickly convince one that he claims for his work a generality which it does not have. The statement of his theorem, and his argument, have a definite meaning only for linear systems.

It has not been possible for me to present all of the material which has been developed since the publication of A. D. E. Thus, I have had to pass by most of Kolchin's study of exponents and a good deal of Levi's work on ideals. Of Strodt's paper, only a sketch is given. My own work on general solutions of equations of the second order in one unknown, and of equations of the first order in two unknowns, is also omitted.

I have tried to give, to the present book, the elementary quality which is possessed by A. D. E. Essentially, no previous knowledge of abstract algebra is necessary. As in A. D. E., a treatment is given of Riquier's existence theorem for orthonomic systems of partial differential equations.

New York, N. Y.
January, 1948.

This page intentionally left blank

CONTENTS

APPENDIX. QUESTIONS FOR INVESTIGATION

Ideals

1. Levi's work shows the nonexistence of a theory of ideals of d.p. possessing the scope of the Lasker-Noether theory of p.i. For d.p., it will be necessary either to use special types of ideals or to use other combinations than intersections and products.

2. Given a finite set of d.p., $F_1, \cdots, F_r$, and a d.p. G, is it possible to determine whether G is contained in $[F_1, \cdots, F_r]$? The methods of Chapter V permit one to decide whether some power of G is in $[F_1, \cdots, F_r]$. It is thus a question of determining a smallest admissible exponent.

3. Kolchin's theory of exponents should admit of extension in several directions. The chief problem examined by Kolchin is that of the exponent of $\{ A \}$ relative to $[A]$, where A is a d.p. in y of the first order. In the theorems obtained by Kolchin, the relative exponents are 1, 2, ∞. For instance, if $A = y^2 + y_1^3$, the exponent is ∞. Now

$$[A] = [y^p] \cdot \Sigma,$$

with p a positive integer and Σ an ideal whose manifold is the general solution of A. One may inquire as to the exponent of $\{ \Sigma \}$ relative to Σ. That exponent may easily be finite. This problem can, of course, be formulated for d.p. A admitting many singular zeros.

The problem of exponents may be examined for d.p. of order higher than the first and for p.d.p.

4. For $F = y^p + y_1^q$, in $\mathfrak{F}\{ y \}$, with $q > p$, what is the smallest integer r such that

$$y^r G \equiv 0, \qquad [F],$$

where G does not vanish for $y = 0$? This problem can be extended to general classes of d.p.

5. For $p > 0$, $i > 0$, what is the least q such that $y_i^q \equiv 0$, $[y^p]$? For $i = 1$, it is not hard to show that $q = 2p - 1$. In $\mathfrak{F}\{ u, v \}$, what is the least power of $u_i v_j$ which is contained in $[uv]$?

6. The ideals generated by various differential expressions may be examined. One may study the wronskian, the jacobian, the expression $EG - F^2$ of differential geometry, etc.

7. One may study d.p. over a field of characteristic p.

The decomposition problem

8. The basic problem has been met in Chapter V. It is that of determining the number of times which the d.p. in a finite system Φ must be differentiated

before eliminations will produce finite systems whose manifolds are the components of Φ. One would hope to secure a bound which depends on the number of d.p. in Φ, their orders and degrees.

9. Attached to the decomposition problem is the first problem of Laplace, mentioned in III, §37. Let F and A be algebraically irreducible and let F hold the general solution of A. It is required to determine whether the general solution of A is contained in that of F. The author has shown how to settle this question for d.p. F of the second order.[1] The methods can perhaps be extended to cover the case in which F, in $\mathfrak{F}\{y\}$, is of order n, and A of order $n - 2$. One might perhaps undertake to develop a test for the presence of $y = 0$ in the general solution of a d.p. of the third order. Other problems of this type will readily suggest themselves.

Intersections

10. One can see from Chapter VII that if there is regularity in the theory of intersections of algebraic differential manifolds, that regularity is not immediately visible. In VII, §1, an anomaly is found in the dimension of the intersection of a general solution with a second irreducible manifold. One might try to use complete manifolds of d.p. rather than general solutions. Thus, let $F_1, \cdots, F_r$ be d.p. in $\mathfrak{F}\{y_1, \cdots, y_n\}$. Suppose that $r < n$. Is every component of the system $F_1, \cdots, F_r$ of dimension at least $n - r$? For $r = 1$, we see from III, §1, that the answer is affirmative.

11. One may seek to extend the result of VII, §6, on Jacobi's bound to systems of n d.p. in n indeterminates.

The anomaly met in connection with the order of a component of the intersection of two general solutions raises the following problem. Let A and B be algebraically irreducible d.p. in y and z. Let $\mathfrak{M}$ be a component of dimension zero in the intersection of the general solutions of A and B. It is required to find a bound for the order of $\mathfrak{M}$ in terms of the orders of A and B in y and z. It is conceivable, of course, that no bound exists.

12. One may generalize the problem of III, §1, as follows. Let Σ be a nontrivial prime ideal in $\mathfrak{F}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$ with the u parametric and with

$$A_1, \cdots, A_p \qquad (1)$$

a characteristic set. Let Σ_0 be the prime p.i. for which (1), with the A considered as polynomials, is a characteristic set. Let Σ' be the system of d.p. obtained from Σ_0 when the polynomials in Σ_0 are regarded as d.p. What are the dimensions of the components of Σ'? Does the low power theorem have a generalization for this situation?

Differential power series

13. This subject has been mentioned in III, §39. Only one paper has been

[1] Ritt, 31. In connection with §65 of this paper, see the final remarks of §51 of Ritt, 32.

written on it. The entire program awaits development, both for ordinary differential equations and for partial. In the analytic case, the procedure will depend on whether one works in the neighborhood of a point in the space of the independent variables or in the neighborhood of a set of functions constituting a point of a manifold.

Birational transformations

14. The theory of the resolvent furnishes an instance of the birational equivalence of two irreducible manifolds. The general problem is that of finding conditions for such equivalence. The results of algebraic geometry should be a guide.

In studying birational transformations, one will meet *differential Cremona transformations*. For instance, let

$$Y = y\frac{d}{dx}\left(\frac{z}{y}\right), \qquad Z = z\frac{d}{dx}\left(\frac{z}{y}\right).$$

We find

$$y = Y\frac{d}{dx}\left(\frac{Z}{Y}\right), \qquad z = Z\frac{d}{dx}\left(\frac{Z}{Y}\right).$$

Is there a theorem on the structure of such transformations of y and z similar to M. Noether's theorem on ordinary Cremona transformations?

The analogue of Lüroth's theorem presented in Chapter II may have an extension to fields formed by the adjunction of two indeterminates.

Singular solutions of partial differential equations

15. For simplicity, we use two independent variables, x and y. Let F be an algebraically irreducible d.p. in $\mathfrak{F}\{z\}$, of order n in z. Let the components of F be $\mathfrak{M}$, $\mathfrak{M}_1, \cdots, \mathfrak{M}_s$, with $\mathfrak{M}$ the general solution. Each $\mathfrak{M}_i$ is the general solution of a d.p. F_i. Suppose that, for some i, F_i is of order $n - 1$ in z. Considering Hamburger's results for ordinary differential equations, one would expect the functions in $\mathfrak{M}_i$ to be envelopes, with a contact of some natural order, of functions in $\mathfrak{M}$. For $n = 1$, this question has been studied by the author.[2] For $n > 1$, the matter should be more difficult, since there is no theory of characteristics.

Difference algebra

16. This subject has been treated in papers of J. L. Doob, W. C. Strodt, F. Herzog, H. W. Raudenbush, Richard Cohn and the author.[3] The theory is open for cultivation.

---

[2] Ritt, 41.

[3] See bibliography.

BIBLIOGRAPHY

1. Cohn, R. M. *On the analog for differential equations of the Hilbert-Netto theorem*, Bulletin of the American Mathematical Society, vol. 47 (1941), pp. 268–270.

2. ——— *Manifolds of difference polynomials*, Transactions of the American Mathematical Society, vol. 64 (1948), pp. 133–172.

3. Doob, J. L., and Ritt, J. F. *Systems of algebraic difference equations*, American Journal of Mathematics, vol. 55 (1933), pp. 505–514.

4. Drach, J. *Essai sur la théorie générale de l'intégration et sur la classification des transcendantes*, Annales de l'Ecole Normale Supérieure, (3), vol. 15 (1898), pp. 245–384.

5. Gourin, E. *On irreducible systems of algebraic differential equations*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 593–595.

6. Hamburger, M. *Ueber die singulären Lösungen der algebraischen Differenzialgleichungen erster Ordnung*, Journal für die reine und angewandte Mathematik, vol. 112 (1893), pp. 205–246. See also ibid., vol. 121 (1899), p. 265, and vol. 122 (1900), p. 322.

7. Herzog, F. *Systems of mixed difference equations*, Transactions of the American Mathematical Society, vol. 37 (1935), pp. 286–300.

8. Kolchin, E. R., and Ritt, J. F. *On certain ideals of differential polynomials*, Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 895–898.

9. Kolchin, E. R. *On the basis theorem for infinite systems of differential polynomials*, Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 923–926.

10. ——— *On the exponents of differential ideals*, Annals of Mathematics, vol. 42 (1941), pp. 740–777.

11. ——— *On the basis theorem for differential systems*, Transactions of the American Mathematical Society, vol. 52 (1942), pp. 115–127.

12. ——— *Extensions of differential fields*, I, II, Annals of Mathematics, vol. 43 (1942), pp. 724–729; vol. 45 (1945), pp. 358–361.

13. ——— *Extensions of differential fields*, III, Bulletin of the American Mathematical Society, vol. 53 (1947), pp. 397–401.

14. ——— *Algebraic matric groups and the Picard-Vessiot theory of homogeneous ordinary linear differential equations*, Annals of Mathematics, vol. 49 (1948), pp. 1–42.

15. Lagrange, J. L. *Sur les solutions particulières des équations différentielles*, Oeuvres Complètes, vol. 4, pp. 5–108.

16. Laplace, P. S. *Mémoire sur les solutions particulières des équations différentielles et sur les inégalités séculaires des planètes*, Oeuvres Complètes, vol. 8, pp. 326–365.

17. Levi, H. *On the structure of differential polynomials and on their theory of ideals*, Transactions of the American Mathematical Society, vol. 51 (1942), pp. 532–568.

18. ——— *The low power theorem for partial differential polynomials*, Annals of Mathematics, vol. 46 (1945), pp. 113–119.

19. Poisson, S. D. *Sur les solutions particulières des équations différentielles et des équations aux différences*, Journal de l'Ecole Polytechnique, vol. 6, no. 13 (1806), pp. 60–125.

20. Raudenbush, H. W. *Differential fields and ideals of differential forms*, Annals of Mathematics, vol. 34 (1933), pp. 509–517.

21. ——— *Ideal theory and algebraic differential equations*, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 361–368.

22. ——— *Hypertranscendental adjunctions to partial differential fields*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 714–720.

23. ——— *On the analog for differential equations of the Hilbert-Netto theorem*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 371–373.

24. RAUDENBUSH, H. W., and RITT, J. F. *Ideal theory and algebraic difference equations*, Transactions of the American Mathematical Society, vol. 46 (1939), pp. 445–453.

25. RITT, J. F. *Manifolds of functions defined by systems of algebraic differential equations*, Transactions of the American Mathematical Society, vol. 32 (1930), pp. 369–398.

26. ——— *Differential equations from the algebraic standpoint*, American Mathematical Society Colloquium Publications, vol. 14, New York, 1932.

27. ——— *Algebraic difference equations*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 303–308.

28. ——— *Systems of algebraic differential equations*, Annals of Mathematics, vol. 36 (1935), pp. 293–302.

29. ——— *Jacobi's problem on the order of a system of differential equations*, Annals of Mathematics, vol. 36 (1935), pp. 303–312.

30. ——— *Indeterminate expressions involving an analytic function and its derivatives*, Monatshefte für Mathematik, vol. 43 (1936), pp. 97–104.

31. ——— *On the singular solutions of algebraic differential equations*, Annals of Mathematics, vol. 37 (1936), pp. 552–617.

32. ——— *On certain points in the theory of algebraic differential equations*, American Journal of Mathematics, vol. 60 (1938), pp. 1–43.

33. ——— *Systems of differential equations*, I. *Theory of ideals*, American Journal of Mathematics, vol. 60 (1938), pp. 535–548.

34. ——— *On ideals of differential polynomials*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 25 (1939), pp. 90–91.

35. ——— *On the intersections of algebraic differential manifolds*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 25 (1939), pp. 214–215.

36. ——— *On the intersections of irreducible components in the manifold of a differential polynomial*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 26 (1940), pp. 354–356.

37. ——— *On a type of algebraic differential manifold*, Transactions of the American Mathematical Society, vol. 48 (1940), pp. 542–552.

38. ——— *Complete difference ideals*, American Journal of Mathematics, vol. 63 (1941), pp. 681–690.

39. ——— *Bézout's theorem and algebraic differential equations*, Transactions of the American Mathematical Society, vol. 53 (1943), pp. 74–82.

40. ——— *On the manifolds of partial differential polynomials*, Annals of Mathematics, vol. 46 (1945), pp. 102–112.

41. ——— *Analytic theory of singular solutions of partial differential equations of the first order*, Annals of Mathematics, vol. 46 (1945), pp. 120–143.

42. ——— *On the singular solutions of certain differential equations of the second order*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 32 (1946), pp. 255–258.

43. STRODT, W. C. *Systems of algebraic partial difference equations*, Unpublished master's essay, Columbia University, 1937.

44. ——— *Irreducible systems of algebraic differential equations*, Transactions of the American Mathematical Society, vol. 45 (1939), pp. 276–297.

This page intentionally left blank

INDEX

The numbers refer to pages.

A gigantic task undertaken by J. F. Ritt and his collaborators in the 1930's was to give the classical theory of nonlinear differential equations, similar to the theory created by Emmy Noether and her school for algebraic equations and algebraic varieties. The current book presents the results of 20 years of work on this problem. The book quickly became a classic, and thus far, it remains one of the most complete and valuable accounts of differential algebra and its applications.

ISBN 978-0-8218-4638-4

9 780821 846384

COLL/33.S