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Structure and Representations of Jordan Algebras

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Nathan Jacobson

American Mathematical Society
Providence, Rhode Island

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Dedicated to

Adrian Albert

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PREFACE

The purpose of this book is to give a comprehensive account of the structure and representation theory of Jordan algebras over a field of characteristic not two. It may be appropriate at this point to indicate the limits we have set ourselves and what lies immediately beyond these limits. In the first place, we note that a substantial part of the theory carries over to algebras over a commutative ring with 1 containing an element $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$. A reader who has need of a result of this generality will have no difficulty in ascertaining whether or not the corresponding result in the field case carries over.

A more serious and significant extension of the theory, which can now be made, is the passage from a linear theory to a quadratic one. In the case of a special Jordan algebra this amounts to the replacement of the Jordan product $a.b = \frac{1}{2}(ab + ba)$, which is bilinear, by the product aba, which is linear in b and quadratic in a. As the Jordan theory has developed it has become increasingly a quadratic theory based on the product $\{aba\} = bU_a = 2(b.a).a - b.a^{\cdot 2}$, where $a.b$ is the given bilinear product and $a^{\cdot 2} = a.a$. In a special Jordan algebra $\{aba\}$ coincides with aba. An important step in the transition to a quadratic theory was taken by the author in a paper which appeared in 1966 ([38] in the Bibliography), in which we gave an Artinian-like structure theory for Jordan algebras founded on axioms on quadratic ideals. These are defined by means of the composition $\{aba\}$. This theory is considered in detail in Chapter IV. The author's structure theory has been extended by McCrimmon in [6] and [14] to quadratic Jordan algebras over an arbitrary commutative ring with 1. A sketch of McCrimmon's theory can be found in our notes on "Further Results and Open Questions" at the end of this book.

A part of the Jordan theory can be founded also on the notion of inverses. This has been done in the book *Jordan Algebras* by Braun and Koecher and more recently by Springer in an unpublished paper which covers also the case of algebras of characteristic two. The connection between inverses and the quadratic composition aba in associative algebras is given by Hua's identity (p. 2). The drawback of a theory based on inverses is that one must be in a situation in which one has an ample supply of invertible elements. The theory seems to work best for finite-dimensional algebras with 1 over an infinite field. In this case the invertible elements form a Zariski open subset of the algebra. On the other hand, the theory does not apply very well to infinite-dimensional algebras or in arithmetic settings in which invertible elements may be scarce. For this reason we have preferred to base the theory on $a.b$ and $\{aba\}$.

A number of important applications of Jordan algebras have been found. The Jordan theory had its birth in an attempt by P. Jordan and subsequently by Jordan, von Neumann and Wigner to formulate the foundation of quantum mechanics in terms of the product $A.B = \frac{1}{2}(AB + BA)$ rather than the associative product AB. A very important area of applications of the Jordan theory, especially of exceptional Jordan algebras, is to exceptional Lie groups and algebras and related geometries. This is discussed in part in Chapter IX. Indications of additional results and the original papers containing these are given in the "Further Results etc." Another extremely interesting and promising area of applications is to real and complex analysis, particularly to homogeneous cones, Siegel half-spaces and automorphic functions. The most complete account of these extremely interesting results presently available is in Koecher's University of Minnesota mimeographed notes [4]. This aspect of the theory has been completely omitted in our discussion.

In the course of writing this book I have had invaluable help from Kevin McCrimmon—first, in carefully reading and criticising several versions of the manuscript, second, in communicating improved proofs of a number of important results and finally, in carefully reading the proofs. I am indebted also to Marshall Osborn for reading the proofs and to Michel Racine for help in compiling the Bibliography. Also I am greatly indebted to Jacques Tits who took time off from his own important researches on algebraic groups to derive, via the theory of algebraic groups, the elegant constructions of exceptional Jordan algebras which we have given in Chapter IX. I wish to record my sincere appreciation for all of these contributions.

The intention of writing this book had its inception with an invitation by the American Mathematical Society to give Colloquium Lectures at the 1955 summer meeting. The subject has changed enormously in the intervening years. Since there remain many natural open questions and possibilities for applications it is likely to change substantially also in the next decade.

Key West,
January 22, 1968

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FURTHER RESULTS AND OPEN QUESTIONS

CHAPTER I

There are a number of important open questions concerning identities and inverses in Jordan algebras. We proceed to indicate some of these.

First, as in the text, let $FJ^{(r)}$ denote the free Jordan algebra with 1 and r (free) generators $x_1, x_2, \cdots, x_r$ (p. 40) and let $FSJ^{(r)}$ be the free special Jordan algebra with 1 and r (free) generators $u_1, u_2, \cdots, u_r$ (p. 7). Let $\mathfrak{K}^{(r)}$ be the kernel of the homomorphism of $FJ^{(r)}$ onto $FSJ^{(r)}$ sending $1 \to 1$, $x_i \to u_i$, $i = 1, 2, \cdots, r$. It is immediate that $\mathfrak{K}^{(r)}$ is a T-ideal in $FJ^{(r)}$, that is, an ideal mapped into itself by every homomorphism of $FJ^{(r)}$ into itself. We know that $\mathfrak{K}^{(r)} \neq 0$ if $r \geqq 3$. A Jordan algebra with 1 and r generators is a homomorphic image of a special Jordan algebra if and only if it satisfies all the identities in $\mathfrak{K}^{(r)}$. The results of Glennie indicated in the text (p. 51) provide nonzero elements of $\mathfrak{K}^{(3)}$. However, a complete determination of $\mathfrak{K}^{(r)}$ for $r \geqq 3$ has not yet been given. It is not known either if $\mathfrak{K}^{(r)}$, $r \geqq 3$, is finitely generated as a T-ideal.

Let $\Phi\{\{x_1, \cdots, x_r\}\}'$ be the free nonassociative algebra generated by $x_1, \cdots, x_r$, $\mathfrak{T}$, the ideal in this algebra generated by all the elements of the form $ab - ba$, $(a^2b)a - a^2(ba)$, a, $b \in \Phi\{\{x_i\}\}$. Assume that Φ is infinite and let $f \in \Phi\{\{x_i\}\}$ be homogeneous of degree n. We claim that if f is an identity for all finite-dimensional Jordan algebras, then f is an identity for all Jordan algebras. (This result has been communicated to us by Koecher.) We have to show that the hypothesis implies that $f \in \mathfrak{T}$. Since Φ is infinite, $\mathfrak{T}$ is a homogeneous ideal, that is, if $f_1 + f_2 + \cdots + f_k \in \mathfrak{T}$ where $\deg f_j = j$, then every $f_j \in \mathfrak{T}$. Let $\mathfrak{T}_{n+1}$ be the ideal generated by $\mathfrak{T}$ and all monomials in the x's of degree $\geqq n+1$. Then $\mathfrak{T}_{n+1}$ is also a homogeneous ideal and the homogeneous part of degree n in this ideal is contained in $\mathfrak{T}$. It is clear that $\Phi\{\{x_i\}\}/\mathfrak{T}_{n+1}$ is a Jordan algebra and since there are only a finite number of distinct monomials in the x_i of degree $\leqq n$, $\Phi\{\{x_i\}\}/\mathfrak{T}_{n+1}$ is finite dimensional. Hence f is an identity for this Jordan algebra. Then $f \in \mathfrak{T}_{n+1}$, which implies that $f \in \mathfrak{T}$.

Let $\{\mathfrak{J}_\alpha \mid \alpha \in \mathrm{A}\}$ be a family of Jordan algebras with 1. We define a *free composition* of the $\mathfrak{J}_\alpha$ to be a Jordan algebra $\mathfrak{P}$ with 1 together with a family $\{\eta_\alpha \mid \alpha \in \mathrm{A}\}$ where η_α is a homomorphism of $\mathfrak{J}_\alpha$ into $\mathfrak{P}$ sending $1 \to 1$ such that, if $\mathfrak{J}$ is any Jordan algebra with 1 and ζ_α is a homomorphism of $\mathfrak{J}_\alpha$ into $\mathfrak{J}$ with $1 \to 1$, then there exists a unique homomorphism λ of $\mathfrak{P}$ into $\mathfrak{J}$ satisfying $1^\lambda = 1$ and $\zeta_\alpha = \eta_\alpha \lambda$, $\alpha \in \mathrm{A}$. It is immediate that any two free compositions are equivalent in the obvious categorical sense. In particular, the algebra $\mathfrak{P}$ is determined up to isomorphism. Also $\mathfrak{P}$ is generated by the subalgebras $\mathfrak{J}_\alpha^{\eta_\alpha}$. It is easy to prove the existence of a free composition for any family of Jordan algebras. For example, let $\mathfrak{J}_1 = FJ^{(r)}/\mathfrak{K}$, $\mathfrak{J}_2 = FJ^{(s)}/\mathfrak{L}$ where $\mathfrak{K}$ and $\mathfrak{L}$ are ideals. Identify $FJ^{(r)}$ and $FJ^{(s)}$ with the subalgebras of $FJ^{(r+s)}$ generated by $x_1,\cdots,x_r$ and $x_{r+1},\cdots,x_{r+s}$ respectively and let $\mathfrak{M}$ be the ideal in $FJ^{(r+s)}$ generated by $\mathfrak{K}$ and $\mathfrak{L}$. Then it is easily seen that $\mathfrak{P} = FJ^{(r+s)}/\mathfrak{M}$ and the homomorphisms $x+\mathfrak{K}\to x+\mathfrak{M}$, $x\in FJ^{(r)}$ and $y+\mathfrak{L}\to y+\mathfrak{M}$, $y\in FJ^{(s)}$ constitute a free composition of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. This procedure is easily generalized (see Cohn's *Universal Algebra*, p. 142 for a general argument). We shall call the free composition $\{\mathfrak{P},\eta_\alpha\}$ the *free product* of the $\mathfrak{J}_\alpha$ if η_α is a monomorphism for every $\alpha\in \mathrm{A}$. In this case, if $\mathrm{A} = \{1,2,\cdots,n\}$, then we write $\mathfrak{P} = \mathfrak{J}_1 * \mathfrak{J}_2 * \cdots * \mathfrak{J}_n$. It is easy to see that the free product of associative algebras (same definitions as for Jordan algebras) exists. This implies that the free product of special Jordan algebras exist. On the other hand, free products may not exist for exceptional Jordan algebras. For example, let $\mathfrak{J}_1$ be a finite-dimensional simple exceptional Jordan algebra. Then it is known that if $\mathfrak{J}$ is any Jordan algebra with 1 containing $\mathfrak{J}_1$ as subalgebra with 1, then $\mathfrak{J}\cong\mathfrak{J}_1\otimes\mathfrak{C}$, where $\mathfrak{C}$ is an associative Jordan algebra with 1 (Jacobson [18]). It follows that $\mathfrak{J}$ satisfies every multilinear identity which holds for $\mathfrak{J}_1$. Now let $\mathfrak{J}_2 = FJ^{(2)}$ and let $\mathfrak{P}$ be a free composition of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Then $\mathfrak{P}$ satisfies all the multilinear identities satisfied by $\mathfrak{J}_1$. Hence $\mathfrak{P}$ cannot contain a subalgebra isomorphic to $\mathfrak{J}_2$ (cf. ex. 4, p. 363). Thus $\mathfrak{P}$ is not the free product of $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be associative Jordan algebras (with 1) and $\mathfrak{J}_3 = FJ^{(1)}$ with generator z. Since the $\mathfrak{J}_i$ are special the free product $\mathfrak{J}_1*\mathfrak{J}_2*\mathfrak{J}_3$ exists. Also the free associative product $\mathfrak{P}'$ of the associative algebras $\mathfrak{J}_i$ exists. Let $\mathfrak{J}_1*'\mathfrak{J}_2*'\mathfrak{J}_3$ be the subalgebra of the special Jordan algebra $\mathfrak{P}'^+$ generated by the images of the $\mathfrak{J}_i$ in $\mathfrak{P}'$. We have the canonical homomorphism σ of $\mathfrak{J}_1*\mathfrak{J}_2*\mathfrak{J}_3$ onto $\mathfrak{J}_1*'\mathfrak{J}_2*'\mathfrak{J}_3$. Let $\mathfrak{K}$ be the kernel of σ and let $\mathfrak{Z}$ be the subspace of $\mathfrak{J}_1*\mathfrak{J}_2*\mathfrak{J}_3$ spanned by products of elements of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ and a single z (that is, degree 1 in z). If $\mathfrak{J}_1 = \mathfrak{J}_2 = FJ^{(1)}$, then Macdonald's theorem states that $\mathfrak{K}\cap\mathfrak{Z}=0$. The same result has been proved by McCrimmon in [9] if $\mathfrak{J}_1 = \mathfrak{J}_2$ is isomorphic to the group algebra of an infinite cyclic group. In both of these cases the universal multiplication envelope of $\mathfrak{J}_i$ is commutative (cf. §2.3 and §2.13). We conjecture that this condition on the $\mathfrak{J}_i$, $i = 1,2$, is sufficient to insure that $\mathfrak{K}\cap\mathfrak{Z} = 0$.* This would constitute a generalization of Macdonald's and McCrimmon's theorems. Another consequence of the conjectured result would be that the free product of two associative Jordan algebras having commutative universal envelopes is special. This would constitute a generalization of Shirshov's theorem.

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* This conjecture was arrived at in a conversation with McCrimmon.

McCrimmon's analogue of Macdonald's theorem can be used to establish identities in x, y, $x^{\cdot -1}$, $y^{\cdot -1}$, z which are of degree one or zero in z by verifying these for special Jordan algebras. Examples of this sort are the identities $\{xyx\}.\{x^{\cdot -1}y^{\cdot -1}x^{\cdot -1}\} = 1$, $\{xyx\}^{\cdot 2}.\{x^{\cdot -1}y^{\cdot -1}x^{\cdot -1}\} = \{xyx\}$, which follow directly from Theorem 1.13(6), p. 52. Another example is

$$\{y\{(x^{\cdot -1} + y^{\cdot -1})\{xzx\}(x^{\cdot -1} + y^{\cdot -1})\}y\} = \{(x + y)z(x + y)\}.$$

On the other hand, McCrimmon's theorem is not adequate for establishing more general rational identities in x, y which involve inverses of polynomials in x and y and more general rational expressions, e.g. $x^{\cdot -1} + (y^{\cdot -1} - x)^{\cdot -1}$ (which occurs in Hua's identity, ex. 3, p. 54). To handle such identities we consider the free Jordan algebra $FJ^{(n+2)}$ generated by $x, y, x_1, \cdots, x_n$. Let $P_i = P_i(x, y, x_1, \cdots, x_i)$ be an element of the subalgebra generated by $1, x, y, x_1, \cdots, x_i$, $i = 0, 1, \cdots, n$. Let $I = I(P_0, \cdots, P_n)$ be the ideal in $FJ^{(n+2)}$ generated by the $2n$ elements $P_{i-1}.x_i - 1$, $P_{i-1}.x_i^{\cdot 2} - x_i$. Now let $\mathfrak{J}$ be a Jordan algebra with 1, a, b elements of $\mathfrak{J}$. Assume $P_0(a, b)$ is invertible in $\mathfrak{J}$ with a_1 as its inverse, $P_1(a, b, a_1)$ is invertible with inverse a_2, and, in general, $P_{i-1}(a, b, a_1, \cdots, a_{i-1})$ is invertible with inverse a_i. Then the homomorphism of $FJ^{(n+2)}$ into $\mathfrak{J}$ such that $1 \to 1$, $x \to a$, $y \to b$, $x_i \to a_i$ maps I into 0, so we have the homomorphism of $FJ^{(n+2)}/I$ into $\mathfrak{J}$ sending $1 \to 1$, $x + I \to a$, $y + I \to b$, $x_i + I \to a_i$. It is clear also that $P_{i-1} + I$ is invertible in $FJ^{(n+2)}/I$ with inverse $x_i + I$. The elements of I may be regarded as rational identities in the two elements x, y. As an example, let $n = 3$, $P_0 = x$, $P_1 = y$, $P_2 = x - x_2$. Then Hua's identity is equivalent to the statement that the following elements are contained in I:

$$(x_1 - x_3).(x - \{xyx\}) - 1, \qquad (x_1 - x_3)^{\cdot 2}.(x - \{xyx\}) - (x_1 - x_3).$$

We conjecture that for any choice of the P_i the subalgebra of $\mathfrak{J}$ generated by 1, a, b, $a_1, \cdots, a_n$ (as indicated) is special (assuming that $P_{i-1}(a, b, a_1, \cdots, a_{i-1})$ is invertible with inverse a_i). This would constitute a generalization of the theorem of Shirshov-Cohn. If $\mathfrak{J}$ is algebraic (that is, every element of $\mathfrak{J}$ is algebraic) then the subalgebra generated by 1, a, b, $a_1, \cdots, a_n$ is generated by $1, a, b$. Hence this is special by the Shirshov-Cohn theorem. In general, if the subalgebra generated by $1, a, b, a_1, \cdots, a_n$ is special, so that this can be identified with a subalgebra of $\mathfrak{A}^+$, $\mathfrak{A}$ associative, then a_i and $P_{i-1}(a, b_1, a_1, \cdots, a_{i-1})$ are inverses in the associative algebra $\mathfrak{A}$. Then the verification of the fact that certain elements of $\mathfrak{J}$ are mapped into 0 under the homomorphism we defined can be reduced to the verification of certain rational identities in associative algebras. In particular, it is clear that the validity of Hua's identity for algebraic Jordan algebras, hence for finite-dimensional Jordan algebras, is a consequence of Hua's identity for associative algebras. In general, if the conjecture we have made is true, then any rational identity in two elements of a Jordan algebra could be established by proving corresponding rational identities in two elements of an associative algebra.

One can also attempt to reduce identities which are rational in two elements and integral and of the first degree in a third element to associative rational identities in a similar manner. Here let $FJ^{(n+3)}$ be the free Jordan algebra generated by $1, x, y, z, x_1, \cdots, x_n$, $P_i(x, y, z, x_1, \cdots, x_i)$ an element of the subalgebra generated by $1, x, y, z, x_1, \cdots, x_i$, I the ideal generated by $P_{i-1} \cdot x_i - 1$, $P_{i-1} \cdot x_i^{\cdot 2} - x_i$. Let $\Phi\{u, v, w, u_1, \cdots, u_n\}$ be the free associative algebra generated by 1 and the indicated elements, I_s the ideal generated by $P_{i-1}u_i - 1$, $u_iP_{i-1} - 1$, where $P_i = P_i(u, v, w, u_1, \cdots, u_n)$. Then we have a homomorphism of $FJ^{(n+3)}$ into the Jordan algebra $(\Phi\{\cdots\}/I_s)^+$ such that $1 \to 1$, $x \to u + I_s$, $y \to v + I_s$, $z \to w + I_s$, $x_i \to u_i + I_s$. We conjecture that if $f \in FJ^{(n+3)}$ is of degree 1 in z and is mapped into 0 under the homomorphism of $FJ^{(n+3)}$ into $(\Phi\{\cdots\}/I_s)^+$, then $f \in I$. If this were the case, then we would have a reduction of identities rational in two elements and integral of the first degree in a third element to associative rational identities.

There are a number of interesting questions on the embeddability of Jordan integral domains in division rings. We recall first a well-known result of Ore's that any associative ring without zero-divisors $\neq 0$ which has the (left) common multiple property can be embedded in a division ring (Jacobson, *Theory of Rings*, p. 118 or Cohn's *Universal Algebra*, p. 275). The common multiple property for an associative ring $\mathfrak{A}$ is that if a and b are nonzero elements of $\mathfrak{A}$ then $\mathfrak{A}a \cap \mathfrak{A}b \neq 0$ for the left ideals $\mathfrak{A}a$, $\mathfrak{A}b$. We have called an element of a Jordan algebra $\mathfrak{J}$ a zero divisor in $\mathfrak{J}$ if the mapping U_a is not injective in $\mathfrak{J}$, that is, there exists a $b \neq 0$ such that $bU_a = 0$. We shall now say that the elements a, $b \in \mathfrak{J}$ *have a common multiple* if $\mathfrak{J}U_a \cap \mathfrak{J}U_b \neq 0$ for the quadratic ideals $\mathfrak{J}U_a$, $\mathfrak{J}U_b$. In view of Ore's result one is tempted to conjecture that if $\mathfrak{J}$ is a Jordan algebra in which any two nonzero elements have a common multiple then $\mathfrak{J}$ can be embedded in a Jordan division algebra.

It is a well-known result, due independently to Malcev and B. H. Neumann, that the free associative algebra $\Phi\{x_1, x_2, \cdots, x_n\}$ can be embedded in a division algebra (Cohn's *Universal Algebra*, p. 276). It follows from this that the free Jordan algebra $FJ^{(2)}$ can be embedded in a division algebra. Hence this is an integral domain. Is the free Jordan algebra $FJ^{(n)}$, $n \geqq 3$, an integral domain and can this be embedded in a division algebra? If the latter statement is true then it would provide examples of Jordan division algebras which are infinite dimensional over their centers. On the other hand, it is also possible that $FJ^{(n)}$ contains nonzero zero-divisors and possibly nonzero nilpotent elements if $n \geqq 3$. This is the situation for free alternative algebras (see Humm and Kleinfeld [7]).

CHAPTER II

The notion of bimodule and universal multiplication envelope for associative and Lie algebras are the starting points of the cohomology theories for these classes of algebras. The theory in the first case was initiated by Hochschild and in the second case by Chevalley and Eilenberg. We recall the basic foundational results in these theories (see Cartan-Eilenberg [1] for the details). Let $\mathfrak{A}$ be an associative algebra with 1, $\mathfrak{M}$ a unital bimodule for $\mathfrak{A}$. The unital universal multiplication envelope for $\mathfrak{A}$ is $U_1(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{A}^0$ and $\mathfrak{A}$ is a bimodule (the regular bimodule) for $\mathfrak{A}$. One defines the nth *cohomology group of* $\mathfrak{A}$ *with coefficients in* $\mathfrak{M}$ for $n = 0,1,2,$ by $H^n(\mathfrak{A},\mathfrak{M}) = \mathrm{Ext}^n_{U_1(\mathfrak{A})}(\mathfrak{A},\mathfrak{M})$. These are vector spaces over the base field Φ, $H^0(\mathfrak{A},\mathfrak{M})$ is isomorphic to the subspace of $\mathfrak{M}$ of elements u such that $ua = au$, $a \in \mathfrak{A}$, $H^1(\mathfrak{A},\mathfrak{M}) \cong \mathrm{Der}(\mathfrak{A},\mathfrak{M})/\mathrm{Inder}(\mathfrak{A},\mathfrak{M})$, where $\mathrm{Inder}(\mathfrak{A},\mathfrak{M})$ is the subspace of derivations of $\mathfrak{A}$ into $\mathfrak{M}$ of the form $a \to [a,u]$, $a \in \mathfrak{A}$, $u \in \mathfrak{M}$. $H^2(\mathfrak{A},\mathfrak{M})$ is isomorphic to the vector space of equivalence classes of null (= singular) extensions $0 \to \mathfrak{M} \to \mathfrak{E} \to \mathfrak{A} \to 0$ where the vector space compositions on such equivalence classes are the Baer compositions. Next let $\mathfrak{L}$ be a Lie algebra, $U(\mathfrak{L})$ the universal multiplication envelope (= Birkhoff-Witt algebra) and regard Φ as a trivial bimodule for $\mathfrak{L}$ so that $\alpha l = 0 = l\alpha$, $\alpha \in \Phi$, $l \in \mathfrak{L}$. If $\mathfrak{M}$ is an $\mathfrak{L}$-bimodule, one defines $H^n(\mathfrak{L},\mathfrak{M}) = \mathrm{Ext}^n_{U(\mathfrak{L})}(\Phi,\mathfrak{M})$, $n = 0,1,2,\cdots$. Then $H^0(\mathfrak{L},\mathfrak{M})$ is isomorphic to the subspace of $\mathfrak{M}$ of u such that $ul = 0$ and $H^1(\mathfrak{L},\mathfrak{M}) \cong \mathrm{Der}(\mathfrak{L},\mathfrak{M})/\mathrm{Inder}(\mathfrak{L},\mathfrak{M})$ where $\mathrm{Inder}(\mathfrak{L},\mathfrak{M})$ is the set of derivations of the form $l \to lu$ $(= -ul)$, $l \in \mathfrak{L}$, $u \in \mathfrak{M}$. Also, as in the associative case $H^2(\mathfrak{L},\mathfrak{M})$ is isomorphic to the vector space of equivalence classes of null extensions of $\mathfrak{L}$ by $\mathfrak{M}$.

In both the associative and Lie cases, one has simple standard resolutions of the bimodules $\mathfrak{A}$ and Φ respectively which lead to determinations of $H^n(-,\mathfrak{M})$ by cochains, cocycles and coboundaries. Also in both cases one has important interpretations of the vanishing of H^n for $n = 1,2,3$.

A definition of cohomology spaces for $n \geqq 2$ in any variety of algebras has been given by Gerstenhaber in [2] as follows. Let $\mathfrak{A} \in \mathscr{V}(I)$, the variety of algebras defined by a set of identities I, $\mathfrak{M}$ an I-bimodule for $\mathfrak{A}$. Define a *singular extension of length two* of $\mathfrak{A}$ by $\mathfrak{M}$ to be a null extension of $\mathfrak{A}$ by $\mathfrak{M}$ as defined on p. 91. If $n > 2$, define a *singular extension of length n* to be an exact sequence of bimodules $0 \to \mathfrak{M} \to \mathfrak{M}_{n-1} \to \cdots \to \mathfrak{M}_2 \to \mathfrak{C} \to 0$ together with a singular extension $0 \to \mathfrak{C} \to \mathfrak{E} \to \mathfrak{A} \to 0$. These give the exact sequence

$$0 \to \mathfrak{M} \to \mathfrak{M}_{n-1} \to \cdots \to \mathfrak{M}_2 \to \mathfrak{E} \to \mathfrak{A} \to 0.$$

Morphisms, equivalences, addition and scalar multiplication of equivalence classes of singular extensions can be defined. Then one defines the nth *cohomology group* $H^n(\mathfrak{A},\mathfrak{M})$ *for* $n \geqq 2$ of $\mathfrak{A} \in \mathscr{V}(I)$ with coefficients in $\mathfrak{M}$ as the vector space of equivalence classes of singular extensions of length n of $\mathfrak{A}$ by $\mathfrak{M}$. These definitions are equivalent to the classical ones in the associative and Lie cases.

Gerstenhaber's definitions of $H^n(\mathfrak{A},\mathfrak{M})$ for $n \geqq 2$ have been recently supplemented by Glassman in [1] to give definitions of $H^0(\mathfrak{A},\mathfrak{M})$ and $H^1(\mathfrak{A},\mathfrak{M})$. We shall now indicate these. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be I-bimodules for $\mathfrak{A} \in \mathscr{V}(I)$, η a homomorphism of $\mathfrak{M}_1$ into $\mathfrak{M}_2$. Then any derivation D of $\mathfrak{A}$ into $\mathfrak{M}_1$ determines the derivation $D\eta$ of $\mathfrak{A}$ into $\mathfrak{M}_2$ so we have the linear mapping $\tilde{\eta}: D \to D\eta$ of $\mathrm{Der}(\mathfrak{A},\mathfrak{M}_1)$ into $\mathrm{Der}(\mathfrak{A},\mathfrak{M}_2)$. In this way one obtains a functor from the category of $\mathfrak{A}$-bimodules to the category of vector spaces so that $\mathfrak{M} \to \mathrm{Der}(\mathfrak{A},\mathfrak{M})$, $\eta \to \tilde{\eta}$. An *inner derivation functor* J: $\mathfrak{M} \to J(\mathfrak{A},\mathfrak{M})$, $\eta \to \tilde{\eta} \mid J(\mathfrak{A},\mathfrak{M})$ is defined to be a subfunctor of the preceding which respects epimorphisms. It can be shown that the inner derivation functors are in 1-1 correspondence with submodules of $\mathrm{Der}(\mathfrak{A}, U(\mathfrak{A}))$ considered in a natural way as left $U(\mathfrak{A})$-module. Relative to the choice of J one defines $H_J^1(\mathfrak{A},\mathfrak{M}) = \mathrm{Der}(\mathfrak{A},\mathfrak{M})/J(\mathfrak{A},\mathfrak{M})$. Now assume the inner derivation functor J is finitely generated in the sense that the left $U(\mathfrak{A})$-module $J(U(\mathfrak{A}))$ is finitely generated. Let $\{d_1, d_2, \cdots, d_k\}$ be a set of generators for this module and let X_i, $1 \leqq i \leqq k$, be a free $\mathfrak{A}$-bimodule with generator x_i corresponding to the generator 1 of $U(\mathfrak{A})$. Let $\bar{d}_i$ be the derivation of $\mathfrak{A}$ into X_i corresponding to d_i and let Y be the submodule of $\sum_1^k \oplus X_i$ generated by the elements $a \sum_1^k \bar{d}_i$, $a \in \mathfrak{A}$. Put $\mathfrak{C}_{\{d_i\}} = (\sum \oplus X_i)/Y$ and define $H^0_{J,\{d_i\}}(\mathfrak{A},\mathfrak{M}) = \mathrm{Hom}_{U(\mathfrak{A})}(\mathfrak{C}_{\{d_i\}},\mathfrak{M})$ for any bimodule $\mathfrak{M}$ of $\mathfrak{A}$. For suitable choices of J and $\{d_i\}$ in the associative and Lie cases the definitions of H^0 and H^1 given by Glassman are equivalent to the usual ones. Also if $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is a short exact sequence of $\mathfrak{A}$-bimodules, then one can define connecting homomorphisms to obtain the usual long exact sequence of cohomology spaces.

All the results just indicated can be developed for algebras over commutative rings. The notion of bimodules, universal multiplication envelopes and extensions have been considered by Knopfmacher in [1] and [2] in this generality.

CHAPTER III

The Corollary (p. 138) to the Coordinatization Theorem has been generalized recently by McCrimmon in [11]. If e_1 and e_2 are nonzero orthogonal idempotents in a Jordan algebra $\mathfrak{J}$, then these are said to be *interconnected* if $e_i \in \mathfrak{J}_{ij}^{\cdot 2} \cdot e_i$, $i \neq j = 1, 2$, $\mathfrak{J}_{ij} = \mathfrak{J}U_{e_i,e_j}$. This is equivalent to: $\mathfrak{J}_{ij}^{\cdot 2} \cdot e_i = \mathfrak{J}_{ii}$, $\mathfrak{J}_{ii} = \mathfrak{J}U_{e_i}$. It is clear that connectedness implies interconnectedness. McCrimmon has proved that if $\mathfrak{J}$ is a Jordan algebra with $1 = \sum_1^n e_i$, where the e_i are nonzero interconnected orthogonal idempotents and $n \geqq 4$, then $\mathfrak{J}$ is special (and hence any Jordan algebra containing $\mathfrak{J}$ as subalgebra with the same identity is special). This theorem had been conjectured by the author and was suggested by Martindale's theorem. It is easy to see that any two nonzero idempotents of a simple Jordan algebra are interconnected. As has been shown by McCrimmon, a consequence of this and a generalization of the foregoing theorem to algebras not necessarily containing 1 is the following striking result: If $\mathfrak{J}$ is a simple Jordan algebra (not necessarily containing 1) which contains three nonzero orthogonal idempotents whose sum is not an identity element for $\mathfrak{J}$, then $\mathfrak{J}$ is special.

CHAPTER IV

The structure theory of this chapter reduces the study of the Jordan algebras which are nondegenerate and satisfy the minimum conditions on quadratic ideals to that of Jordan division algebras. Thus the situation is quite similar to that which obtains for Artinian semisimple algebras. At the present time the relation between Jordan and associative division algebras remains to be clarified. A natural question here is the following: Is every special Jordan division algebra of one of the following types: (1) a Jordan algebra of a symmetric bilinear form, (2) Δ^+ where Δ is an associative division algebra, (3) $\mathfrak{H}(\Delta, J)$ where Δ is a division algebra with involution J? At the present no examples of exceptional Jordan division algebras which are infinite dimensional over their centers are known.

There is no satisfactory analysis as yet of degenerate Jordan algebras satisfying appropriate minimum conditions on quadratic ideals. In particular, it would be interesting to have a definition of a radical for a Jordan algebra satisfying the minimum conditions which is analogous to one of the definitions of the radical of an Artinian associative algebras.

It would be interesting to extend to Jordan algebras other aspects of the associative structure theory. One interesting direction would be the development of a theory of Jordan *PI*-algebras, that is, Jordan algebras satisfying a polynomial identity. Since there exist nontrivial identities which are satisfied by all special Jordan algebras, the so-called *s*-identities (p. 49) it is natural to define a *PI*-Jordan algebra to be one which satisfies an identity p which is not an *s*-identity. For example, the Jordan algebra of a nondegenerate symmetric bilinear form satisfies the identity $p = [[x_1, x_2, x_3]^{\cdot 2}, x_4, x_5]$ (ex. 6, p. 364), and it is easily seen that this is not an *s*-identity. Are these and simple Jordan algebras which are finite dimensional over their centers the only simple *PI*-Jordan algebras? The same question can be asked for Jordan division algebras.

The structure theory of this chapter has been extended by McCrimmon ([6] and [14]) to algebras over an arbitrary commutative ring with 1. As we indicated in Chapter I, in considering special Jordan algebras, to obtain a general theory one has to replace the bilinear composition $a \cdot b$ by the composition bU_a which is linear in b and quadratic in a, and, if the existence of 1 is not assumed, then one must consider also the unary composition $a^{\cdot 2}$. To simplify the discussion we stick to the case of algebras with 1. It is convenient to formulate the axioms in terms of the mappings U_a. For the sake of comparison with the Jordan algebras considered in this book, we give first an analogous definition of the usual ones in operator form. Accordingly, we *define a Jordan algebra with* 1 *over a field* Φ *of characteristic not two* to be a triple $(\mathfrak{J}, R, 1)$, where $\mathfrak{J}$ is a vector space over Φ, 1 a particular element of $\mathfrak{J}$, and R a mapping of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ satisfying the following conditions:

1. R is Φ-linear.
2. $R_1 = 1$.
3. $[R_a, R_{aR_a}] = 0$.
4. $R_a = L_a$ if $xL_a \equiv aR_x$.

It is clear that this definition is equivalent to the usual one and that if P is an extension field of Φ then $\mathfrak{J}_P$, 1 and the extension R of R to a linear mapping of $\mathfrak{J}_P$ satisfies conditions 1–4. Hence we obtain the Jordan algebra $(\mathfrak{J}_P, R, 1)$. The foregoing definition can also be given for left unital Φ-modules over an arbitrary commutative ring with 1, provided that Φ contains an element $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$. In this case also $\mathfrak{J}_P$, R and 1 would satisfy the same axioms for any commutative ring extension P of Φ (with the same 1). A *homomorphism* η of a Jordan algebra $(\mathfrak{J}, R, 1)$ into a Jordan algebra $(\mathfrak{J}', R', 1')$ is a Φ-homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$ such that $1^\eta = 1'$ and $(bR_a)^\eta = b^\eta R'_{a^\eta}$.

We now give McCrimmon's definition of a *quadratic Jordan algebra with* 1 *over an arbitrary commutative ring* Φ *with* 1 as a triple $(\mathfrak{J}, U, 1)$ where $\mathfrak{J}$ is a unital left Φ-module, 1 a distinguished element of $\mathfrak{J}$, and U is a mapping $a \to U_a$ of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ satisfying the following axioms:

1. U is Φ-quadratic, that is, $U_{\alpha a} = \alpha^2 U_a$, $\alpha \in \Phi$, $a \in \mathfrak{J}$ and $U_{a,b} \equiv U_{a+b} - U_a - U_b$ is Φ-bilinear in a and b.
2. $U_1 = 1$.
3. $U_a U_b U_a = U_{bU_a}$.
4. $V_{a,b} U_b = U_b V_{b,a}$ where $xV_{a,b} = bU_{a,x}$.

If P is a commutative ring extension with 1 of Φ, then U has a unique extension to a quadratic mapping of $\mathfrak{J}_P = P \otimes_\Phi \mathfrak{J}$ into $\mathrm{Hom}_P(\mathfrak{J}_P, \mathfrak{J}_P)$. Here if $\rho_i \in P$, $a_i \in \mathfrak{J}$ then $U_{\Sigma \rho_i a_i} = \sum \rho_i^2 U_{a_i} + \sum_{i<j} \rho_i \rho_j U_{a_i, a_j}$. We require also

5. Conditions 3 and 4 hold in every $\mathfrak{J}_P$.

Axiom 5 is equivalent to intrinsic conditions on $\mathfrak{J}$, namely, the validity of certain linearizations of 3 and 4. It is easy to see that if Φ is a field with more than three elements, then these linearizations are consequences of 3 and 4. Hence 5 is superfluous in this case.

We put $a^{\cdot 2} = 1U_a$, $a \circ b = (a+b)^{\cdot 2} - a^{\cdot 2} - b^{\cdot 2} = 1U_{a,b}$, $V_a = U_{a,1}$. Then $a \circ b = b \circ a$ and $a \circ a = 2a^{\cdot 2}$. Also $V_1 = U_{1,1} = 2$. We write $\{abc\} = bU_{a,c}$, so $\{aba\} = 2bU_{a,a}$ since $U_{a,a} = 2U_a$. Also, $\{abc\} = cV_{a,b}$.

As a special case of the notion of homomorphism for general algebras with finitary compositions (cf. Cohn's *Universal Algebra*, p. 49), one defines a *homomorphism* η of $(\mathfrak{J}, U, 1)$ into the quadratic Jordan algebra $(\mathfrak{J}', U', 1')$ to be a Φ-homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$ such that $1^\eta = 1'$ and $(bU_a)^\eta = b^\eta U_{a^\eta}$. The class of quadratic Jordan algebras with 1 over Φ is a category whose morphisms are homomorphisms. If Φ is a field of characteristic not two and $\mathfrak{J}$ is a Jordan algebra with 1 over Φ then $\mathfrak{J}$ defines a quadratic Jordan algebra with 1, $(\mathfrak{J}, U, 1)$

in which $1 = 1$, $U_a = 2R_a^2 - R_{a^{\cdot 2}}$. All the conditions except 4 are clear and this has been proved on p. 328. Conversely, let $(\mathfrak{J}, U, 1)$ be a quadratic Jordan algebra with 1 over a field of characteristic not two. Then it can be shown that, if we put $R_a = \frac{1}{2}V_a$, we obtain a Jordan algebra with 1, $(\mathfrak{J}, R, 1)$. One can show also that the two constructions are inverses and homomorphisms of $(\mathfrak{J}, U, 1)$ coincide with homomorphisms of $(\mathfrak{J}, R, 1)$. One obtains in this way a category isomorphism of the category of quadratic Jordan algebras with 1 over Φ with the category of Jordan algebras with 1 over Φ. The same result holds for any ring Φ which contains an element $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$, if one defines a Jordan algebra over Φ as indicated above (that is, as in the field case).

Let $\mathfrak{A}$ be an associative algebra with 1 over a commutative ring Φ (with 1). Define $xU_a = axa$. Then $xU_{a,b} = axb + bxa$, $xV_{a,b} = abx + xba$. It is immediate that 1–5 hold, so $\mathfrak{A}$ with 1 and the indicated U is a quadratic Jordan algebra with 1, $\mathfrak{A}^{(q)} = (\mathfrak{A}, U, 1)$. If $(\mathfrak{J}, U, 1)$ is a quadratic Jordan algebra with 1, a *subalgebra* $\mathfrak{K}$ is a Φ-submodule containing 1 and closed under bU_a. Subalgebras of algebras $\mathfrak{A}^{(q)}$, $\mathfrak{A}$ associative with 1, are called *special quadratic Jordan algebras with* 1. If $(\mathfrak{A}, J)$ is an associative algebra with involution and 1 then $\mathfrak{H}(\mathfrak{A}, J)$, the set of J-symmetric elements, is a subalgebra of $\mathfrak{A}^{(q)}$. Let $\mathfrak{B}$ be a vector space over a field of any characteristic, f a quadratic form. Then it is readily verified that the subspace $\mathfrak{J} = \Phi 1 + \mathfrak{B}$ of the Clifford algebra $C(\mathfrak{B}, f)$ is a subalgebra of $C(\mathfrak{B}, f)^{(q)}$. We call this the *quadratic Jordan algebra* of f.

We shall see in a moment that the standard exceptional Jordan algebras also have analogues in quadratic Jordan algebras (see also the notes on Chapter IX).

It seems likely that much of the usual Jordan theory can be carried over to quadratic Jordan algebras. We proceed to indicate that this is indeed the case with the structure theory as developed in this chapter. In the notes on the remaining chapters, extensions of some other parts of the theory will be indicated.

We consider first the basic concepts of the structure theory. One defines an *inner* (= quadratic) *ideal* $\mathfrak{B}$ of $(\mathfrak{J}, U, 1)$ to be a Φ-submodule of $\mathfrak{J}$ such that $\mathfrak{J}U_b \subseteq \mathfrak{B}$, $b \in \mathfrak{B}$. An *outer ideal* $\mathfrak{B}$ is a Φ-submodule such that $bU_a \in \mathfrak{B}$, $b \in \mathfrak{B}$, $a \in \mathfrak{J}$. An *ideal* is a subset which is both an inner and an outer ideal. If $\mathfrak{B}$ is an ideal it is easily seen that if $b \in \mathfrak{B}$ and $a, c \in \mathfrak{J}$, then $bU_{a,c}$, $b^{\cdot 2}$, $b \circ c$, $bV_{a,c} \in \mathfrak{B}$. It follows that the Φ-module $\mathfrak{J}/\mathfrak{B}$ is a quadratic Jordan algebra with 1 in the obvious way. The kernel of a homomorphism is an ideal. If $b \in \mathfrak{J}$ then $\mathfrak{J}U_b$ is an inner ideal called the *principal inner ideal* generated by b. If Φ contains $\frac{1}{2}$, then any outer ideal is an ideal. Otherwise, this may not be the case, as the following examples show: (1) Let $\mathfrak{J}$ be the special quadratic Jordan algebra $\mathfrak{H}(Z_n)$ $n \times n$ symmetric matrices over the ring Z of integers, and let $\mathfrak{B}$ be the subset of integral matrices with even diagonal elements. Then $\mathfrak{B}$ is an outer ideal which is not an ideal. (2) Let Φ be a field of characteristic two, P an extension field such that the subfield $\Phi(\mathrm{P}^2)$ over Φ generated by the squares of the elements of P does not coincide with P. Let $\mathfrak{H}(\mathrm{P}_n)$ be the special quadratic Φ-algebra o

$n \times n$ symmetric matrices over P. Let $\mathfrak{B}$ be the subset consisting of the matrices with diagonal entries in $\Phi(\mathrm{P}^2)$. Then $\mathfrak{B}$ is an outer ideal which is not an ideal.

The outer ideal $\mathfrak{J}'$ of $(\mathfrak{J}, U, 1)$ generated by 1 is a subalgebra which we shall call the *core* of $\mathfrak{J}$. Any subalgebra of $\mathfrak{J}$ containing $\mathfrak{J}'$ will be called an *ample* subalgebra of $\mathfrak{J}$.

In an associative algebra with 1 the relations $aba = a$ and $ab^2a = 1$ imply $ab = 1 = ba$. This suggests the definition: $a \in (\mathfrak{J}, U, 1)$ is *invertible* with b as an *inverse* if $bU_a = a$, $b^{\cdot 2} U_a = 1$. This is equivalent to the usual notions in a Jordan algebra and one has a close analogue of Theorem 1.13 (on inverses). For example, the inverse is unique and $U_a U_b = 1 = U_b U_a$ for b the inverse $a^{\cdot -1}$. A quadratic Jordan algebra with 1 is a *division algebra* if every nonzero element of the algebra is invertible.

If u is an invertible element of $(\mathfrak{J}, U, 1)$, then we put $1^{(u)} = u^{\cdot -1}$, $U_a^{(u)} = U_u U_a$. Then $(\mathfrak{J}, U^{(u)}, 1^{(u)})$ is a quadratic Jordan algebra called the *u-isotope* of $(\mathfrak{J}, U, 1)$. As in the case of Jordan algebra, isotopy is an equivalence relation.

An element e of a quadratic Jordan algebra $(\mathfrak{J}, U, 1)$ is *idempotent* if $e^{\cdot 2} = e$ and the idempotents e, f are *orthogonal* if $fU_e = e \circ f = eU_f = 0$. Let $\{e_i \mid i = 1, \cdots, n\}$ be orthogonal idempotents such that $\sum_1^n e_i = 1$. Put $E_{ii} = U_{e_i}$, $E_{ij} = U_{e_i, e_j}$ for $i \neq j$. Then $1 = \sum_{i \leq j} E_{ij}$ and the E_{ij} are orthogonal projections. Hence $\mathfrak{J} = \sum \oplus \mathfrak{J}_{ij}$. This is called the *Peirce decomposition* of $\mathfrak{J}$ relative to the e_i. The $\mathfrak{J}_{ii}$ are inner ideals. There are many important relations connecting the Peirce components $\mathfrak{J}_{ij}$ analogous to those we derived for Jordan algebras. One defines connectedness and strong connectedness for orthogonal idempotents in a quadratic Jordan algebra with 1 as for Jordan algebras. The principal results carry over: transitivity holds and one can pass from connected idempotents to strongly connected idempotents in an isotope as in Lemma 5 of §3.1 (p. 123).

We consider next the definition of quadratic Jordan matrix algebras. For the sake of simplicity we stick to the case of standard involutions; the more general case of canonical involutions is reducible to this via isotopy by a diagonal matrix. First, let $(\mathfrak{D}, j)$ be an associative algebra with involution and 1. For any $n = 1, 2, 3, \cdots$ we define the standard quadratic Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ to be the set of $n \times n$ matrices over $\mathfrak{D}$ which are symmetric under the standard involution $X \to \bar{X}^t$ $(\bar{d} = d^j)$ where $AU_B = BAB$ and 1 and the Φ-module structure are as usual. In studying these algebras it is convenient to modify our usual definition of $a[ij]$, $a \in \mathfrak{D}$ by putting $a[ii] = ae_{ii}$, $a \in \mathfrak{H}(\mathfrak{D}, j)$, $a[ij] = ae_{ij} + \bar{a}e_{ji}$, $a \in \mathfrak{D}$. Using these one can develop formulas for the U operator analogous to (18′)–(23′) of p. 126. These and the formulas which one obtains from the Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_3)$, where $(\mathfrak{D}, j)$ is alternative over a field of characteristic $\neq 2$, lead to the definition of $\mathfrak{H}(\mathfrak{D}_3)$ for $(\mathfrak{D}, j)$ alternative. Here one begins with an alternative algebra with involution $(\mathfrak{D}, j)$ and 1 satisfying the condition that $d\bar{d} \in N(\mathfrak{D})$, the nucleus of $\mathfrak{D}$, for every $d \in \mathfrak{D}$. Then it can be shown that $\mathfrak{D}' = \mathfrak{H}(\mathfrak{D}, j) \cap N(\mathfrak{D})$ is a Φ-submodule of $\mathfrak{D}$ such that $dd'\bar{d} \in \mathfrak{D}'$ for all $d \in \mathfrak{D}$, $d' \in \mathfrak{D}'$. One defines $\mathfrak{H}(\mathfrak{D}_3)$

to be the set of matrices which are symmetric under the standard involution in $\mathfrak{D}_3$ and *which have diagonal elements in* $\mathfrak{D}'$. (Note that if $\mathfrak{D}$ is associative this is the same as our former definition.) Equivalently, we let $\mathfrak{H}(\mathfrak{D}_3)$ be the Φ-module direct sum of three copies of $\mathfrak{D}$ and three copies of $\mathfrak{D}'$ denoted respectively as $\mathfrak{D}[ij]$, $i<j$, $\mathfrak{D}'[ii]$, $i,j = 1,2,3$. If $i>j$, we put $d[ij] = \bar{d}[ji]$, where $\bar{d}[ji]$ is the image in $\mathfrak{D}[ji]$ of $\bar{d} \in \mathfrak{D}$. It is easy to see that there is a unique quadratic mapping $a \to U_a$ of $\mathfrak{H}(\mathfrak{D}_3) = \mathfrak{D}[12] \oplus \mathfrak{D}[23] \oplus \mathfrak{D}[13] \oplus \mathfrak{D}'[11] \oplus \mathfrak{D}'[22] \oplus \mathfrak{D}'[33]$ into $\mathrm{Hom}_\Phi(\mathfrak{H}(\mathfrak{D}_3),\mathfrak{H}(\mathfrak{D}_3))$ such that the following formulas hold:

$$b'[ii]\,U_{a'[ii]} = a'b'a'[ii], \tag{1}$$

$$a'[ii]\,U_{a[ij]} = \bar{a}a'a[jj], \tag{2}$$

$$b[ij]\,U_{a[ij]} = a\bar{b}a[ij], \tag{3}$$

$$\{a'[ii]b[ij]c'[jj]\} = a'bc'[ij], \tag{4}$$

$$\{a'[ii]b[ij]c[ji]\} = (a'bc + \overline{a'bc})[ii], \tag{5}$$

$$\{a'[ii]b[ij]c[jk]\} = a'bc[ik], \tag{6}$$

$$\{a'[ii]b'[ii]c[ij]\} = a'b'c[ij], \tag{7}$$

$$\{a[ij]b'[jj]c[jk]\} = ab'c[ik], \tag{8}$$

$$\{a[ij]b[ji]c[ik]\} = a(bc)[ik], \tag{9}$$

$$\{a[ij]b[jk]c[ki]\} = (a(bc) + \overline{a(bc)})[ii], \tag{10}$$

where $\{abc\} = bU_{a,c}$ and it is understood that all the U formulas not covered by these and $a[ji] = \bar{a}[ij]$ are 0 (e.g. $a'[ii]U_{b'[jj]} = 0$ if $i \neq j$). We remark that the condition that $d\bar{d} \in N(\mathfrak{D})$ implies that $d\bar{d}$ and $d + \bar{d} \in \mathfrak{D}'$ for $d \in \mathfrak{D}$. Hence the right-hand sides of (1)–(10) are contained in $\mathfrak{H}(\mathfrak{D}_3)$. It is a formidable task which has been carried out by McCrimmon to show that $(\mathfrak{H}(\mathfrak{D}_3), U, 1)$, where $1 = \sum_1^3 1[ii]$ satisfies the axioms for a quadratic Jordan. We shall now define a *standard quadratic Jordan matrix algebra* to be either an algebra of the form $(\mathfrak{H}(\mathfrak{D}_n), U, 1)$, where $(\mathfrak{D}, j)$ is associative with involution and 1, or one of the algebras $(\mathfrak{H}(\mathfrak{D}_3), U, 1)$ just defined.

Let $\mathfrak{J}$ be such a standard quadratic Jordan matrix algebra and let $\mathfrak{D}_0$ be the Φ-submodule of $\mathfrak{D}$ generated by the elements $d\bar{d}$, $d \in \mathfrak{D}$. Then it is easily seen that if we identify $d'[ii]$ with $d'e_{ii}$, $d[ij]$ with $de_{ij} + \bar{d}e_{ji}$, $i<j$, then the core $\mathfrak{J}' = \mathfrak{H}(\mathfrak{D}_n)'$, $n \geqq 3$ is the subset of matrices having diagonal elements in $\mathfrak{D}_0$. The elements $e_i = 1[ii]$ and $1[ij]$, $i<j$, are contained in $\mathfrak{J}'$. The e_i are orthogonal idempotents such that $\sum e_i = 1$ and e_i and e_j, $i \neq j$, are strongly connected by $1[ij]$. As for ordinary Jordan algebras, one has a Strong Coordinatization Theorem which is a structure theorem for quadratic Jordan algebras $(\mathfrak{J}, U, 1)$ such that $1 = \sum_1^n e_i$, where the e_i are nonzero strongly connected orthogonal idempotents and $n \geqq 3$

The result states that under these conditions one has a homomorphism η of $(\mathfrak{J}, U, 1)$ into a standard quadratic Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ such that $\mathfrak{J}^\eta$ is an ample subalgebra of $\mathfrak{H}(\mathfrak{D}_n)$ and the kernel of η is an ideal consisting of absolute zero divisors z (that is, $U_z = 0$) such that $2z = 0$.

We can now state McCrimmon's extensions of our structure theorems. The class of algebras to which these apply are the quadratic Jordan algebras with 1, $(\mathfrak{J}, U, 1)$ which contain no absolute zero-divisors $\neq 0$, and which satisfy the minimum conditions for inner (= quadratic) ideals (p. 157). Such an algebra is a direct sum of simple quadratic Jordan algebras satisfying the same conditions. A determination of these can be given by the following theorem: A quadratic Jordan algebra $(\mathfrak{J}, U, 1)$ is simple with minimum conditions for inner ideals if and only if it is of one of the following types: I. a quadratic Jordan division algebra, II. an ample subalgebra of a quadratic Jordan algebra of a nondegenerate quadratic form, III. an ample subalgebra of a quadratic Jordan matrix algebra $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $(\mathfrak{O}, j)$ is an octonion algebra with standard involution, IV. an ample subalgebra of a quadratic Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$, where $(\mathfrak{A}, J)$ is a simple Artinian associative algebra with involution.

We note finally that McCrimmon has succeeded also in extending the theory to quadratic Jordan algebras without 1 by giving axioms in terms of the U operator and a unary composition $a \rightarrow a^{\cdot 2}$, which insure that such a system can be imbedded in a quadratic Jordan algebra with 1.

Chapter V

Braun and Koecher [1] have defined the radical of finite-dimensional Jordan algebra $\mathfrak{J}/\Phi$ with 1 to be the intersection of the radicals of all the symmetric bilinear forms f on $\mathfrak{J}$ into extension fields of Φ such that $f(z, 1) = 0$ for all nilpotent z. It is clear that $\operatorname{rad}\mathfrak{J}$ as defined in this chapter is contained in the Braun-Koecher radical $\mathfrak{R}$. Conversely, let $a \notin \operatorname{rad}\mathfrak{J}$, so a has a nonzero image $\bar{a}$ in some simple homomorphic image $\bar{\mathfrak{J}}$ of $\mathfrak{J}$. Let $\bar{\mathrm{P}}$ be the center of $\bar{\mathfrak{J}}$ so $\bar{\mathrm{P}}$ is a finite-dimensional extension field of Φ. Let $\bar{t}$ be the generic trace bilinear form on $\bar{\mathfrak{J}}/\bar{\mathrm{P}}$ so $\bar{t}$ is nondegenerate (by Chapter VI, p. 240 or Chapter VIII, p. 353), $\bar{t}$ has values in $\bar{\mathrm{P}}$ and $\bar{t}(\bar{z}, \bar{1}) = 0$ for $\bar{z}$ nilpotent. Since $\bar{a} \neq 0$, $\bar{a}$ is not contained in the radical of $\bar{t}$. Define f by $f(x, y) = \bar{t}(\bar{x}, \bar{y})$. Then it is immediate that f is a form of the type considered by Braun and Koecher and a is not contained in the radical of f. Hence $a \notin \mathfrak{R}$. Thus $\mathfrak{R} \subseteq \operatorname{rad}\mathfrak{J}$ and $\mathfrak{R} = \operatorname{rad}\mathfrak{J}$.

Chapter VI

The author has proved in [32] that the generic norm $n(x)$, $x = \sum_1^n \xi_i u_i$, of any finite-dimensional simple Jordan algebra is an irreducible polynomial in $\Phi[\xi_1, \cdots, \xi_n]$. The method of proof of this result is an extension of one used by Dieudonné in [2] to prove the same result for simple associative algebras. Since any simple associative algebra $\mathfrak{A}$ over a field of characteristic not two determines the simple Jordan algebra $\mathfrak{A}^+$ with the same generic norm, the theorem for associative algebras of characteristic not two follows from the Jordan algebra result. The Jordan theorem gives the following consequence of Theorem 3 (iv): If $\mathfrak{A}$ is a simple Jordan algebra and the remaining hypotheses are as in Theorem 3 (iv) then $Q(\xi_1,\cdots,\xi_n)$ is a power of the generic norm. It is easy to extend this result to the semisimple case.

The classical theorem on composition of quadratic forms (see Theorem 4.5, Jacobson [22], and the references given there) has been generalized by Schafer in [14]. His results are that if $\mathfrak{A}$ is an algebra which possesses a nondegenerate homogeneous form $Q(x)$ of degree n permitting the composition $Q(xy) = Q(x)Q(y)$ and the characteristic is 0 or $p > n$, then $\mathfrak{A}$ is alternative. Moreover, if $\mathfrak{A}$ is finite dimensional, then $\mathfrak{A}$ is separable. Nondegeneracy of Q in Schafer's sense means that if $Q(x_1, x_2, \cdots, x_n)$ is the symmetric multilinear form associated with Q as usual so that $Q(x,\cdots,x) = Q(x)$, then the only z such that $Q(z, x_2,\cdots,x_n) = 0$ for all x_i is $z = 0$. Schafer's result, along with Theorem 3, can be used to give a complete determination of the nondegenerate Q which permit composition on a finite-dimensional algebra. Schafer conjectured that his conclusions are valid without the assumption of finiteness of dimensionality. In another paper [13] Schafer considered the problem of characterizing Jordan algebras by commutativity and the existence of a nondegenerate form Q satisfying the Jordan composition $Q(\{xyx\}) = Q(x)^2Q(y)$. He was able to treat this problem only for $\deg Q = 2$ and 3.

An extensive generalization of these results of Schafer's has been given by McCrimmon in [2]. He considers finite-dimensional algebras $\mathfrak{A}/\Phi$ and defines a form (homogeneous polynomial function of degree > 0) Q on $\mathfrak{A}$ to *admit composition* if there exists two rational mappings E annd F of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{A},\mathfrak{A})$ such that: 1. $E(1) = 1 = F(1)$, 2. $\Delta_1^a\ E = \alpha a_L$, $\Delta_1^a\ F = \beta a_R$ where α,β are nonzero elements of Φ and a_L and a_R are respectively the left and right multiplications by a, 3. $Q(aE_b) = Q(a)e(b)$, $Q(aF_b) = Q(a)f(b)$ for some rational functions e and f in $\mathfrak{A}$ (whenever the various functions are defined). A form Q is called *nondegenerate* if the corresponding *trace form* $\tau(a,b) = -\Delta_1^a\Delta^b \log Q$ is a nondegenerate symmetric bilinear form in a and b. If Q admits composition and the characteristic is 0 or a prime p exceeding the degree of Q, then this condition is equivalent to Schafer's conditions for nondegeneracy. With these definitions, McCrimmon has proved that if $\mathfrak{A}$ is a finite-dimensional algebra with 1 over an infinite field which possesses a nondegenerate form Q admitting composition, then $\mathfrak{A}$ is a separable noncommutative Jordan algebra (definition in ex. 8, p. 33) and $\mathfrak{A}^+$ is a separable (commutative) Jordan algebra. The converse is also valid as is shown by McCrimmon in [2] and Osborn in [7]. McCrimmon's theorem implies Schafer's theorem on composition in the usual sense with the improvement that the condition on the characteristic can be replaced by the assumption that the number of elements in the base field exceeds the degree of Q. McCrimmon's theorem yields also the generalization of Schafer's results on Jordan composition to forms of arbitrary degree.

In [8] McCrimmon generalized the foregoing results to algebras for which finiteness of dimensionality is not assumed and has obtained an affirmative answer to Schafer's conjecture noted above. In [10] the same author has given an extension of the theory of the generic minimum polynomial to strictly power associative algebras with 1 which may be infinite dimensional but are "generically algebraic" in a certain sense.

The basic tool in these papers of McCrimmon's is that of the differential calculus of rational mappings. This method was used systematically first by Koecher in applying the structure theory of Jordan algebras over the real field to determine the homogeneous domains of positivity and more general "ω-domains" (see Koecher [4] and also Vinberg [1]).

The following theorem has been proved by Meyberg in [3]: Let $\mathfrak{J}$ and $\mathfrak{J}'$ be finite-dimensional central simple Jordan algebras with the same underlying vector space of characteristic $\neq 3$ (and $\neq 2$) and assume that $\mathfrak{J}$ and $\mathfrak{J}'$ have the same structure group (equivalently, in view of Theorems 6.6 and 6.7, if $|\Phi|$ is large enough, the same groups of norm similarities), then $\mathfrak{J}$ and $\mathfrak{J}'$ are isotopes.

Chapter VII

It would be interesting to obtain proof of the main results of this chapter (the separability of $U(\mathfrak{J})$ for $\mathfrak{J}$ separable, the Albert-Penico-Taft theorem, Harris' theorem) without using the classification of simple Jordan algebras. For characteristic 0, such proofs are given in Chapter VIII. Perhaps it may be possible to transfer the results of this case to characteristic p by a method of reduction modulo p.

It is unknown at the present time whether or not anything like Theorem 6.16 is valid in the characteristic p case. In particular, the following question appears to be unsettled: Let $\mathfrak{J} = \mathfrak{S} \oplus \mathfrak{R}$ where $\mathfrak{S}$ is separable and $\mathfrak{R}$ is the radical. Let $\mathfrak{B}$ be a separable subalgebra. Then does there exist an automorphism η such that $\mathfrak{B}^{\eta} \subseteq \mathfrak{S}$? (Theorem 6.15 gives an affirmative answer in a special case.)

Chapter VIII

We shall indicate the group theoretic background for some of the Lie algebras considered in this chapter and the extensions of these notions to quadratic Jordan algebras (cf. Koecher [8] and McCrimmon [6]).

We consider first the structure group $\Gamma(\mathfrak{J})$. We defined this to be the group of isotopies of $\mathfrak{J}$ onto $\mathfrak{J}$ and we saw in §1.12 that $\Gamma(\mathfrak{J})$ is the set of bijective linear mappings α in $\mathfrak{J}$ for which there exists a linear mapping β such that $U_x\alpha = \beta U_{x^\alpha}$, $x \in \mathfrak{J}$. Then 1^α is invertible and $\beta = \alpha(U_{1^\alpha})^{-1}$. Hence the condition is:

$U_{x^\alpha} = U_{1^\alpha}\alpha^{-1} U_{x^\alpha}$ or, with a slight change of notation, we can define $\Gamma(\mathfrak{J})$ to be the set of bijective linear mappings W of $\mathfrak{J}$ such that

$$(1) \qquad U_{xW} = W^*U_xW, \qquad W^* = U_{1W}W^{-1}, \quad x \in \mathfrak{J}.$$

It follows that $W \in \Gamma(\mathfrak{J})$ if and only if W is a bijective linear mapping in $\mathfrak{J}$ such that there exists a bijective linear mapping W^* in $\mathfrak{J}$ satisfying $U_{xW} = W^*U_xW$. Then necessarily $W^* = U_{1W}W^{-1}$.

These considerations carry over to quadratic Jordan algebras $(\mathfrak{J}, U, 1)$ with 1 over an arbitrary commutative ring Φ with 1. We define the *structure group* $\Gamma(\mathfrak{J})$ to be the group of bijective Φ-endomorphisms W of $\mathfrak{J}$ for which there exists a bijective Φ-endormorphism W^* such that $U_{xW} = W^*U_xW$, $x \in \mathfrak{J}$. It is clear from the axiom 3 for quadratic Jordan algebras with 1 that if a is invertible, then $U_a \in \Gamma(\mathfrak{J})$. Also, $U_{1W} = W^*W$ is bijective, so $1W$ is invertible and $W^* = U_{1W}W^{-1} \in \Gamma(\mathfrak{J})$. It is immediate that $W \to W^*$ is an anti-automorphism in $\Gamma(\mathfrak{J})$ and $W^{**} = W$. If a is invertible and $W \in \Gamma(\mathfrak{J})$, then $U_{aW} = W^*U_aW$ is invertible. Hence aW is invertible in $\mathfrak{J}$ and $(aW)^{\cdot -1} = (aW)U_{aW}^{-1} = a^{\cdot -1}(W^*)^{-1}$. If $W \in \Gamma(\mathfrak{J})$ and $1W = 1$ then $W^* = W^{-1}$ and $U_xW = WU_{xW}$. This states that W is a homomorphism of $(\mathfrak{J}, U, 1)$ so W is an automorphism. Hence the group of automorphisms $\mathrm{Aut}\,\mathfrak{J} \subseteq \Gamma(\mathfrak{J})$ and $\mathrm{Aut}\,\mathfrak{J}$ is the subgroup of $\Gamma(\mathfrak{J})$ fixing the element 1. The subgroup $\Gamma_1(\mathfrak{J})$ of $\Gamma(\mathfrak{J})$ generated by the invertible U_a is called the *inner structure group*. The elements of $\Gamma_1(\mathfrak{J}) \cap \mathrm{Aut}\,\mathfrak{J}$ are called *inner automorphisms*.

Let $\mathfrak{D}$ be the algebra over Φ with basis $(1, t)$, where $t^2 = 1$ (cf. the proof of Theorem 2.1(7) on p. 67). Consider $(\mathfrak{J}_\mathfrak{D}, U, 1)$ and let $\mathfrak{G}(\mathfrak{J})$ be the set Φ of endomorphisms A of $\mathfrak{J}$ such that $1 + At \in \Gamma(\mathfrak{J}_\mathfrak{D})$. Here A is the $\mathfrak{D}$-endomorphism in $\mathfrak{J}_\mathfrak{D}$ extending the given A. The condition that $W = 1 + At \in \Gamma(\mathfrak{J}_\mathfrak{D})$ is that (1) holds for $x \in \mathfrak{J}$. Since $W^{-1} = 1 - At$, this gives the condition that $A \in \mathfrak{G}(\mathfrak{J})$ if and only if

$$(2) \qquad U_{x,xA} = U_{1,1A}U_x + [U_x, A], \qquad x \in \mathfrak{J}.$$

It follows from this, or directly from the definition, that $\mathfrak{G}(\mathfrak{J})$ is a Lie algebra (subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})^-$) which is restricted if $p\Phi = 0$ for a prime p (cf. Serre, *Lie Algebras and Lie Groups*, New York, 1965, p. L.A. 1.3). We call $\mathfrak{G}(\mathfrak{J})$ the *structure Lie algebra* of $(\mathfrak{J}, U, 1)$. It is easily seen that $A \in \mathfrak{G}(\mathfrak{J})$ if and only if there exists an $A' \in \mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ such that

$$(3) \qquad U_{x,xA} = A'U_x + U_xA, \qquad x \in \mathfrak{J}.$$

Then A' is uniquely determined and in fact, $A' = U_{1A} - A$. One can deduce from the axioms for quadratic Jordan algebras with 1 the identity $U_xV_{a,b} + V_{b,a}U_x = U_{x,xV_{a,b}}$ which shows that $V_{a,b} \in \mathfrak{G}(\mathfrak{J})$. Since $U_{1,1A} = V_{1,1A}$ we see that $A' \in \mathfrak{G}(\mathfrak{J})$. Hence $A \to A'$ is an anti-automorphism of the Lie algebra $\mathfrak{G}(\mathfrak{J})$ and $\varepsilon: A \to \bar{A} \equiv -A'$ is an automorphism satisfying $\varepsilon^2 = 1$. Also we have $\bar{V}_{a,b} = -V_{b,a}$.

If $A \in \mathfrak{G}(\mathfrak{J})$ and $W \in \Gamma(\mathfrak{J})$, then $W^{-1}(1 + At)W \in \Gamma(\mathfrak{J}_{\mathfrak{D}})$. Hence $W^{-1}AW \in \mathfrak{G}(\mathfrak{J})$. Thus $\Gamma(\mathfrak{J})$ acts on $\mathfrak{G}(\mathfrak{J})$ by means of $A \to W^{-1}AW$ and this is an automorphism of $\mathfrak{G}(\mathfrak{J})$. One can establish easily from the definitions that $(W^{-1}AW)' = W^*A'(W^{-1})^*$.

The defining condition (3) for $A \in \mathfrak{G}(\mathfrak{J})$ gives $U_{x,yA} + U_{y,xA} = A'U_{x,y} + U_{x,y}A$ which implies that $AV_{x,z} + V_{xA,z} = V_{x,zA'} + V_{x,z}A$. Hence $[V_{a,b}, A] = V_{aA,b} + V_{a,b\bar{A}}$. This implies that the subset $\mathfrak{L}(\mathfrak{J})$ of sums of the operators $V_{a,b}$ is an ideal in $\mathfrak{G}(\mathfrak{J})$. We shall call this ideal the *inner structure Lie algebra* of $(\mathfrak{J}, U, 1)$.

We define a *derivation* D of $(\mathfrak{J}, U, 1)$ to be a Φ-endomorphism such that $1D = 0$. and $[U_x, D] = U_{x,xD}$, $x \in \mathfrak{J}$. It is clear from (3) that $D \in \mathfrak{G}(\mathfrak{J})$ and $D' = -D$ or $\bar{D} = D$. The set of derivations Der $\mathfrak{J}$ is the subalgebra of $\mathfrak{G}(\mathfrak{J})$ of mappings satisfying $1D = 0$. This is clear from (2). The elements of $\mathfrak{L}(\mathfrak{J}) \cap \operatorname{Der}\mathfrak{J}$ are called *inner derivations*. Among these one has the *strictly inner derivations* $V_{a,b} - V_{b,a}$. The inner derivations and strictly inner derivations constitute ideals Inder $\mathfrak{J}$ and Strinder $\mathfrak{J}$ in Der $\mathfrak{J}$.

We consider next the extension of the Tits-Koecher construction of the Lie algebra $\mathfrak{R}(\mathfrak{J})$ to quadratic Jordan algebras. Since $V_{a,b} = 2a \triangle b$ if $\mathfrak{J}$ is a Jordan algebra and $V_{a,b}$ is defined as in $(\mathfrak{J}, U, 1)$ with $U_a = 2R_a^2 - R_{a^2}$, one is lead to define $\mathfrak{R}(\mathfrak{J}) = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$, where $\bar{\mathfrak{J}}$ is a copy of $\mathfrak{J}$ and $\mathfrak{L}(\mathfrak{J})$ is the inner structure Lie algebra. We define a product $[\ ,\ ]$ in $\mathfrak{R}(\mathfrak{J})$ as in (26) on p. 325 with the modification that $a \triangle b$ is replaced by $V_{a,b}$ (cf. Koecher [8]). As in ex. 1, p. 329, we can introduce the same bilinear product $[\ ,\ ]$ in $\mathfrak{F}(\mathfrak{J}) \equiv \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{G}(\mathfrak{J})$, $\mathfrak{G}(\mathfrak{J})$ the structure Lie algebra. The verification given in the text carries over to show that $\mathfrak{F}(\mathfrak{J})$ and $\mathfrak{R}(\mathfrak{J})$ are Lie algebras. We have the automorphism ε in $\mathfrak{F}(\mathfrak{J})$: $a + \bar{b} + A \to b + \bar{a} + \bar{A}$ (cf. p. 327) and $\varepsilon^2 = 1$.

We now return to Jordan algebras with 1 (over a field of characteristic $\neq 2$, though everything is valid for a commutative ring Φ containing $\frac{1}{2}$). Along with the Jordan algebra $\mathfrak{J}$ we consider the associated quadratic Jordan algebra $(\mathfrak{J}, U, 1)$, where $U_a = 2R_a^2 - R_{a^2}$. Then $V_{a,b} = 2a \triangle b = 2(R_{a.b} - [R_aR_b])$. A linear mapping D is a derivation of the Jordan algebra if and only if it is a derivation of $(\mathfrak{J}, U, 1)$. Since $\mathfrak{G}(\mathfrak{J})$ includes the $R_a = \frac{1}{2}V_{a,1}$, it is immediate that $\mathfrak{G}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \operatorname{Der}\mathfrak{J}$ and $\mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$.

We shall now prove a recent theorem of Koecher's which is an important addition to the theory of $\mathfrak{R}(\mathfrak{J})$ given in the text. This states that any derivation D of $\mathfrak{R}(\mathfrak{J})$ into $\mathfrak{F}(\mathfrak{J})$ can be extended to an inner derivation D of $\mathfrak{F}(\mathfrak{J})$ and every derivation of $\mathfrak{F}(\mathfrak{J})$ is inner. To be consistent with the text, we use the definitions given in the text rather than the modifications just indicated for quadratic Jordan algebras. We note first that if A is a linear transformation in $\mathfrak{J}$ such that there exists a linear transformation $\tilde{A}$ such that $[a \triangle b, A] = aA \triangle b + a \triangle b\tilde{A}$, then $A \in \mathfrak{G}(\mathfrak{J})$ and $\tilde{A} = \bar{A}$. This follows by retracing the steps of the argument given above to show that $\mathfrak{L}(\mathfrak{J})$ is an ideal in $\mathfrak{G}(\mathfrak{J})$. Now let D be a derivation of the ideal $\mathfrak{R}(\mathfrak{J})$ into $\mathfrak{F}(\mathfrak{J})$. Since $[a + \bar{b} + L, R_1] = a - \bar{b}$ we can subtract an inner derivation from D to obtain a derivation mapping R_1 into $\mathfrak{G}(\mathfrak{J})$. Hence we may assume $R_1 D \in \mathfrak{G}(\mathfrak{J})$. Since $[a, R_1] = a$ for $a \in \mathfrak{J}$, we have $[aD, R_1] + [a, R_1 D] = aD$, and since $[aD, R_1] \in \mathfrak{J} + \bar{\mathfrak{J}}$ and $R_1 D \in \mathfrak{G}(\mathfrak{J})$, so $[a, R_1 D] \in \mathfrak{J}$, we have $aD \in \mathfrak{J} + \bar{\mathfrak{J}}$. Hence we can write $aD = aD_1 + aD_2$, where D_1 and D_2 are linear mappings of $\mathfrak{J}$ into $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ respectively. Then $[a, R_1] = a$ gives $aD_1 + aD_2 = aD_1 - aD_2 + a(R_1 D)$. Since $aR_1 D \in \mathfrak{J}$, we obtain $D_2 = 0$, so $\mathfrak{J}D \subseteq \mathfrak{J}$. Similarly, $\bar{\mathfrak{J}}D \subseteq \bar{\mathfrak{J}}$. Then $[\mathfrak{J}\bar{\mathfrak{J}}]D \subseteq [\mathfrak{J}\bar{\mathfrak{J}}]$, and since $[\mathfrak{J}\bar{\mathfrak{J}}] = \mathfrak{L}$, we see that also $\mathfrak{L}D \subseteq \mathfrak{L}$. Let D_1 be the restriction of D to $\mathfrak{J}$ and write the restriction of D to $\bar{\mathfrak{J}}$ as $\bar{x} \to \overline{xD_2}$, $x \in \mathfrak{J}$. Let D_3 be the restriction of D to $\mathfrak{L}$. If $a \in \mathfrak{J}$ and $L \in \mathfrak{L}$, then $[a, L] = aL$ and $aLD_1 = [aD_1, L] + [a, LD_3] = aD_1 L + aLD_3$. Hence $LD_3 = [L, D_1]$. If $b \in \mathfrak{J}$, then $[a \triangle b, D_1] = (a \triangle b)D_3 = [a, \bar{b}]D = [aD_1, \bar{b}] + [a, \overline{bD_2}] = aD_1 \triangle b + a \triangle bD_2$. By the result noted at the beginning, we have $D_1 \in \mathfrak{G}(\mathfrak{J})$ and $D_2 = \bar{D}_1$. It follows that the derivation D coincides with the restrictions to $\mathfrak{K}(\mathfrak{J})$ of the inner derivation $X \to [X, D_1]$ determined by $D_1 \in \mathfrak{G}(\mathfrak{J})$. It follows easily that every derivation of $\mathfrak{F}(\mathfrak{J})$ is inner and that $\mathrm{Der}\,\mathfrak{F}(\mathfrak{J}) \cong \mathfrak{F}(\mathfrak{J}) \cong \mathrm{Der}\,\mathfrak{K}(\mathfrak{J})$.

We consider next the automorphisms of $\mathfrak{F}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$. We have already defined the automorphism ε: $a + \bar{b} + L \to b + \bar{a} + \bar{L}$ of $\mathfrak{F}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$. Let $W \in \Gamma(\mathfrak{J})$ the structure group and $A \in \mathfrak{G}(\mathfrak{J})$. Then $W^{-1}AW \in \mathfrak{G}(\mathfrak{J})$. The definition of $\Gamma(\mathfrak{J})$ implies that $W^{-1}(a \triangle b)W = aW \triangle b(W^*)^{-1}$. Hence $A \to W^{-1}AW$ is an automorphism of $\mathfrak{G}(\mathfrak{J})$ which maps $\mathfrak{L}(\mathfrak{J})$ into itself. Also, $a + \bar{b} + A \to aW + \overline{b(W^*)^{-1}} + W^{-1}AW$ is an automorphism α_W of $\mathfrak{F}(\mathfrak{J})$ which maps $\mathfrak{K}(\mathfrak{J})$ into itself. It can be shown that the α_W can be characterized as the automorphisms of $\mathfrak{F}(\mathfrak{J})$ ($\mathfrak{K}(\mathfrak{J})$) which map $\mathfrak{J}$ into itself. If $a \in \mathfrak{J}$, then $(\mathrm{ad}\, a)^3 = 0$ in $\mathfrak{F}(\mathfrak{J})$ (cf. ex. 4, p. 329). It follows that $\tau_a = \exp(\mathrm{ad}\, a)$ where $X \exp(\mathrm{ad}\, a)) = 1 + [Xa] + \frac{1}{2}[[Xa]a]$ is an automorphism in $\mathfrak{F}(\mathfrak{J})$ mapping $\mathfrak{K}(\mathfrak{J})$ into itself. The automorphism ε, α_W for $W \in \Gamma(\mathfrak{J}), \tau_a$ for $a \in \mathfrak{J}$ generate a subgroup $\mathrm{Aut}'\,\mathfrak{F}(\mathfrak{J})$ of the group of automorphisms of $\mathfrak{F}(\mathfrak{J})$. Relations between these generators have been considered by Koecher. There exist Jordan algebras $\mathfrak{J}$ such that $\mathrm{Aut}'\, \mathfrak{F}(\mathfrak{J}) \neq \mathrm{Aut}\, \mathfrak{F}(\mathfrak{J})$.

For finite-dimensional Jordan algebras, Koecher has defined in a recent paper [10] a group of rational mappings which is closely related to the foregoing groups. This is the group $\Xi(\mathfrak{J})$ generated by the structure group $\Gamma(\mathfrak{J})$, the translations $t_a: x \to x + a$ and the mapping $j: x \to -x^{\cdot -1}$ defined on the set of invertible elements. In general, the mappings of Ξ are rational defined on Zariski open subsets of $\mathfrak{J}$ and the composition is the usual composition $\circ$ (cf. §6.2). The Hua identity can be written in operator form as $j \circ t_a \circ j \circ t_{a^{\cdot -1}} \circ j \circ t_a = U_a$. This shows that the inner structure group $\Gamma_1(\mathfrak{J})$ is contained in the subgroup $\Xi_1(\mathfrak{J})$ generated by j and the translations. If T denotes the subgroup of translations, then it follows from Hua's identity that $\Xi = \Gamma \circ T \circ j \circ T \circ j \circ T$ (with obvious meaning). Koecher has given a beautiful characterization of the group Ξ by a differential equation such that a rational mapping is contained in Ξ if and only if it satisfies the equation. Using this, he has proved that Ξ is an algebraic group (not linear). Also he has studied the question of the uniqueness of the representation of an element according to the decomposition of Ξ as $\Gamma \circ T \circ j \circ T \circ j \circ T$. One obtains a linear representation of Ξ onto $\mathrm{Aut}'\mathfrak{F}(j)$ sending $W \to \alpha_W$, $t_a \to \tau_a$, $j \to \varepsilon$.

Chapter IX

A number of characterizations of finite-dimensional exceptional central simple Jordan algebras based on forms have been given. We have already noted Schafer's characterization in [13] by means of cubic forms satisfying the Jordan composition (see the notes on Chapter VI). Another important characterization is one due to Springer in [1]. In this, one assumes that $\mathfrak{J}$ is a finite-dimensional algebra with 1 over a field of characteristic $\neq 2, 3$ equipped with a quadratic form Q with non-degenerate associated bilinear form (x, y) such that (1) $Q(x^2) = Q(x)^2$ if $(x, 1) = 0$, (2) $(xy, z) = (x, yz)$, (3) $Q(1) = 3/2$. Then Springer has shown that $\mathfrak{J}$ is a central simple Jordan algebra of degree three with $Q(x) = \frac{1}{2}t(x^{\cdot 2})(x^{\cdot 2} = x^2)$ and conversely.

Another characterization based on cubic forms is one which was first suggested by Freudenthal (cf. [6]) and was established by Springer in [5] for finite-dimensional algebras over fields of characteristic $\neq 2, 3$. Recently, this has been extended by McCrimmon in [15] to quadratic Jordan algebras over arbitrary fields. The Springer-McCrimmon axioms are the following. Let $\mathfrak{J}$ be a finite-dimensional vector space over an arbitrary field Φ equipped with a cubic form n and a distinguished element $l \in \mathfrak{J}$ satisfying the following conditions:

1. $n(1) = 1$.
2. $t(x, y) = -\Delta_x^1 \Delta_y \log n$ is a nondegenerate symmetric bilinear form.
3. If $x^\#$ is defined by $t(x^\#, y) = \Delta_x^y n$, then $x^{\#\#} = n(x)x$.

Now define $x \times y = (x + y)^\# - x^\# - y^\#$ and

$$yU_x = t(x, y)x - x^\# \times y. \qquad (1)$$

Then McCrimmon has shown, using the techniques of the differential calculus of rational mappings, that $(\mathfrak{J}, U, 1)$ is a quadratic Jordan algebra with 1. (Actually, the hypothesis of finiteness of dimensionality is not essential. One requires only a formulation which permits the application of the differential calculus.) Let $(\mathfrak{D}, j)$ be a composition algebra over an arbitrary field Φ, $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ the space of 3×3 matrices over $\mathfrak{D}$ which are symmetric under the canonical involution $X \to \gamma^{-1}\bar{X}^t\gamma$, $\gamma = \mathrm{diag}\,\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_i \neq 0$ in Φ. Let $n(X)$ be defined as usual (equation (50), p. 232) and let 1 be the usual identity matrix. Then it can be shown that the conditions 1–3 are satisfied, so if U is defined by (1), then $(\mathfrak{H}(\mathfrak{D}_3, J_\gamma), U, 1)$ is a quadratic Jordan algebra with 1. The structure then defined is an isotope of a standard quadratic Jordan matrix algebra as defined in the notes on Chapter IV by an octonion algebra.

Similarly, one can extend Tits' constructions to quadratic Jordan algebras. We indicate only the first of these. Hence let $\mathfrak{A}$ be a central simple associative algebra (with 1) of degree three. Then the generic norm n on $\mathfrak{A}$ is a cubic form and

$t(a, b) = -\Delta_a^1 \Delta_b \log n$ is a nondegenerate symmetric bilinear form on $\mathfrak{A}$. Let $\mathfrak{J} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ a direct sum of three copies of $\mathfrak{A}$, $a \to a_i$ a linear isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_i$. Let μ be a nonzero element of Φ and define

$$n(x) = n(a) + \mu n(b) + \mu^{-1} n(c)$$

for $x = a_0 + b_1 + c_2$, $a,b,c \in \mathfrak{A}$. Then it can be shown that this and $1 = 1_0$ satisfy the condition 1–3 so (1) defines a quadratic Jordan algebra with 1. If the characteristic is not two, this coincides with the quadratic Jordan algebra defined by Tits.

There is an extensive literature on exceptional algebraic groups, Lie algebras and geometries connected with exceptional Jordan algebras. We indicate this briefly. Tae-Il Suh in [1] has proved that any isomorphism between the little projective groups of Moufang projective planes is induced by an isomorphism or correlation between the planes. His proof is based on a classification of involutions contained in the little projective group. Another method of obtaining Suh's result and an extension of this to the middle projective group has been given by Veldkamp in [2]. An extensive study of elliptic and hyperbolic Moufang planes, that is, the geometry of Moufang planes relative to certain types of polarities, has been made by Springer and Veldkamp in [1]. Recently, Springer and Veldkamp in [2] and Veldkamp in [3], [4] and [5] have developed a geometry of split Jordan algebras. This is again based on the elements of rank one.

Additional results on the groups of automorphisms and of norm preserving transformations of reduced exceptional simple Jordan algebras are given in Jacobson [26] and [27]. These are algebraic groups of types F_4 and E_6 respectively. Soda in [1] has studied the groups $\mathrm{Aut}\,\mathfrak{J} \mid \mathfrak{K}$, where $\mathfrak{J}$ is reduced simple exceptional and $\mathfrak{K}$ is a cubic subfield. These are exceptional simple algebraic groups of type D_4, which had been studied previously in a geometric fashion by Tits [3]. Soda's results are based in part on some given by Springer in [9].

In a series of papers [6]–[14], Freudenthal has studied certain real forms of the Lie algebras E_7 and E_8 and related these to symplectic and metasymplectic geometries. Certain forms of Lie algebras of types D_4 and E_6 have been studied respectively by Allen in [1] and by Ferrar in [1].

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SUBJECT INDEX

The theory of Jordan algebras has played important roles behind the scenes of several areas of mathematics. Jacobson's book has long been the definitive treatment of the subject. It covers foundational material, structure theory, and representation theory for Jordan algebras. Of course, there are immediate connections with Lie algebras, which Jacobson details in Chapter 8. Of particular continuing interest is the discussion of exceptional Jordan algebras, which serve to explain the exceptional Lie algebras and Lie groups.

Jordan algebras originally arose in the attempts by Jordan, von Neumann, and Wigner to formulate the foundations of quantum mechanics. They are still useful and important in modern mathematical physics, as well as in Lie theory, geometry, and certain areas of analysis.

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