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E. B. Dynkin

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Diffusions, Superdiffusions and Partial Differential Equations

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ABSTRACT. The subject of this book is connections between linear and semilinear differential equations and the corresponding Markov processes called diffusions and superdiffusions. Most of the book is devoted to a systematic presentation of the results obtained by the author and his collaborators since 1988. Many results obtained originally by using superdiffusions are extended in the book to more general equations by applying a combination of diffusions with purely analytic methods. Almost all chapters involve a mixture of probability and analysis.

For researchers and graduate students working in probability theory and theory of partial differential equations.

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Preface

Interactions between the theory of partial differential equations of elliptic and parabolic types and the theory of stochastic processes are beneficial for both probability theory and analysis. At the beginning, mostly analytic results were used by probabilists. More recently, analysts (and physicists) took inspiration from the probabilistic approach. Of course, the development of analysis in general and of theory of partial differential equations in particular, was motivated to a great extent by problems in physics. A difference between physics and probability is that the latter provides not only an intuition but also rigorous mathematical tools for proving theorems.

The subject of this book is connections between linear and semilinear differential equations and the corresponding Markov processes called diffusions and superdiffusions. A diffusion is a model of a random motion of a single particle. It is characterized by a second order elliptic differential operator L. A special case is the Brownian motion corresponding to the Laplacian Δ. A superdiffusion describes a random evolution of a cloud of particles. It is closely related to equations involving an operator $Lu - \psi(u)$. Here ψ belongs to a class of functions which contains, in particular, $\psi(u) = u^\alpha$ with $\alpha > 1$. Fundamental contributions to the analytic theory of equations

$$Lu = \psi(u) \tag{0.1}$$

and

$$\dot{u} + Lu = \psi(u) \tag{0.2}$$

were made by Keller, Osserman, Brezis and Strauss, Loewner and Nirenberg, Brezis and Véron, Baras and Pierre, Marcus and Véron.

A relation between the equation (0.1) and superdiffusions was established, first, by S. Watanabe. Dawson and Perkins obtained deep results on the path behavior of the super-Brownian motion. For applying a superdiffusion to partial differential equations it is insufficient to consider the mass distribution of a random cloud at fixed times t. A model of a superdiffusion as a system of exit measures from time-space open sets was developed in [**Dyn91c**], [**Dyn92**], [**Dyn93**]. In particular, a branching property and a Markov property of such system were established and used to investigate boundary value problems for semilinear equations. In the present book we deduce the entire theory of superdiffusion from these properties.

We use a combination of probabilistic and analytic tools to investigate positive solutions of equations (0.1) and (0.2). In particular, we study removable singularities of such solutions and a characterization of a solution by its trace on the boundary. These problems were investigated recently by a number of authors. Marcus and Véron used purely analytic methods. Le Gall, Dynkin and Kuznetsov combined probabilistic and analytic approach. Le Gall invented a new powerful probabilistic tool — a path-valued Markov process called the Brownian snake. In his pioneering work he used this tool to describe all solutions of the equation $\Delta u = u^2$ in a bounded smooth planar domain.

Most of the book is devoted to a systematic presentation (in a more general setting, with simplified proofs) of the results obtained since 1988 in a series of papers of Dynkin and Dynkin and Kuznetsov. Many results obtained originally by using superdiffusions are extended in the book to more general equations by applying a combination of diffusions with purely analytic methods. Almost all chapters involve a mixture of probability and analysis. Exceptions are Chapters **7** and **9** where the probability prevails and Chapter **13** where it is absent. Independently of the rest of the book, Chapter **7** can serve as an introduction to the Martin boundary theory for diffusions based on Hunt's ideas. A contribution to the theory of Markov processes is also a new form of the strong Markov property in a time inhomogeneous setting.

The theory of parabolic partial differential equations has a lot of similarities with the theory of elliptic equations. Many results on elliptic equations can be easily deduced from the results on parabolic equations. On the other hand, the analytic technique needed in the parabolic setting is more complicated and the most results are easier to describe in the elliptic case.

We consider a parabolic setting in Part 1 of the book. This is necessary for constructing our principal probabilistic model — branching exit Markov systems. Superprocesses (including superdiffusions) are treated as a special case of such systems. We discuss connections between linear parabolic differential equations and diffusions and between semilinear parabolic equations and superdiffusions. (Diffusions and superdiffusions in Part 1 are time inhomogeneous processes.)

In Part 2 we deal with elliptic differential equations and with time-homogeneous diffusions and superdiffusions. We apply, when it is possible, the results of Part 1. The most of Part 2 is devoted to the characterization of positive solutions of equation (0.1) by their traces on the boundary and to the study of the boundary singularities of such solutions (from both analytic and probabilistic point of view). Parabolic counterparts of these results are less complete. Some references to them can be found in bibliographical notes in which we describe the relation of the material presented in each chapter to the literature on the subject.

Chapter 1 is an informal introduction where we present some of the basic ideas and tools used in the rest of the book. We consider an elliptic setting and, to simplify the presentation, we restrict ourselves to a particular case of the Laplacian Δ (for L) and to the Brownian and super-Brownian motions instead of general diffusions and superdiffusions.

In the concluding chapter, we give a brief description of some results not included into the book. In particular, we describe briefly Le Gall's approach to superprocesses via random snakes (path-valued Markov processes). For a systematic presentation of this approach we refer to [**Le 99a**]. We do not touch some other important recent directions in the theory of measure-valued processes: the Fleming-Viot model, interactive measure-valued models... We refer on these subjects to Lecture Notes of Dawson [**Daw93**] and Perkins [**Per01**]. A wide range of topics is covered (mostly, in an expository form) in "An introduction to Superprocesses" by Etheridge [**Eth00**].

Appendix A and Appendix B contain a survey of basic facts about Markov processes, martingales and elliptic differential equations. A few open problems are suggested in the Epilogue.

I am grateful to S. E. Kuznetsov for many discussions which lead to the clarification of a number of points in the presentation. I am indebted to him for providing me his notes on relations between removable boundary singularities and the Poisson capacity. (They were used in the work on Chapter 13.) I am also indebted to P. J. Fitzsimmons for the notes on his approach to the construction of superprocesses (used in Chapter 4) and to J.-F. Le Gall whose comments helped to fill some gaps in the expository part of the book.

I take this opportunity to thank experts on PDEs who gladly advised me on the literature in their field. Especially important was the assistance of N. V. Krylov and V. G. Maz'ya.

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