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Centre de Recherches Mathématiques Proceedings

SCHOLAR—a Scientific Celebration Highlighting Open Lines of Arithmetic Research

Conference in Honour of M. Ram Murty's Mathematical Legacy on his 60th Birthday
October 15–17, 2013
Centre de Recherches Mathématiques,
Université de Montréal, Québec, Canada

A. C. Cojocaru
C. David
F. Pappalardi
Editors

American Mathematical Society
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M. Ram Murty

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Preface

1. Conference overview

"*There seem to be at least three layers of understanding: the first is* ***acquaintance*** *with syntax, words, and symbols; the second is* ***meaning****; and the third can be called* ***the meaning of meaning****. The means of entering the deeper layers are to* ***cogitate****,* ***research****,* ***reflect*** *what we know at the first layer.*" (M. Ram Murty, "The Art of Research", Colloquium Lecture, Queen's University at Kingston, Canada, 2001.)

This is how M. Ram Murty, Professor and Head of the Department of Mathematics and Statistics at Queen's University in Kingston, Canada, might sound when talking to a friend, a colleague, or a student.

A refined scholar, an intellectual with an unquenchable thirst for all facets of truth, scientific and philosophic, M. Ram Murty has had a profound impact on the development of number theory throughout the world. To honour his mathematical legacy, a conference focused on new research directions in number theory inspired by Murty's most significant achievements was organized by Alina Carmen Cojocaru (University of Illinois at Chicago, USA), Chantal David (Concordia University, Montréal, Canada), Hershy Kisilevsky (Concordia University, Montréal, Canada) and Francesco Pappalardi (Università degli Studi Roma Tre, Rome, Italy) at the Centre de Recherches Mathématiques in Montréal. The conference was titled *SCHOLAR—a Scientific Celebration Highlighting Open Lines of Arithmetic Research* and took place during October 15–17 of 2013.

The conference hosted about 100 participants representing several generations of research mathematicians, from beginning graduate students to emeritus professors. The speakers included some of the most prominent researchers in number theory from around the world: Henri Darmon (McGill University, Montréal, Canada); Étienne Fouvry (Université Paris-Sud, Orsay, France); John Friedlander (University of Toronto, Canada); Dorian Goldfeld (Columbia University, New York City, USA); Henryk Iwaniec (Rutgers University, New Brunswick, USA); Ernst Kani (Queen's University, Kingston, Canada); Wen-Ching Winnie Li (Pennsylvania State University, State College, USA); Kumar Murty (University of Toronto, Canada); Yiannis Petridis (University College London, UK); Carl Pomerance (Dartmouth College, Hanover, USA); Dinakar Ramakrishnan (California Institute of Technology, Pasadena, USA); Michael I. Rosen (Brown University, Providence, USA); N. Saradha (Tata Institute of Fundamental Research, Mumbai, India); Joseph Silverman (Brown University, Providence, USA); Cam Stewart (University of Waterloo, Canada); Dinesh S. Thakur (University of Rochester, USA); Yitang Zhang (University of New Hampshire, Durham, USA). Their presentations illustrated research of highest caliber on a broad spectrum of arithmetic topics: abelian varieties; function

field arithmetic; Galois representations; L-functions; modular forms; sieve methods; transcendental number theory. A special lecture was given by Yitang Zhang, who had just been awarded the Cole Prize for his contributions to the Twin Prime Conjecture. Zhang's lecture alone, titled "*A new method applying to small gaps between primes*", attracted more than 130 attendees (conference participants and others from the Montréal mathematical community). Associated with the meeting was also a distinguished lecture, "*The Sato–Tate Conjecture*", given by M. Ram Murty himself on October 18 within the CRM–ISM Colloquium Series.

During the conference, a celebratory dinner was held for all participants. A special guest was Cynthia Fekken, Professor of Psychology and Associate Vice-Principal (Research) at Queen's University, who travelled to the conference venue in order to personally convey Queen's University's respect and admiration for Murty's scholarly legacy. Heartfelt stories were shared by several conference participants who lauded Murty's exceptional contributions and commitment to research, education and service, both at his home institution and worldwide. The evening was filled with deep feelings of gratitude, respect and affection for M. Ram Murty and inspired the participants towards great pursuits.

The conference was generously supported by the Centre de Recherches Mathématiques, the Fields Institute for Research in Mathematical Sciences, the Number Theory Foundation, and the University of Montréal. It is culminating with the present volume, published in the CRM Proceedings Series together with the American Mathematical Society as part of the Contemporary Mathematics Series.

2. M. Ram Murty's mathematical legacy

2.1. Biography. Born on October 16, 1953, in Guntur, Andhra Pradesh, India, M. Ram Murty moved at the age of 8 with his family to Ottawa, Ontario, Canada, maintaining throughout his life tight connections to both his Indian and Canadian homes. He studied science and philosophy at Carleton University in Ottawa, obtaining a BSc degree in 1976, and studied advanced number theory and Indian philosophy in the Boston area in Massachusetts, USA, obtaining a PhD degree in mathematics from MIT in 1980, under the direction of Harold Stark.

As a young adult, Murty was profoundly influenced by mathematics books of Z.I. Borevich & I.R. Shafarevich, G.H. Hardy & E.M. Wright, S. Ramanujan, and E.C. Titchmarsh; by philosophical visions of Sri Aurobindo, Mahatma Gandhi, and Swami Vivekananda; and by mathematical mentors Dorian Goldfeld, Paul Erdős, Atle Selberg, Jean-Pierre Serre, and Harold Stark. These influences have formed a continuous thread throughout his life, setting the highest mathematical and ethical standards that we now associate with M. Ram Murty and paving the way to the numerous awards and honours that he has received. Among others: the Coxeter–James Prize (1998) and the Jeffrey–Williams Prize (2003) awarded by the Canadian Mathematical Society; the E. W. R. Steacie Fellowship (1991–1993) awarded by the National Sciences and Engineering Research Council of Canada; the Killam Research Fellowship (1998–2000) awarded by the Canada Council for the Arts; the Balaguer Prize (1996) awarded by the Ferran Sunyer i Balaguer Foundation in Spain; the elections as Fellow of the Royal Society of Canada (1990), the Fields Institute for Research in Mathematical Sciences (2003), the Indian National Academy of Sciences (2007), and the Indian National Science Academy (2008).

During 1982–1996, M. Ram Murty held a permanent academic position at McGill University in Montréal, Québec, Canada. In 1996, he took a permanent academic position at Queen's University at Kingston, Ontario, Canada, where he is now a distinguished Queen's Research Chair and is cross-appointed to the Department of Mathematics & Statistics and the Department of Philosophy. In addition, he holds adjunct academic positions at: the Harish-Chandra Research Institute in Allahabad, India; the Institute for Mathematical Sciences and the Chennai Mathematical Institute in Chennai, India; the Indian Institute of Technology and the Tata Institute for Fundamental Research in Mumbai, India.

2.2. Most significant research accomplishments. Over 35 years, M. Ram Murty has made fundamental contributions in the field of number theory, a few of which are summarized below.

2.2.1. *Applications of sieve methods.* M. Ram Murty's mathematical research started during his undergraduate studies and focused on classical problems in number theory. One such problem is the infinitude of primes in arithmetic progressions, which can be solved in an elementary way for *certain* arithmetic progressions. As a young student, Murty investigated what stays at the core of such proofs, opening the door to a vast world of mathematics in which ancient methods from the time of Eratosthenes play vigorously alongside current mathematical developments.

A celebrated problem originating in such basic questions about primes is Artin's Primitive Root Conjecture, formulated by E. Artin in 1927 and asserting that every integer $a \neq \pm 1$, which is not a perfect square, is a generator of the multiplicative group $\mathbb{F}_p^*$ for infinitely many primes p. In 1967, this conjecture was proven by C. Hooley under the Generalized Riemann Hypothesis (GRH), marking a breakthrough in the area. This was followed by R. Gupta and M. Ram Murty's own breakthrough, published in the paper *A remark on Artin's conjecture*, Inventiones Mathematicae 78, 1984, pp. 127–130. In this paper the authors showed the existence of a finite set of 13 numbers such that, for at least one of these numbers, Artin's Primitive Root Conjecture is true. Thanks to subsequent advances in sieve methods, in particular the Chen–Iwaniec switching and the celebrated theorem of E. Bombieri, J.B. Friedlander and H. Iwaniec about primes in arithmetic progressions with large moduli, the above result was improved to one about a set of 3 numbers by R.D. Heath-Brown in 1986.

Along the lines of investigating primes in arithmetic progressions for large moduli, in the paper *A variant of the Bombieri–Vinogradov Theorem*, CMS Proceedings, AMS, 1987, pp. 243–272, M. Ram Murty and V. Kumar Murty proved a variant of the Bombieri–Vinogradov Theorem in the context of algebraic number fields. This was consequently applied to improving known estimates of Hecke eigenvalues of automorphic L-functions.

Additionally, this work has had ramifications to other parts of mathematics, most notably to the classification of euclidean rings. Building on his prior work with D. Clark, R. Gupta, and V. Kumar Murty, in the joint paper *Euclidean rings of algebraic integers*, Canadian Journal of Mathematics 56, 2004, pp. 71–76, M. Ram Murty and his doctoral student M. Harper classified all euclidean rings arising from an algebraic number field whose ring of integers has unit rank greater than 3. In particular, an outcome of this work is that $\mathbb{Z}[\sqrt{14}]$ is euclidean, previously a long-standing conjecture in algebraic number theory.

More recently, together with his postdoctoral fellow K. Petersen, Murty showed that his results on variations of the Bombieri–Vinogradov Theorem can also be used to study subgroups of $\mathrm{PSL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a number field K.

In several other papers, M. Ram Murty and his co-authors have explored the potential for powerful applications of sieves which were little known or believed obsolete. In particular, these explorations have led to the revival of the sieve of Erathostenes and the development of the sieve of Turán as techniques leading to modern applications.

2.2.2. *Elliptic curves.* Elliptic curves play a fundamental role in pure mathematics, as well as in applied sciences such as data encryption and internet security. In several papers, M. Ram Murty and his collaborators have made progress on major conjectures about the reductions modulo primes of an elliptic curve, formulated by S. Lang and H. Trotter in the 1970s, and on conjectures about the rank of an elliptic curve, formulated by B. Birch and P. Swinnerton-Dyer in the 1960s.

One of the Lang-Trotter Conjectures may be viewed as an elliptic curve analogue of Artin's Primitive Root Conjecture, asserting that, for any elliptic curve $E/\mathbb{Q}$ with positive (arithmetic) rank and for any fixed point $P \in E(\mathbb{Q})$ of infinite order, the density of primes p for which the group $E(\mathbb{F}_p)$ is generated by the reduction of P modulo p exists.

In his PhD thesis at MIT, M. Ram Murty considered a relaxed version of this conjecture, that of proving the existence of infinitely many primes p for which the group $E(\mathbb{F}_p)$ is cyclic, after imposing the necessary (and sufficient) hypothesis that $\mathbb{Q}(E[2]) \neq \mathbb{Q}$. This problem had already been investigated in the late 1970s by J-P. Serre under GRH, in analogy with Hooley's conditional investigation of Artin's Primitive Root Conjecture. In his thesis, M. Ram Murty placed the Cyclicity Conjecture in a more conceptual abstract setting, coherent with that of Artin's Primitive Root Conjecture; he then provided a conditional proof of this generalized conjecture following the spirit of Serre's proof, and provided the first unconditional proof of the existence of infinitely many primes p for which $E(\mathbb{F}_p)$ is cyclic in the case of an elliptic curve $E/\mathbb{Q}$ with complex multiplication.

These results have paved the way to Murty's subsequent breakthroughs, obtained with R. Gupta in the papers *Primitive points on elliptic curves*, Compositio Mathematicae 58, 1986, pp. 13–44, and *Cyclicity and generation of points mod p on elliptic curves*, Inventiones Mathematicae 101, 1990, pp. 225–235, in which they proved additional strong results supporting the Lang-Trotter Primitive Point Conjecture.

Another one of the Lang-Trotter Conjectures investigated by M. Ram Murty concerns the supersingular primes of an elliptic curve E over $\mathbb{Q}$, that is, the primes p for which $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is an order in a quaternion algebra. It is conjectured that, provided E has no complex multiplication, the number of such primes $p < x$ behaves asymptotically as $c(E)\sqrt{x}/\log x$ for some constant $c(E)$, depending on E. The first breakthrough on this problem is that of N. Elkies, who proved the infinitude of such primes in 1987. Subsequently, É. Fouvry and M. Ram Murty established an unconditional lower bound for the number of supersingular $p < x$, still the best known so far, and established an average version of the conjecture over an arbitrary two-parameter family of elliptic curves. This latter result has set the stage for future

notable results in the area, leading to an entire current line of research focused on studying families of elliptic curves, and, more recently, of abelian varieties.

The reductions of an elliptic curve allow for an intimate interplay between the global and local properties of the curve. In this direction, the famous Birch and Swinnerton-Dyer Conjecture relates the arithmetic rank of an elliptic curve $E/\mathbb{Q}$ with its analytic rank, and predicts the finiteness of an important group associated to E, its Tate–Shafarevich group. In the late 1980s, V. Kolyvagin showed that, as predicted by the Birch and Swinnerton-Dyer Conjecture, the existence of a quadratic twist of the L-function of E with a simple zero at $s = 1$ implies the finiteness of the Tate–Shafarevich group of E. In their joint paper *Mean values of derivatives of modular L-series*, Annals of Mathematics 133, 1991, pp. 447–475, M. Ram Murty and V. Kumar Murty proved the existence of such a quadratic twist, setting an important milestone in the field of arithmetic geometry. This piece of work further motivated the writing of their monograph *Non-vanishing of L-functions and applications*, published by Birkhauser in 1995, which was awarded the Balaguer Prize and which has become a standard reference in the subject. It has also solidified the philosophy held by leading analytic number theorists that classical methods are indeed powerful and meaningful in current research, and it has inspired further significant advances in the area.

2.2.3. *Modular forms.* Companions to elliptic curves, modular forms stay at the core of the theory of automorphic representations. Since the beginning of his career, M. Ram Murty has contributed to the study of modular forms in significant ways.

In a series of papers, starting with the joint paper with R. Balasubramanian, *An Ω-theorem for Ramanujan's tau function*, Inventiones Mathematicae 68, 1982, pp. 241–252, M. Ram Murty investigated refinements of P. Deligne's results about the Ramanujan Conjecture on bounds for the Fourier coefficients of a modular form. Indeed, building on his work with Balasubramanian and on fundamental results by F. Shahidi on the symmetric fourth power L-function associated to automorphic forms, in the paper *Oscillations of Fourier coefficients of modular forms*, Mathematische Annalen 262, 1983, pp. 431–446, M. Ram Murty proved that, for a normalized Hecke eigenform f of integral weight k and level 1, the n-th Fourier coefficient satisfies

$$a_n = \Omega_{\pm}\left(n^{(k-1)/2}\exp\left(\frac{c\log n}{\log\log n}\right)\right)$$

for some constant $c > 0$. At the same time, in the paper *Prime divisors of Fourier coefficients of modular forms*, Duke Math. Journal 51, 1984, pp. 57–76, M. Ram Murty and V. Kumar Murty studied the normal order of the prime factors of a_n for a normalized cusp form of integral weight k for $\Gamma_0(N)$.

Such results have inspired important further research by Murty and by several other number theorists from a variety of branches (automorphic, geometric, probabilistic). In particular, they have been useful in shedding new light on the congruence number of a newform and on the degree conjecture for an elliptic curve, as also investigated by M. Ram Murty in the paper *Congruences between modular forms*, Analytic Number Theory, LMS Lecture Notes 247, 1997, pp. 313–320, and *Bounds for congruence primes*, Automorphic Forms, Automorphic Representations and Arithmetic, Proceedings of Symposia in Pure Mathematics 66, AMS, 1999, pp. 177–192.

In more recent work, M. Ram Murty has continued his investigations of Fourier coefficients of modular forms, establishing, with V. Kumar Murty, a part of a general conjecture of S. Lang and H. Trotter that for a Hecke eigenform of integral weight $k \geq 12$ and level 1, there are only finitely many Fourier coefficients taking on a fixed odd value, as well as establishing, with K. Sinha, a general equidistribution theorem for Hecke eigenvalues. This latter result originates in work of J-P. Serre from the 1990s and has applications to the study of the splitting of the Jacobian variety $J_0(N)$, and also to the study of the eigenvalues of adjacency matrices of regular graphs.

2.2.4. *Artin L-functions.* Artin L-functions play a fundamental role in the general study of primes, similar to the one played by Dirichlet L-functions in the study of primes in arithmetic progressions. A central open problem in their theory is the holomorphy of non-abelian Artin L-functions; this was also one of the motivating problems for the creation of the Langlands Program in the late 1960s. M. Ram Murty and his collaborators have brought important contributions to the study of these L-functions.

The generalization of Dirichlet's Theorem on primes in an arithmetic progression is the ubiquitous Chebotarev Density Theorem, proven by N. Chebotarev in the 1920s, and encompassing, among others, the non-abelian analogue of arithmetic progressions. Effective versions of this theorem were first proven by J. Lagarias and A. Odlyzko in the 1970s and have played a major role in many of the principal research advances in number theory, from work of G. Faltings to work of J-P. Serre. In the joint paper with V. Kumar Murty and N. Saradha, *Modular forms and the Chebotarev density theorem*, American Journal of Mathematics 110, 1988, pp. 253–281, the authors proved effective versions of the Chebotarev Density Theorem with sharp error terms, by relying on a deep understanding of the role played by the holomorphy of the Artin L-functions in the study of the distribution of Frobenius automorphisms. These results have led to important advances in the investigations of the reductions of an elliptic curve over $\mathbb{Q}$ and of the Fourier coefficients of modular forms. In particular, M. Ram Murty, V. Kumar Murty and N. Saradha proved that, for a non-CM Hecke eigenform of integral weight k and for an integer a, the number of primes $p < x$ for which the p-th Fourier coefficient a_p equals a is $O(x^{4/5}/(\log x)^{1/5})$ if $a \neq 0$, and $O(x^{3/4})$ if $a = 0$, under GRH. These are substantial improvements of prior results of Serre from 1981, and represent important progress towards the Lang–Trotter Conjectures on Frobenius traces.

Another important consequence of the holomorphy of non-abelian Artin L-functions is Dedekind's Conjecture that the quotient of Dedekind zeta functions $\zeta_L(s)/\zeta_K(s)$ of an extension of number fields L/K is entire. A. Raghuram and M. Ram Murty made progress towards this conjecture by proving the holomorphy of certain products of Artin L-functions and extending prior work by K. Uchida and R. van der Waall.

In a different direction, in 1989 A. Selberg proposed general conjectures about the factorization of Dirichlet series with functional equations and Euler products. In the semiexpository paper *Selberg's conjectures and Artin L-functions*, Bulletin of the American Mathematical Society 31, 1994, pp. 1–14, Murty proved that the Selberg conjectures imply the Holomorphy Conjecture for Artin L-functions, as well as the Langlands Reciprocity Conjecture. These connections provide a better understanding of the underlying core characteristics of general L-functions.

2.2.5. *Transcendental numbers.* It is a classical conjecture that Euler's constant,

$$\gamma := \lim_{x\to\infty} \sum_{n\le x} \frac{1}{n} - \log x,$$

is a transcendental number. The generalized Euler constants,

$$\gamma(a,q) := \lim_{x\to\infty} \sum_{\substack{n\le x \\ n\equiv a \,(\mathrm{mod}\ q)}} \frac{1}{n} - \frac{\log x}{q},$$

are also conjectured to be transcendental.

M. Ram Murty's most recent work has focused on these conjectures, leading to the development of a whole research program which is being carried out by M. Ram Murty himself, V. Kumar Murty, N. Saradha, and their postdoctoral fellows. Among the several interesting recent results in this area is the one by M. Ram Murty and N. Saradha, published in the paper *Euler–Lehmer constants and a conjecture of Erdős*, Journal of Number Theory 130, 2010, pp. 2671–2682, that there is at most one algebraic number among $\gamma(a,q)$ for $q \ge 2$ and $1 \le a < q$. Other recent interesting results by Sanoli Gun, M. Ram Murty, V. Kumar Murty, and Purusottam Rath, pertain to the transcendental nature of special values of class group L-functions and of zeroes of modular forms.

2.3. Mentoring, training and broader impacts. Throughout his career, M. Ram Murty has played a leadership role in higher education by training over 40 Master's and PhD students, by sponsoring and mentoring over 30 postdoctoral fellows, and by writing didactic monographs which describe, in accessible terms, frontiers of current research. Several of Murty's former students have embarked on solid independent research careers in academia, such as three of the organizers of this event (Cojocaru, David and Pappalardi). At the same time, Murty's books have steadily gained popularity among graduate students, bridging the gap between undergraduate curriculum and current research, and bridging cultural and economical gaps between students all over the world.

3. Acknowledgements

The editors express their most sincere thanks to the Centre de Recherches Mathématiques, the Fields Institute for Research in Mathematical Sciences, the Number Theory Foundation, and Concordia University for generously supporting the SCHOLAR conference. They express most sincere thanks to Galia Dafni, André Montpetit, Louis Pelletier, Mike Saitas, Christine M. Thivierge, and the anonymous referees for generously lending their time and expertise towards bringing the SCHOLAR conference and this volume to fruition.

A. C. Cojocaru
C. David
F. Pappalardi

The Centre de Recherches Mathématiques (CRM) was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM receives funding from the Natural Sciences and Engineering Research Council (Canada), the FRQNT (Québec), the NSF (USA), and its partner universities (Université de Montréal, McGill, UQAM, Concordia, Université Laval, Université de Sherbrooke and University of Ottawa).

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SCHOLAR—Arithmetic Research • Cojocaru et al., Editors

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This proceedings volume is representative of the broad spectrum of topics that were addressed at the conference, such as elliptic curves, function field arithmetic, Galois representations, L-functions, modular forms and automorphic forms, sieve methods, and transcendental number theory.

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