CONTEMPORARY MATHEMATICS

689

Knots, Links, Spatial Graphs, and Algebraic Invariants

AMS Special Session on
Algebraic and Combinatorial Structures in Knot Theory

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Spatial Graphs

October 24–25, 2015
California State University, Fullerton, CA

Erica Flapan
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Part I: Knot Theoretic Structures

The field of knot theory has come a long way since Kurt Reidemeister [**19**] wrote his groundbreaking book on the subject in 1932. For most of the last century, the focus of research in this area was dominated by classical knot theory, which studies simple closed curves in $\mathbb{R}^3$ or $\mathbb{S}^3$, perhaps equipped with an orientation or a framing. With the introduction of virtual knot theory in the late 1990's [**14**] came an explosion of combinatorial generalizations of knot theory that continues to this day. Each of these combinatorial theories focuses on equivalence classes of diagrams that are preserved under a set of Reidemeister-style moves. Examples include, but are not limited to, the following.

- *Virtual Knots* are defined in terms of knots in thickened orientable surfaces. Virtual knots and links have diagrams with extra *virtual* crossings representing genus in the supporting surface of the diagrams [**4**, **13**, **14**].
- *Twisted Virtual Knots* are an extension of virtual knots in which the supporting surface is allowed to be non-orientable [**1**].
- *Classical and Virtual Pseudoknots* are equivalence classes of classical or virtual knot diagrams that may be missing certain classical crossing information [**9**, **10**].
- *Flat Knots* are homotopy classes of virtual knots, equivalent to virtual knots modulo a classical crossing change move [**7**, **11**].
- *Free Knots* are equivalence classes of flat knots with an extra *virtualization* move [**16**].
- *Singular Knots* are rigid isotopy classes of knotted 4-regular graphs [**21**].
- *Handlebody-Knots* are ambient isotopy classes of handlebodies embedded in $\mathbb{R}^3$, and are equivalent to embeddings of trivalent graphs in $\mathbb{R}^3$ together with an extra move [**12**].
- *Marked Graph Diagrams* are knot diagrams with an extra crossing type representing knotted surfaces in $\mathbb{R}^4$ [**23**].
- *Ribbon Knots* are knots that can be represented by thin strips of paper folded flat in the plane [**15**].

Many of these combinatorial knot theories have associated algebraic structures such as *quandles*, *biquandles*, *racks*, etc., whose axioms are derived from the appropriate set of Reidemeister-type moves. We can think of these algebraic structures as *colorings* of the relevant diagrams, and then count homomorphisms between them to get integer-valued invariants. Such invariants can be enhanced with information from homology theories, such as quandle and biquandle homology [**2**, **3**], to obtain stronger invariants. Recently, such algebraic structures and their homologies have

been generalized to structures including *unital shelves* and *Yang-Baxter operators* [**17**].

The first part of this volume is devoted to current results on algebraic, combinatorial, and geometric structures associated with knot theory and its combinatorial generalizations. On the algebraic side, the volume includes the work of J. Prztycki on knot polynomial complexity, results of M. Elhamdadi and J. Kerr on quandle colorings, and a paper of E. Clark and M. Saito on quandle identities and homology. From the combinatorial and geometric domain, we include a paper by E. Denne, M Kamp, R. Terry, and C. Zhu's on folded ribbon knots, an article by H. Dye which generalizes classical checkerboard colorings to virtual knots and links, and a paper by M. Chrisman and A. Kaestner which connects virtual knot theory and classical knot theory by using geometric techniques.

Part II: Spatial Graph Theory

Spatial graph theory developed in the 1980's when topologists began using the tools of knot theory to study ambient isotopy classes of graphs embedded in $\mathbb{R}^3$ or S^3. Later, this area came to be known as *spatial graph theory* to distinguish it from the study of abstract graphs. Much of the current work in spatial graph theory can trace its roots back either to the ground breaking results of John Conway and Cameron Gordon on intrinsic knotting and linking of graphs in S^3 or to the topology of non-rigid molecules [**8**, **20**, **22**]. In particular, in 1983, Conway and Gordon [**6**] proved that every embedding of the complete graph K_6 in S^3 contains a non-trivial link and every embedding of the complete graph K_7 contains a non-trivial knot. Since such links and knots exist for every embedding of the graphs in $\mathbb{R}^3$, K_6 is said to be *intrinsically linked* and K_7 is said to be *intrinsically knotted*. Conway and Gordon's results sparked widespread interest in intrinsic linking and knotting more generally, leading to numerous results that have been obtained over the past 30 years and that continue at the present time.

Independent of Conway and Gordon's result, in 1986 Jonathan Simon [**20**] used techniques from knot theory to prove that the molecule known as the *molecular Möbius ladder* is not ambient isotopic to its mirror image, and hence cannot be chemically equivalent to its mirror image. This answered a question raised by the chemist David Walba [**22**] who first synthesized this molecule in 1983. Simon's work led to the use of spatial graph theory to study symmetries of non-rigid molecules more generally (see for example [**18**], [**5**]). Then in 1990, Claus Ernst and De Witt Sumners [**8**] introduced the *tangle theory of site specific recombination* to explain the behavior of the recombinase enzyme Tn3 Resolvase. This topological approach to the study of site specific recombination has blossomed into a fruitful ongoing collaboration between knot theorists, spatial graph theorists, and molecular biologists studying the behavior of DNA, RNA, proteins, and other biopolymers.

The second half of this volume is devoted to new results in spatial graph theory. It begins with a survey of recent developments in spatial graph theory and its applications written by E. Flapan, T. Mattman, B. Mellor, and R. Naimi. Subsequent papers include an article by T. Mattman, C. Morris, and J. Ryker which classifies all graphs with 9 vertices which are intrinsically knotted; an article by A. Kawauchi exploring the connection between trivalent graphs embedded in $\mathbb{R}^3$ and ribbon surface-links in $\mathbb{R}^4$; and a paper by T. Mattman and M. Pierce about

graphs with the property that deleting n or fewer vertices results in a planar graph. Finally, there are two articles which connect spatial graphs with some of the combinatorial knot theories discussed above. In particular, we present a paper by S. Nelson and A. Ishii relating biquandles and spatial trivalent graphs as well as a paper by L. Kauffman and A. Henrich exploring the connection between 4-valent rigid vertex graphs, pseudoknots, and singular knots.

We hope that the reader gets as much enjoyment out of reading the articles in both parts of this volume as was had in researching, presenting, and writing the articles.

Finally, we would like to acknowledge Sergei Gelfand, who encouraged us to compile this collection at the 2015 Fall Western Sectional Meeting.

Erica Flapan
Allison Henrich
Aaron Kaestner
Sam Nelson

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