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Moufang and commutant elements in magmas

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Abstract. The interplay of commutativity and associativity in Moufang loops is well known, e.g., the many associator identities and inner mapping identities involving commutant elements, especially those involving the exponent three. Previously [Comment. Math. Univ. Carolin. 57 (2016), pp. 555–565], we investigated all of this in the variety of Bol loops. Here, we investigate this in the much more general setting of arbitrary magmas. We also investigate the notion of Moufang elements in an arbitrary groupoid, thus generalizing prior work [J. Algebra Appl. 8 (2009), pp. 477–492]. Along the way, we introduce a number of new submagmas of any given magma.

1. Introduction and a note on Prover9

A set, Q, with a binary operation $\cdot$ is called a *groupoid*. A *magma* is a groupoid with a (necessarily) unique two-sided identity element, denoted by e. We usually write xy instead of $x \cdot y$, and reserve $\cdot$ to have lower priority than juxtaposition among factors to be multiplied; for instance, $x \cdot yz$ stands for $x(yz)$. A *loop* is a magma in which, in the equation $xy = z$, knowledge of any two of x, y and z specifies the third uniquely. Important varieties of loops include groups (associative loops) and Moufang loops—these are loops that satisfy the identity $(xy \cdot x)z = x(y \cdot xz)$.

The *commutant*, $C(Q)$, of a groupoid Q is the set of those elements which commute with each element in the groupoid. That is, $C(Q) = \{c : \forall x \in Q, cx = xc\}$. Even if Q is a (highly structured) loop, $C(Q)$ need not be a subgroupoid [**1**], [**2**]. We say that the set $\{a, b, c\}$ *associates* if the elements a, b, and c associate in any order, that is, if each of the following six identities holds: $ab \cdot c = a \cdot bc, ac \cdot b = a \cdot cb, ba \cdot c = b \cdot ac, bc \cdot a = b \cdot ca, ca \cdot b = c \cdot ab$, and $cb \cdot a = c \cdot ba$. For an element a in a magma Q, we say that *a has a two-sided inverse* if there is an $a^{-1} \in Q$ such that $aa^{-1} = a^{-1}a = e$. The next example shows that inverses need not be unique in magmas.

Example 1.1. In this example, both 2 and 3 are two-sided inverses of 2 (2 is also a commutant element).

1	2	3
2	1	1
3	1	1

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Our investigations were aided by the automated reasoning tool Prover9 [**3**] and by the finite model builder Mace4 [**4**]. Many authors simply use the Prover9 output file as the proof of a theorem; it is common practice to publish untranslated Prover9 proofs [**5**]. This is mathematically sound since the program can be made to output a simple *proof object*, which can be independently verified by a short `lisp` program. We have chosen to translate some, but not all, of the Prover9 proofs we generated for theorems in this paper. The proofs we have not translated are not particularly complicated, although some are hundreds of lines long. The input files for these proofs are also direct, and require no advanced techniques, (they require no "hints" techniques, for example). We have included one input file in the body of this paper (Theorem 2.7) as an illustrative example; the others are of similar setup. You may find Prover9 input files and output files (proofs) for each of the theorems that appears without proof in this paper, here: `http://euclid.nmu.edu/ jophilli/paper-supplements.html`.

2. Moufang elements and two new submagmas

In the variety of loops, each of the following four identities implies the other three:

$$(A): \quad z(xy \cdot z) = zx \cdot yz \qquad\qquad (C): \quad z(x \cdot zy) = (zx \cdot z)y$$
$$(B): \quad (z \cdot xy)z = zx \cdot yz \qquad\qquad (D): \quad (xz \cdot y)z = x(z \cdot yz)$$

A loop, L, that satisfies any one (hence, all four) of these identities is called a *Moufang loop*. Thus, there are 12 possible ways to "localize" the Moufang laws ($a \in L$ is a constant in each equation; x, and y are universally quantified):

$$(A2): \quad a(xy \cdot a) = ax \cdot ya \qquad\qquad (C2): \quad a(x \cdot ay) = (ax \cdot a)y$$
$$(A1x): \quad z(ay \cdot z) = za \cdot yz \qquad\qquad (C1x): \quad z(a \cdot zy) = (za \cdot z)y$$
$$(A1y): \quad z(xa \cdot z) = zx \cdot az \qquad\qquad (C1y): \quad z(x \cdot za) = (zx \cdot z)a$$

$$(B2): \quad (a \cdot xy)a = ax \cdot ya \qquad\qquad (D2): \quad (xa \cdot y)a = x(a \cdot ya)$$
$$(B1x): \quad (z \cdot ay)z = za \cdot yz \qquad\qquad (D1x): \quad (az \cdot y)z = a(z \cdot yz)$$
$$(B1y): \quad (z \cdot xa)z = zx \cdot az \qquad\qquad (D1y): \quad (xz \cdot a)z = x(z \cdot az)$$

We investigate these identities in the general settings of groupoids and magmas, rather than restricting our attention to loops. We will call an element a that satisfies the identity $(A1x)$ *an* $(A1x)$ *element*, and we use $(A1x)_Q$ to denote the set of all $(A1x)$ elements in a given groupoid, Q. The analogous definitions for the other 11 identities are now clear. $(A2)$ is equivalent to $(B2)$ in the variety of magmas; other than this trivial equivalence, no one of these identities implies any other in the variety of magmas (distinguishing examples are small and easy to generate with Mace4). Thus, for a given magma, these 12 different "local Moufang laws" axiomatize up to 11 different "Moufang subsets;" none of them has to be a submagma. Distinguishing examples are, again, small and easy to generate; the next two examples give a taste (we chose two examples because the equivalence of $(A2)$ and $(B2)$ set them apart, to some extent). See [**8**] for a full account of this in arbitrary loops.

EXAMPLE 2.1. In this example, 2 is an $(A2)$ element but not any of the 10 other Moufang elements. For example 2 is not an $(A1x)$ element since $3 \cdot ((2 \cdot 2) \cdot 3) = 3 \cdot (1 \cdot 3) = 3 \cdot 3 = 3 \neq 1 = 2 \cdot 2 = (3 \cdot 2) \cdot (2 \cdot 3)$.

1	2	3
2	1	2
3	2	3

EXAMPLE 2.2. In this example, 2 is an $(A1y)$ element but not any of the 11 other Moufang elements. For example 2 is not an $(A1x)$ element since $2 \cdot ((2 \cdot 1) \cdot 2) = 2 \cdot (2 \cdot 2) = 2 \cdot 3 = 2 \neq 3 = 3 \cdot 2 = (2 \cdot 2) \cdot (1 \cdot 2)$.

1	2	3
2	3	2
3	3	2

Clearly, then, the question of what to use as the definition of "Moufang element" is a nontrivial one. In loops, the best definition turns out to be the most algebraic one [**8**]. This definition is grounded in the algebraic notion of autotopism. A triple of bijections, (f, g, h) on a groupoid Q is called an *autotopism* if $\forall x, y \in Q$ we have $xf \cdot yg = (x \cdot y)h$; as you can see, we follow the convention of writing functions on the right of their arguments. It's easy to see that $(A2)$ and $(B2)$ can be given via autotopisms (we use the standard notation for the right and left translations: $xR(y) = yL(x) = xy$):

$$(A2) : (L(a), R(a), R(a)L(a)) \qquad (B2) : (L(a), R(a), L(a)R(a))$$

If the left translation $L(a)$ has an inverse, then $(C2)$ can also be given via an autotopism: $(C2) : (L(a)R(a), L(a)^{-1}, L(a))$. Similarly, if the right translation $R(a)$ has an inverse, then $(D2)$ can also be given via an autotopism: $(D2) : (R(a)^{-1}, R(a)L(a), R(a))$.

Thus, since $(A2)$ and $(B2)$ are equivalent, we see that there are three algebraically appealing possible definitions of Moufang element. We give each of them in the following definition (note the obvious analog with the left, middle, and right nuclei), which we recall from [**8**], but here, applied more broadly to groupoids.

DEFINITION 2.3. An element a in a groupoid Q is called

(1) a *left Moufang element* if for all $x, y \in Q, a(x \cdot ay) = (ax \cdot a)y$, *i.e.*, if a is (C2),
(2) a *middle Moufang element* if for all $x, y \in Q, a(xy \cdot a) = ax \cdot ya$, *i.e.*, if a is (A2),
(3) a *right Moufang element* if for all $x, y \in Q, (xa \cdot y)a = x(a \cdot ya)$, *i.e.*, if a is (D2).

We use more evocative notation for these three important subsets rather than the $(C2)_Q, (A2)_Q$ and $(D2)_Q$ introduced above—we denote the set of all left Moufang elements $\mathrm{M}_\lambda(Q)$; $\mathrm{M}_\mu(Q)$ and $\mathrm{M}_\rho(Q)$ are defined analogously; i.e., $\mathrm{M}_\lambda(Q) = (C2)_Q, \mathrm{M}_\mu(Q) = (A2)_Q$ and $\mathrm{M}_\rho(Q) = (D2)_Q$. We are now ready to give the definition of Moufang element.

DEFINITION 2.4. An element a in a groupoid Q is called a *Moufang element* if it is a left, a middle, and a right Moufang element. We call the set of all Moufang elements of Q the *Moufang core of* Q, and we denote it by $\mathrm{M}(Q)$; that is, $\mathrm{M}(Q) = \mathrm{M}_\lambda(Q) \cap \mathrm{M}_\mu(Q) \cap \mathrm{M}_\rho(Q)$.

We record the following useful lemma:

LEMMA 2.5. *Let Q be a magma, let $a \in Q$ be a Moufang element, let $b \in Q$ be a Moufang element with a two-sided inverse, and let $c \in C(Q)$ be a Moufang element with a two-sided inverse. Then*

(1) $a \cdot ax = a^2 \cdot x$
(2) $x \cdot a^2 = xa \cdot a$
(3) $ax \cdot a = a \cdot xa$
(4) $b^{-1} \cdot xb = b^{-1}x \cdot b$
(5) $b^{-1} \cdot bx = xb^{-1} \cdot b$
(6) $c(c \cdot xy) = cx \cdot cy$
(7) $cx \cdot c^{-1} = c \cdot c^{-1}x$
(8) $c^{-1}x = xc^{-1}$
(9) $c \cdot c^{-1}x = x$

PROOF. The proofs are direct, if a bit tedious. But they should "feel" familiar, at least to the reader who is familiar with Moufang loops—all but (6) is a simple consequence of diassociative, in Moufang loops, and (6) is a simple consequence of $(A2)$ and commutativity.

(1) $a \cdot ax = a(e \cdot ax) = (ae \cdot a)x = a^2 \cdot x$.
(2) dual to the the proof of (1).
(3) $ax \cdot a = (ax \cdot a)e = a(x \cdot ae) = a \cdot xa$.
(4) $b^{-1} \cdot xb = b^{-1}(bb^{-1} \cdot xb) = b^{-1} \cdot b(b^{-1}x \cdot b) = (b^{-1}b \cdot b^{-1}x)b = b^{-1}x \cdot b$.
(5) $b^{-1} \cdot bx = b^{-1}(bx \cdot b^{-1}b) = b^{-1} \cdot b(xb^{-1} \cdot b) = (b^{-1}b \cdot xb^{-1})b = xb^{-1} \cdot b$
(6) $c(c \cdot xy) = (c \cdot xy)c = cx \cdot yc = cx \cdot cy$.
(7) $cx \cdot c^{-1} = xc \cdot c^{-1} = (cc^{-1} \cdot xc)c^{-1} = (c \cdot c^{-1}x)c \cdot c^{-1} = c(c^{-1}x \cdot cc^{-1}) = c \cdot c^{-1}x$.
(8) $c^{-1}x = ec^{-1} \cdot x = (c^{-1}c \cdot c^{-1})x = c^{-1}(c \cdot c^{-1}x) = c^{-1}(cx \cdot c^{-1}) = cc^{-1} \cdot xc^{-1} = xc^{-1}$.
(9) $xc \cdot y = (cc^{-1} \cdot xc)y = (c \cdot c^{-1}x)c \cdot y = c(c^{-1}x \cdot cy) = c \cdot c^{-1}x(cc^{-1} \cdot cy) = c{\cdot}c^{-1}x(cc{\cdot}c^{-1}y) = c(c^{-1}x{\cdot}c(c{\cdot}c^{-1}y)) = (c{\cdot}c^{-1}x)c{\cdot}(c{\cdot}c^{-1}y) = xc{\cdot}(c{\cdot}c^{-1}y) = cx \cdot (c^{-1}y \cdot c) = c(x \cdot c^{-1}y) \cdot c = c \cdot c(x \cdot c^{-1}y) = cx \cdot (c \cdot c^{-1}y) = xc \cdot (c \cdot c^{-1}y)$ Cancellation gives the desired $y = (c \cdot c^{-1}y)$.

□

Since each identity in Lemma 2.5 is familiar from the variety of Moufang loops (and thus, easy to apply in subsequent proofs), we use these identities without comment in the balance of the paper.

The next theorem is a slight variant of Theorem 4.2 in [**9**].

THEOREM 2.6. *Let Q be an arbitrary magma in which each element has a two-sided inverse. If each element is also a $(C2)$ element (or a $(D2)$ element), then the magma is a loop.*

The proof of Theorem 2.6 may be found in [**9**]. The next theorem, together with the autotopic flavor of the definition suggests that ours is a good definition of Moufang element.

THEOREM 2.7. *Let Q be an arbitrary magma. The set of those Moufang core elements that also have two-sided inverses is a submagma of Q. Thus, by Theorem* 2.6, *it is a loop.*

PROOF. The untranslated proof that this set is a submagma may be found at the author's website, url given above. Here is the Prover9 input file for this theorem:

```
formulas(assumptions).
% identity element
0 * x = x.
x * 0 = x.

% C has an inverse
C * C' = 0.
C' * C = 0.

% C is a Moufang element
% A2
C * ((x * y) * C) = (C * x) * (y * C).
% C2
C * (x * (C * y)) = ((C * x) * C) * y.
% D2
((x * C) * y) * C = x * (C * (y * C)).

% A has an inverse
A * A' = 0.
A' * A = 0.

% A is a Moufang element
% A2
A * ((x * y) * A) = (A * x) * (y * A).
% C2
A * (x * (A * y)) = ((A * x) * A) * y.
% D2
((x * A) * y) * A = x * (A * (y * A)).
end_of_list.

formulas(goals).
% (C * A) has an inverse
(C * A) * (A' * C') = 0.
(A' * C') * (C * A) = 0.

% (C * A) is (C * A) Moufang element
% A2
(C * A) * ((x * y) * (C * A)) = ((C * A) * x) * (y * (C * A)).
% C2
(C * A) * (x * ((C * A) * y)) = (((C * A) * x) * (C * A)) * y.
% D2
((x * (C * A)) * y) * (C * A) = x * ((C * A) * (y * (C * A))).
end_of_list.
```

□

We illustrate Theorem 2.7 by the following example.

EXAMPLE 2.8. In this example, the subloop guaranteed by Theorem 2.7 is $\{1, 2\}$. The magma itself, though, is clearly not a loop. We note also that it is not commutative.

1	2	3	4
2	1	3	4
3	4	1	1
4	3	2	2

The following example shows that the existence of two-sided inverses is necessary in Theorem 2.7.

EXAMPLE 2.9. In this example, 2 and 3 are Moufang elements, neither has a two-sided inverse, and $3 \cdot 2$ is not a Moufang element, since, $((3 \cdot 2) \cdot 3) \cdot (2 \cdot (3 \cdot 2)) = (5 \cdot 3) \cdot (2 \cdot 5) = 4 \cdot 4 = 4 \neq 5 = 5 \cdot 5 = 5 \cdot (5 \cdot 5) = (3 \cdot 2) \cdot ((3 \cdot 2) \cdot (3 \cdot 2))$.

1	2	3	4	5
2	4	4	4	4
3	5	4	4	4
4	4	4	4	4
5	4	4	4	5

As we noted above, there are many other possible definitions for Moufang element. Here is a natural one.

DEFINITION 2.10. An element a in a magma Q is called a *totally Moufang element* if $a \in (A2) \cap (A1x) \cap (A1y) \cap (B2) \cap (B1x) \cap (B1y) \cap (C2) \cap (C1x) \cap (C1y) \cap (D2) \cap (D1x) \cap (D1y)$.

The next theorem gives a second submagma in an arbitrary magma.

THEOREM 2.11. *Let Q be a magma. Then the set of those totally Moufang elements that also have two-sided inverses is a submagma of Q. Thus, by Theorem* 2.6, *it is a loop.*

PROOF. The untranslated proof that this set is a submagma may be found at the author's website, url given above. □

The following example shows that the existence of two-sided inverses is necessary in Theorem 2.11.

EXAMPLE 2.12. In this example, 2 and 3 are totally Moufang elements, neither has a two-sided inverse, and $3 \cdot 2$ is not totally Moufang , since, $((3 \cdot 2) \cdot 4) \cdot (4 \cdot (3 \cdot 2)) = (4 \cdot 4) \cdot (4 \cdot 4) = 6 \cdot 6 = 4 \neq 5 = 4 \cdot 5 = 4 \cdot (6 \cdot 4) = (3 \cdot 2) \cdot ((4 \cdot 4) \cdot (3 \cdot 2))$.

1	2	3	4	5	6
2	5	5	5	5	5
3	4	5	5	5	5
4	5	5	6	5	5
5	5	5	5	5	5
6	5	5	5	5	4

Obviously, totally Moufang elements are also Moufang elements. The next example shows that Moufang elements need not be totally Moufang, i.e., the two submagmas we have defined can be distinct.

EXAMPLE 2.13. In this example, 2 is a Moufang element, but it does not satisfy any of the other eight local versional of the Moufang law. For example, $(3 \cdot 2) \cdot (1 \cdot 3) = 4 \cdot 3 = 2 \neq 1 = 3 \cdot 3 = 3 \cdot (2 \cdot 3) = 3 \cdot ((2 \cdot 1) \cdot 3)$; that is, 2 is not an $(A1x)$ element. The other seven are similar.

1	2	3	4
2	1	3	4
3	4	1	1
4	3	2	2

3. The left semimedial law

We recall the following basic fact about commutative Moufang loops [**7**]:

THEOREM 3.1. *The left semimedial law, $xx \cdot yz = xy \cdot xz$, axiomatizes—in the variety of loops—the (sub)variety of commutative Moufang loops.*

Theorem 3.1 generalizes, in various ways, to Moufang loops [**8**]. Our aim in this section is to generalize this theorem to the much broader setting of arbitrary magmas.We begin by noting the three local versions of the left semimedial law:

$$(LS2) : a^2 \cdot xy = ax \cdot ay \qquad (LSy) : x^2 \cdot ay = xa \cdot xy \qquad (LSz) : x^2 \cdot ya = xy \cdot xa$$

Note that $(LS2)$ can be given by an autotopism: $(L(a), L(a), L(a^2))$. Next, we recall the following theorem from [**7**], which is a generalization of Theorem 3.1 to arbitrary Moufang loops.

THEOREM 3.2 (Moufang version of Theorem 3.1). *Let L be a Moufang loop. Then $C(L) = (LS2)_L = (LSy)_L$.*

Note that Theorem 3.2 involves only two of the three localized versions of the left semimedial law. To see why, let L be an arbitrary Moufang loop with identity element e. Obviously e is a commutant element. If e is also an (LSz) element, then $x^2 \cdot y = x^2 \cdot ye = xy \cdot xe = xy \cdot x = x \cdot yx$. And so by cancellation, L is commutative. Thus, in any noncommutative Moufang loop, L, $C(L)$ and $(LSz)_L$ are distinct. Note that in this derivation we used both the left alternative law $(x^2 \cdot y = x \cdot xy)$ and the flexible law $(x \cdot yx = xy \cdot x)$, both of which hold in Moufang loops [**6**].

Here, then, is Theorem 3.2's generalization to arbitrary magmas.

THEOREM 3.3 (Magma version of Theorem 3.2). *Let Q be a magma. For a Moufang element $c \in Q$ with a two-sided inverse, the following are equivalent:*

(1) $c \in C(Q)$,
(2) *c is an $(LS2)$ element.*

PROOF. (1) $\Longrightarrow$ (2) First note that $c \cdot z(c \cdot xy) = (cz \cdot c)(xy) = (c \cdot cz)(xy)$. Now, let $z = e$ to obtain $c(c \cdot xy) = c^2 \cdot xy$. Thus, $c^2 \cdot xy = c(c \cdot xy) = c(xy \cdot c) = cx \cdot yc = cx \cdot cy$.

(2) $\implies$ (1). First note that $c \cdot cx = ce \cdot cx = cc \cdot ex = cc \cdot xe = cx \cdot ce = cx \cdot ec = c(xe \cdot c) = c \cdot xc$. Thus, setting $x = c^{-1}y$, we have $c(c \cdot c^{-1}y) = c(c^{-1}y \cdot c)$. Using this we obtain the desired: $xc = e \cdot xc = cc^{-1} \cdot xc = c(c^{-1}x \cdot c) = c(c \cdot c^{-1}x) = ce \cdot (c \cdot c^{-1}x) = c^2(e \cdot c^{-1}x) = c^2 \cdot c^{-1}x = cc^{-1} \cdot cx = e \cdot cx = cx$.

□

Note, that in the proof of (1) $\implies$ (2) in Theorem 3.3 we did not assume that c has a two-sided inverse. The next example shows that the existence of a two-sided inverse is necessary in (2) $\implies$ (1) in Theorem 3.3.

EXAMPLE 3.4. In this example, 2 is both an $(LS2)$ element and a Moufang element, but 2, which clearly does not have a two-sided inverse, is not a commutant element since $2 \cdot 3 = 2 \neq 3 = 3 \cdot 2$.

1	2	3
2	2	2
3	3	3

Note that Theorem 3.3 involves only one of the three localized versions of the left semimedial law, namely $(LS2)$, which is the one that can be expressed as an autotopism.

The next two examples show that $(LS2)$ is independent from (LSy) in this context.

EXAMPLE 3.5. In this example, 1 is a both Moufang element that is its own two-sided inverse and a commutant—hence, also an $(LS2)$—element, but $(2 \cdot 2) \cdot (1 \cdot 2) = 3 \cdot 2 = 2 \neq 1 = 2 \cdot 3 = (2 \cdot 1) \cdot (2 \cdot 2)$; i.e., 1 is not an (LSy) element.

1	2	3
2	3	1
3	2	1

EXAMPLE 3.6. In this example, 2 is both a Moufang element that is a two-sided inverse of itself and an (LSy) element, but $2 \cdot 3 = 4 \neq 3 = 3 \cdot 2$, and hence 2 is not a commutant—hence, also, not an $(LS2)$—element.

1	2	3	4
2	1	4	3
3	3	3	3
4	4	4	4

Next, we examine (LSz) in this context.

THEOREM 3.7. *Let Q be a magma, and let $c \in Q$ be a Moufang element with a two-sided inverse. If c is an (LSz) element, then it is also an $(LS2)$ element.*

PROOF. First note that $x \cdot c^2 = xc \cdot c = cc^{-1} \cdot (xc \cdot c) = c \cdot (c^{-1} \cdot xc)c = c \cdot (c^{-1}x \cdot c)c = (c \cdot c^{-1}x)c^2 = c^2(c^{-1}x \cdot c) = c \cdot c(c^{-1}x \cdot c) = c(cc^{-1} \cdot xc) = c \cdot xc$. Here, we used (LSz) in the sixth equality.

Next, we use this (in the third equality in what follows) in noting that: $cx = cx \cdot c^{-1}c = c(xc^{-1} \cdot c) = xc^{-1} \cdot c^2 = (xc^{-1} \cdot c)c = (c^{-1} \cdot cx)c = c^{-1}(cx \cdot c) = c^{-1}(c \cdot xc) = (c^{-1}c \cdot x)c = xc$. Thus, $c \in C(Q)$; hence, by Theorem 3.3 c is $(LS2)$.

□

The next example shows that the existence of a two-sided inverse is necessary in Theorem 3.7.

EXAMPLE 3.8. In this example, 2 is an (LSz) and a Moufang element, but 2, which clearly does not have a two-sided identity element, is not an $(LS2)$ element since $(2 \cdot 2) \cdot (3 \cdot 1) = 2 \cdot 3 = 3 \neq 2 = 3 \cdot 2 = (2 \cdot 3) \cdot (2 \cdot 1)$.

1	2	3
2	2	3
3	2	3

The next example shows that the converse of Theorem 3.7 does not hold.

EXAMPLE 3.9. In this example, 1 is a a Moufang element that is a two-sided inverse of itself and an $(LS2)$ element, but $(2 \cdot 2) \cdot (3 \cdot 1) = 1 \cdot 3 = 3 \neq 2 = 1 \cdot 2 = (2 \cdot 3) \cdot (2 \cdot 1)$, hence 1 is not an (LSz) element.

1	2	3
2	1	1
3	1	1

THEOREM 3.10. *Let Q be a magma. Then the set of those (LSz) elements that are both Moufang and have two-sided inverses is a submagma of Q. Thus, by Theorem* 2.6*, it is a loop, and by Theorem* 3.7 *it is contained in $C(Q)$.*

PROOF. The untranslated proof that this set is a submagma may be found at the author's website, url given above. □

The following example shows that the existence of two-sided inverses is necessary in Theorem 3.10.

EXAMPLE 3.11. In this example, 2 is an (LSz) and a Moufang element which clearly does not have a two-sided inverse, but $2 \cdot 2$ is not an (LSz) element, since, $(3 \cdot 3) \cdot (1 \cdot (2 \cdot 2)) = 5 \cdot (1 \cdot 4) = 5 \cdot 4 = 6 \neq 3 = 3 \cdot 6 = 3 \cdot (3 \cdot 4) = (3 \cdot 1) \cdot (3 \cdot (2 \cdot 2))$.

1	2	3	4	5	6
2	4	2	4	4	4
3	5	5	6	6	3
4	4	4	4	4	4
5	6	5	6	6	6
6	6	5	6	6	6

The next example is of a magma in which $(LSz)_Q$ is nontrivial and the containment in 3.10 is proper.

EXAMPLE 3.12. In this example, 3 is an (LSz) Moufang element that is its own two-sided inverse. Also, 2 is a commutant element that is not an (LSz) element, since, $(4 \cdot 5) \cdot (4 \cdot 2) = 4 \cdot 2 = 2 \neq 4 = 4 \cdot 5 = (4 \cdot 4) \cdot (5 \cdot 1)$. Hence, $1 < (LSz)_Q < C(Q)$.

1	2	3	4	5
2	2	2	2	5
3	2	1	4	5
4	2	4	4	4
5	5	5	4	5

The next example shows that the set of those (LSy) elements that are both Moufang and have two-sided inverses need not be a submagma of Q.

Example 3.13. In this example, 2 is a Moufang element, it is its own two-sided inverse, and it is an (LSy) element. But $2 \cdot 2$ is not an (LSy) element since, $(3 \cdot 3) \cdot ((2 \cdot 2) \cdot 4) = 1 \cdot (1 \cdot 4) = 1 \cdot 4 = 4 \neq 7 = 3 \cdot 6 = (3 \cdot 1) \cdot 6 = (3 \cdot (2 \cdot 2)) \cdot (3 \cdot 4)$.

1	2	3	4	5	6	7
2	1	5	4	3	6	7
3	5	1	6	2	7	4
4	4	4	4	4	4	4
5	3	2	7	1	4	6
6	7	4	4	4	4	4
7	6	4	4	4	4	4

The next two examples complete the analysis of the magma version of Theorem 3.2, by showing that (LSy) is independent from (LSz).

Example 3.14. In this example, 1 is both a Moufang element that is its own two-sided inverse and an (LSy) element, but $(2 \cdot 2) \cdot (3 \cdot 1) = 1 \cdot 3 = 3 \neq 2 = 3 \cdot 2 = (2 \cdot 3) \cdot (2 \cdot 1)$, hence 1 is not an (LSz) element.

1	2	3
2	1	3
3	2	1

Example 3.15. In this example, 1 is both a Moufang element that is its own two-sided inverse and an (LSz) element, but $(2 \cdot 1) \cdot (2 \cdot 2) = 2 \cdot 3 = 2 \neq 3 = 3 \cdot 2 = (2 \cdot 2) \cdot (1 \cdot 2)$, hence 1 is not an (LSy) element.

1	2	3
2	3	2
3	3	3

We end this section by summarizing all of this for totally Moufang elements; we leave the straightforward proof to the reader.

Theorem 3.16. *Let Q be a magma. For a totally Moufang element $c \in Q$ with a two-sided inverse, we have*

(1) *$c \in C(Q)$ if and only if $c \in (LS2)_Q$,*
(2) *if $c \in (LSz)_Q$, then $c \in (LS2)_Q$, and*
(3) *if $c \in (LS2)_Q$, then $c \in (LSy)_Q$.*

There are no other implications.

4. The middle semimedial law and a new submagma

Obviously, the results in the previous section dualize to results about the *right semimedial law*: $zy \cdot x^2 = zx \cdot yx$. In this brief section, we investigate the third, and final, semimedial law, namely, the *middle semimeidal law*: $xy \cdot zx = xz \cdot yx$. Clearly, the middle semimedial law has two local versions:

$$(MS1) : ax \cdot ya = ay \cdot xa \qquad (MS2) : xa \cdot yx = xy \cdot ax$$

We call an element that is both an $(MS1)$ and an $(MS2)$ element a *middle semimedial element.*

THEOREM 4.1. *Let Q be a magma. The set of those elements that have two-sided inverses and are both middle semimedial elements and Moufang elements forms a submagma of Q. Thus, by Theorem 2.6, it is a loop. This subloop is contained in $C(Q)$.*

PROOF. The untranslated proof that this set is a submagma may be found at the author's website, url given above. □

The following example illustrates that the existence of two-sided inverses is necessary in Theorem 4.1.

EXAMPLE 4.2. In this example, 2 and 3 are both middle semimedial and Moufang elements, but neither has a two-sided inverse. Note also that $2 \cdot 3$ is not middle semimedial since $((2 \cdot 3) \cdot 4) \cdot (1 \cdot (2 \cdot 3)) = (6 \cdot 4) \cdot (1 \cdot 6) = 6 \cdot 6 = 5 \neq 6 = 6 \cdot 4 = (6 \cdot 1) \cdot (4 \cdot 6) = ((2 \cdot 3) \cdot 1) \cdot (4 \cdot (2 \cdot 3))$.

1	2	3	4	5	6
2	5	6	5	5	5
3	6	5	5	5	5
4	5	5	1	5	4
5	5	5	5	5	5
6	5	5	6	5	5

5. A general setting for Moufang theorems

Since the exponent three plays a prominent role in this section, we begin by recording the following easy lemma:

LEMMA 5.1. *If a is a right or left Moufang element in a magma, then $a \cdot a^2 = a^2 \cdot a$, i.e., a^3 is unambiguous.*

PROOF. If a is a right Moufang element, we have $a \cdot a^2 = a(a \cdot ea) = a^2 \cdot e = a^2 = a$. If a is a left Moufang element, we have, $a^2 \cdot a = (ae \cdot a)a = a(e \cdot a^2) = a \cdot a^2$. □

In loops, we define the right and left inner mappings as follows:

$$R(x,y) = R(x)R(y)R(xy)^{-1}$$
$$L(x,y) = L(x)L(y)L(yx)^{-1}.$$

Next, we recall the following foundational theorem from [**7**] about commutative Moufang loops.

THEOREM 5.2. *Let L be a commutative Moufang loop. For all x, y and $z \in L$:*
(1) *If any one of x, y or z is a cube, then $\{x, y, z\}$ associates,*
(2) *$(x, y, z)^3$ vanishes, and*
(3) *$R(x, y)^3 = 1$.*

Like Theorem 3.1, Theorem 5.2 also generalizes, in various ways, to Moufang loops [**7**]. Our aim in this section is to generalize Theorem 5.2 to the much broader setting of arbitrary groupoids and magmas, thus emphasizing that these identities are not "about" Moufang loops at all, but rather, they are much more general results, about Moufang and commutant elements in arbitrary groupoids and magmas. As in the previous section, we preface each new result with the analogous, and aforementioned, theorem from Moufang loops that it generalizes. Thus, we recall from [**7**] the following generalization to Moufang loops of (1) in Theorem 5.2:

THEOREM 5.3 (First Moufang version of (1) in Theorem 5.2). *Let L be a Moufang loop, with $c \in C(L)$. Then c^3 is nuclear.*

Here, then, is this theorem's generalization to arbitrary groupoids.

THEOREM 5.4 (Groupoid version of Theorem 5.3). *Let Q be a groupoid, with $c \in C(Q)$. If c is a Moufang element, then c^3 is nuclear.*

PROOF. Since c is in the commutant, $c^2 \cdot c = c \cdot c^2$, i.e., c^3 is unambiguous. Next, note that $c^3 \cdot xy = (cc \cdot c)(xy) = c \cdot c(c \cdot xy) = c \cdot (c \cdot xy)c = c(cx \cdot yc) = c(cx \cdot cy) = (c \cdot cx)c \cdot y = c(c \cdot cx) \cdot y = c^3 x \cdot y$, i.e., c^3 is left nuclear. Right nuclearity is proven analogously. For middle nuclearity note that $x \cdot c^3 y = x \cdot c(c \cdot cy) = x \cdot c(c \cdot yc) = x \cdot (c \cdot cy)c = x \cdot c(cy \cdot c) = (xc \cdot cy)c = c(cx \cdot yc) = c \cdot (c \cdot xy)c = c \cdot c(c \cdot xy) = c^3 \cdot xy = c^3 x \cdot y = xc^3 \cdot y$. □

The next example is a nontrivial instance of Theorem 5.4.

EXAMPLE 5.5. In this example, 2 is a commutant element, and $2^3 = 2$ is also nuclear, as per the theorem. By inspection, this groupoid is neither commutative, nor is it a magma (hence, not a loop). Also note that $((1 \cdot 1) \cdot 1) \cdot 1 = (3 \cdot 1) \cdot 1 = 2 \cdot 1 = 1 \neq 3 = 1 \cdot 1 = 1 \cdot (1 \cdot 3) = 1 \cdot (1 \cdot (1 \cdot 1))$; similarly, the other three Moufang laws fail.

3	1	1
1	2	3
2	3	2

Next, we recall from [**7**] another generalization to Moufang loops of (1) in Theorem 5.2:

THEOREM 5.6 (Second Moufang version of (1) in Theorem 5.2). *Let L be a Moufang loop, with $c \in C(L)$. Then $\forall x, y \in L$, $\{c, x^3, y\}$ associates.*

Here, then, is this theorem's generalization to arbitrary magmas.

THEOREM 5.7 (Magma version of Theorem 5.6). *Let Q be a magma, and let $c \in C(L)$ be Moufang element with a two-sided inverse. If a is a Moufang element, then $\{c, a^3, x\}$ associates for every $x \in Q$.*

PROOF. First note that $a(a \cdot cx) = a^2 \cdot cx = c(c^{-1} \cdot a^2) \cdot cx = c(a^2 \cdot c^{-1}) = c(a^2 \cdot c^{-1}) \cdot cx = c(a \cdot ac^{-1}) \cdot xc = c((a \cdot ac^{-1})x \cdot c) = c(c \cdot (a \cdot ac^{-1})x) = c(a \cdot ac^{-1}) \cdot cx = c(a \cdot c^{-1}a) \cdot cx = c(ac^{-1} \cdot a) \cdot cx = c(c \cdot (ac^{-1} \cdot a)x)) = c(c \cdot a(c^{-1} \cdot ax)) = ca \cdot c(c^{-1} \cdot ax) = ca \cdot ax$.

We are now ready to prove the first of the six required identities. In what follows, the fourth equality is obtained by applying the identity derived in the previous paragraph. $a^3 \cdot cx = (a \cdot a^2)(cx) = (a^2 \cdot a)(cx) = a \cdot (a(a \cdot cx)) = a(ca \cdot ax) = a(ac \cdot ax) = (a \cdot ac)a \cdot x = a(ac \cdot a) \cdot x = a(ca \cdot a) \cdot x = (ac \cdot a^2)x = (ca \cdot a^2)x = ca(a \cdot ea) \cdot x = ((ca \cdot a)e \cdot a)x = (ca \cdot a)a \cdot x = c(a \cdot a^2) \cdot x = (a \cdot a^2)c \cdot x = a^3 c \cdot x$.

The five other identities are proved analogously; their untranslated Prover9 proofs may be found at the author's website, url given above.

□

The next example is a nontrivial instance of Theorem 5.7.

EXAMPLE 5.8. In this example, 2 and 3 are Moufang elements; 2 is also a commutant element that is its own inverse. By inspection, this magma is not commutative, nor is it a loop. Also note that, $((4 \cdot 3) \cdot 4) \cdot 5 = (4 \cdot 4) \cdot 5 = 4 \cdot 5 = 5 \neq 4 = 4 \cdot 3 = 4 \cdot (3 \cdot 5) = 4 \cdot (3 \cdot (4 \cdot 5))$; similarly, the other three Moufang laws fail. Finally, one may check that $(2 \cdot 3^3) \cdot x = 2 \cdot (3^3 \cdot x)$ for all x in this magma.

1	2	3	4	5
2	1	3	4	5
3	3	3	3	3
4	4	4	4	5
5	5	4	3	3

In loops, we define the *associator*, (x, y, z) of x, y, and z, as follows: $xy \cdot z = (x \cdot yz)(x, y, z)$. Next, we recall from [**7**] the following generalization to Moufang loops of (2) in Theorem 5.2:

THEOREM 5.9 (Moufang version of (2) in Theorem 5.2). *Let L be a Moufang loop, with $c \in C(L)$. Then $\forall x, y \in L$, each of $(c, x, y)^3, (x, c, y)^3$, and $(x, y, c)^3$ vanishes.*

Here, then, is this theorem's generalization to abritrary magmas.

THEOREM 5.10 (Magma version of Theorem 5.9). *Let Q be a magma with $c \in C(Q)$. If a, b and c are Moufang elements with inverses, then each of $(a, b, c)^3$, $(a, c, b)^3$, $(b, a, c)^3$, $(b, c, a)^3$, $(c, a, b)^3$ and $(c, b, a)^3$ vanishes.*

PROOF. By Theorem 2.7, a, b, and c are contained in the subloop of the Moufang core consisting of those core elements with two-sided inverses. Clearly, this subloop is itself a Moufang loop. Thus, by Theorem 2.2 in [**7**], the cubes of all six associators vanish. □

Next, we recall the following generalization from [**7**] to Moufang loops of (3) in Theorem 5.2:

THEOREM 5.11 (Moufang version of (3) in Theorem 5.2). *Let L be a Moufang loop, with $c \in C(L)$. Then $\forall x, y \in L$ we have $R(x, c)^3 = R(c, x)^3 = L(x, c)^3 = L(c, x)^3 = 1$.*

In magmas, the right and left inner mappings need not exist (since inverses of translations need not exist). Here, we're interested in triples a, b, c for which $cR(a, b)$ *does* exist. (Because our definition of Moufang element is symmetric, we do not need to analyze left inner mappings separately.) Explicitly, we'll say that $cR(a, b)$ *makes sense* if there is a $d \in Q$ such that $ca \cdot b = d \cdot ab$, i.e., $cR(a, b) = d$.

Here, then, is this theorem's generalization to abritrary magmas.

THEOREM 5.12 (Magma version of Theorem 5.11). *Let Q be a magma with $c \in C(Q)$. If a and c are Moufang elements with inverses and if $bR(a, c)$ makes sense for some $b \in Q$, then $bR(a, c)^3 = bR(c, a)^3 = b$.*

PROOF. The untranslated proof of this theorem may be found at the author's website, url given above. □

We conclude with an example of a nontrivial instance of Theorem 5.12.

EXAMPLE 5.13. In this example, 3 is a Moufang commutant element that is its own inverse. Clearly this magma is not a loop. Also, it is not Moufang, since $((2\cdot 1)\cdot 2)\cdot 2 = (2\cdot 2)\cdot 2 = 4\cdot 2 = 4 \neq 2 = 2\cdot 4 = 2\cdot(1\cdot 4) = 2\cdot(1\cdot(2\cdot 2))$. Finally, we note that $2R(3,3) = 1$ (i.e., it "makes sense"); and thus, $2R(3,3)^3 = 2$. Examples of the other equality is similar.

1	2	3	4
2	4	2	2
3	2	1	4
4	4	4	2

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