

A family of nonnormal double planes associated to hyperelliptic curves

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ABSTRACT. Let C be the affine hyperelliptic curve defined by $y^2 = g(x)$, where $g(x)$ is a polynomial of degree at least three over a field k . Starting with an example that originally appeared in an article by the author and F. DeMeyer, a nonnormal rational affine double plane $X \rightarrow \mathbb{A}_k^2$ is constructed together with a one-to-one homomorphism from the subgroup of torsion elements in the Picard group of C to the Brauer group of X . This construction is generalized to the situation where C is an arbitrary affine variety.

1. Introduction

One theme of these proceedings is the application of divisor class groups of algebraic curves, with particular emphasis on hyperelliptic curves. The motivation for the main theorem of this article is a construction that permits us to compute the Brauer group invariant for a large class of algebraic surfaces in terms of the Picard group of certain hyperelliptic curves. This example, which originally appeared in [5], shows that given an affine hyperelliptic curve C over a field k defined by an equation of the form $y^2 = g(x)$, there is an integral nonnormal rational affine double plane $X \rightarrow \mathbb{A}_k^2$ together with an embedding $\text{tors}(\text{Pic}(C)) \rightarrow \text{B}(X)$ from the subgroup of torsion elements in the Picard group of C to the Brauer group of X . See Example 2.3 below for the construction of C and X , where it is shown that the affine coordinate ring $\mathcal{O}(X)$ can be embedded as a subring of $k[x, y]$. The Azumaya $\mathcal{O}(X)$ -algebras are described by pullbacks of matrix algebras. Hence, the existence of curves C such that $\text{Pic}(C)$ contains nontrivial torsion provides examples of subrings of $k[x, y]$ which have nontrivial Azumaya algebras.

Before continuing we review for the benefit of the reader the definitions of the Brauer group and Picard group of a commutative ring R . For an introduction to these topics, the reader is referred to [7]. If A is an R -algebra and A^o denotes the opposite ring, then A can be made into a left $A \otimes_R A^o$ -module by the action $a \otimes b \cdot x = axb$. By definition, the algebra A is R -separable if A is projective as a left $A \otimes_R A^o$ -module. If A is both central and separable, then A is called an Azumaya R -algebra. The Brauer group of R , denoted $\text{B}(R)$, parametrizes the Azumaya R -algebras up to Brauer equivalence. The importance of the Brauer group invariant lies in its subtle connections to the algebraic, arithmetic, and topological properties of R . The Picard group of R , written $\text{Pic}(R)$, parametrizes the projective R -modules of

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constant rank one. When R is an integrally closed noetherian integral domain, the Picard group can be identified with a subgroup of the class group of Weil divisors, $\text{Cl}(R)$. If R is a regular noetherian integral domain, then $\text{Pic}(R) \rightarrow \text{Cl}(R)$ is an isomorphism (see, for example, [7, Theorem 6.5.11]). If R is not regular, then from a computational point of view, the Picard group tends to be more elusive than the class group. It is fair to say that the family of rings for which the two invariants $B(R)$, $\text{Pic}(R)$ are either known or not difficult to compute is quite small. In the last fifty years much progress has been made on the problem of getting a better understanding of these two invariants, especially when R is the coordinate ring of an algebraic variety of low dimension. Curves and surfaces receive a lot of attention because in many cases the computations can be completely carried out.

As stated above, Example 2.3 below is an elaboration of an example from [5] which exhibits a class of algebraic surfaces for which the Brauer group invariant is computed in terms of the Picard group of certain hyperelliptic curves. To carry out this computation, the connecting homomorphism in a Mayer-Vietoris sequence is employed. This is but one example appearing in the literature that relates the group of divisors on a hyperelliptic curve to the Brauer group of an algebraic surface. We take the opportunity here to mention a few more. This is not intended to be an exhaustive list. In [3] an algebraic surface X is constructed such that X contains a nonrational singular point P corresponding to the blowing-down of a nonsingular hyperelliptic curve C on a rational surface Y . It is shown that the Brauer group of the local ring $\mathcal{O}_{X,P}$ contains a subgroup corresponding to the subgroup of torsion elements in the class group $\text{Cl}(C)$. A similar but slightly more involved computation is the subject of [8]. In [6] the Brauer groups of two classes of double planes $\pi : X \rightarrow \mathbb{A}^2$ related to hyperelliptic curves C are studied. In the first case, the surface X is a ruled surface of the form $C \times \mathbb{A}^1$. In the second case, the ramification locus of π is a hyperelliptic curve C . In both sets of examples, the class groups of C and X are exploited to construct Azumaya algebras on X as well as on the open complements on X and \mathbb{A}^2 of the ramification locus of π . Finally, if $C \subseteq \mathbb{P}^2$ is a nonsingular plane hyperelliptic curve and $U = \mathbb{P}^2 - C$ is the open complement, then the Brauer group of U is the subject of the papers [10], [11], and [16]. In this example, torsion elements in $\text{Cl}(C)$ give rise to nontrivial Azumaya $\mathcal{O}(U)$ -algebras.

Of the examples mentioned in the previous paragraph, only the double plane X associated to a hyperelliptic curve C that arises in Example 2.3 is nonnormal. In all of the other instances, the surface is either nonsingular or normal. For a nonnormal algebraic variety X , relatively few examples exist for which $B(X)$ is known. Comparatively little is known about the structure of an Azumaya $\mathcal{O}(X)$ -algebra, if X is nonnormal. The main theorem of this article (Theorem 3.1) shows that for an arbitrary algebraic variety C there is associated a variety X such that $B(X)$ can be computed in terms of the torsion subgroup of $\text{Pic}(C)$. Here C is not assumed to be irreducible, and in general the variety X that we construct is integral but nonnormal. The Azumaya algebras on X are described in terms of pullback diagrams involving rings of matrices. Hence, Theorem 3.1 allows us to exhibit a large class of nonnormal algebraic varieties X for which the Brauer group is nontrivial and moreover Azumaya $\mathcal{O}(X)$ -algebras are presented that generate $B(X)$. The Picard group of X is described in Proposition 3.3.

2. Background material

As a general reference for all unexplained notation and terminology, [7] is recommended. Because all of the examples constructed below involve rings that are defined as pullbacks, we begin with a quick review of the necessary properties associated to cartesian squares of rings. Let

$$(2.1) \quad \begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ h_1 \downarrow & & \downarrow f_2 \\ A_1 & \xrightarrow{f_1} & A_3 \end{array}$$

be a commutative diagram of homomorphisms of rings. Then (2.1) is called a *cartesian square* if $A \cong \{(x_1, x_2) \in A_1 \oplus A_2 \mid f_1(x_1) = f_2(x_2)\}$ and h_1, h_2 are induced by the coordinate projections. In this case we say A is the *pullback* of f_1 and f_2 . For example, if B is a ring, I a two-sided ideal of B , and $A \subseteq B$ a subring such that $I \subseteq A$, then

$$(2.2) \quad \begin{array}{ccc} A & \xrightarrow{h_2} & A/I \\ h_1 \downarrow & & \downarrow f_2 \\ B & \xrightarrow{f_1} & B/I \end{array}$$

is a cartesian square where all of the maps are the natural maps (see [7, Exercise 1.4.20]). Propositions 2.1 and 2.2 contain some useful facts about finitely generated k -algebras and rings that are defined by pullback diagrams.

PROPOSITION 2.1. *Let*

$$\begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ h_1 \downarrow & & \downarrow f_2 \\ A_1 & \xrightarrow{f_1} & A_3 \end{array}$$

be a cartesian square of rings.

- (1) *The kernel of h_2 is isomorphic to the kernel of f_1 .*
- (2) *If f_1 is onto, then h_2 is onto.*
- (3) *If f_1 is onto, and f_2 makes A_3 into a finitely generated left A_2 -module, then h_1 makes A_1 into a finitely generated left A -module.*

PROOF. Parts (1) and (2) are left to the reader (see [7, Exercise 1.4.18]).

(3): Since f_1 is onto, it follows that h_2 is onto. Hence A_3 is finitely generated as an A -module. Let $\{z_1, \dots, z_n\}$ be a subset of A_1 such that $\{f_1(z_1), \dots, f_1(z_n)\}$ is a set of generators for A_3 as an A -module. Then $A_1 = Az_1 + \dots + Az_n + I$, where I denotes the kernel of f_1 . Identify A with $\{(a_1, a_2) \in A_1 \oplus A_2 \mid f_1(a_1) = f_2(a_2)\}$. Then $I \oplus (0)$ is a subset of A . Hence I is in the image of h_1 . This proves A_1 is generated as an A -module by $z_1, \dots, z_n, 1$. \square

A proof of Proposition 2.2 can be found in [1, Proposition 7.8].

PROPOSITION 2.2. *Let k be a commutative noetherian ring and B a finitely generated commutative k -algebra. Let A be a k -subalgebra of B with the property that B is a finitely generated A -module. Then A is a finitely generated k -algebra.*

EXAMPLE 2.3. As mentioned in Section 1, we present here an example that originally appeared in [5] which associates to an affine hyperelliptic curve C an affine double plane X . This example is also explored in [7, Section 14.3.3]. Both of these references have the added hypothesis that the ground field k is algebraically closed and has characteristic zero. In the following, these restrictions on k are relaxed. This example provides the motivation for Theorem 3.1 below.

Before continuing, we remark that the terminology “hyperelliptic curve” appearing in the title of this article is applied in a very loose sense. It will describe any affine plane curve defined by an equation of the form $y^2 = g(x)$. This includes the possibilities that C is reducible, or nonnormal. Therefore the genus of an irreducible component of C can be any nonnegative integer.

Let k be any field of characteristic p , where $p = 0$ is allowed. Let $g(x)$ be any polynomial in $k[x]$. The hyperelliptic curve mentioned in the title of this article is the affine plane curve C defined by $y^2 - g(x) = 0$. Let

$$S = k[x, y^2, y(y^2 - g(x))]$$

which is a subring of the polynomial ring $k[x, y]$. The double plane mentioned in the title that is associated to C is the surface X with affine coordinate ring S . To realize X as an affine double plane, define a k -algebra homomorphism

$$k[u, v, w] \xrightarrow{\theta} S = k[x, y^2, y(y^2 - g(x))]$$

by $u \mapsto x, v \mapsto y^2, w \mapsto y(y^2 - g(x))$. Then θ is onto, and one checks that the kernel of θ is the principal ideal generated by $w^2 - v(v - g(u))^2$. Thus, S is isomorphic to $k[u, v, w]/(w^2 - v(v - g(u))^2)$ which is a ramified quadratic extension of $k[u, v]$. Hence, S is a quadratic extension of $k[x, y^2]$.

If we set $\bar{S} = k[x, y]$, then \bar{S} is integral over S and the quotient field of S is $k(x, y)$. It follows that \bar{S} is the integral closure of S in $k(x, y)$. Let

$$c = S : \bar{S} = \{\alpha \in \bar{S} \mid \alpha \bar{S} \subseteq S\}$$

be the conductor ideal from \bar{S} to S . One computes that $c = (y^2 - g(x), y(y^2 - g(x)))$. Notice that c is an ideal of \bar{S} contained in S . As an ideal in \bar{S} we have $c = (y^2 - g(x))$. Then $S/c \cong k[x]$ and $\bar{S}/c \cong k[x, y]/(y^2 - g(x))$. As in Eq. (2.2) there is a cartesian square

$$(2.3) \quad \begin{array}{ccc} S & \xrightarrow{h_2} & k[x] \\ h_1 \downarrow & & \downarrow \phi_2 \\ k[x, y] & \xrightarrow{\phi_1} & k[x, y]/(y^2 - g(x)) = \mathcal{O}(C) \end{array}$$

of rings with ϕ_1 onto and ϕ_2 one-to-one. We are in the context of Theorem 3.1 below from which we conclude that there is an exact sequence of abelian groups

$$(2.4) \quad 0 \rightarrow \text{tors}(\text{Pic}(C)) \xrightarrow{\partial_1} \text{B}(S) \rightarrow \text{B}(k[x, y]) \oplus \text{B}(k[x])$$

and a sequence of abelian groups

$$(2.5) \quad 0 \rightarrow \text{tors}(\text{Pic}(C)) \xrightarrow{\partial_1} \text{B}(S) \rightarrow \text{B}(k) \rightarrow 0$$

which is split exact if $p = 0$ and split exact modulo p -torsion subgroups if $p > 0$. For instance, if the degree of $g(x)$ is at least three and C is nonsingular, the group $\text{tors}(\text{Pic}(C))$ corresponds to a subgroup of the jacobian variety of C . Since the jacobian variety is an abelian variety (see [14]), for any $n \geq 2$ that is invertible in

k , there is a finite extension field F/k such that $\text{Pic}(C \times F)$ contains an element of order n . Upon extension of the base field of X to F , it follows that the Brauer group $B(X \times F)$ also contains an element of order n . Thus Eq. (2.4) shows that $B(S)$ is nontrivial for a sufficiently general choice of $g(x)$. The reader is referred to [2], [4] and other articles in these proceedings for a more specific treatment of the subject of torsion in $\text{Cl}(C)$. For an example when C is not irreducible, see Example 4.3.

3. The main theorem

The proof of Theorem 3.1 is based on the Mayer-Vietoris sequence for Brauer groups that was constructed by Knus and Ojanguren (see [7, Section 14.2]). The cartesian square of rings appearing in Theorem 3.1 is a generalization of Eq. (2.3). Thus, Theorem 3.1 is a generalization of Example 2.3 to the situation where the affine coordinate ring of C is an arbitrary finitely generated commutative k -algebra.

THEOREM 3.1. *Let k be a field of characteristic p , where $p = 0$ is allowed. Let R be a finitely generated commutative k -algebra of Krull dimension $m \geq 0$. Let $r > 0$ and*

$$0 \rightarrow I \rightarrow k[u_1, \dots, u_r] \xrightarrow{f_1} R \rightarrow 0$$

any finite presentation of R . Let $V = \{v_1, \dots, v_m\}$ be a set of indeterminates, X any subset of V , and $f_2 : k[X] \rightarrow R$ any one-to-one homomorphism of k -algebras. Consider the cartesian square of commutative k -algebras

$$\begin{array}{ccc} S & \xrightarrow{h_2} & k[X] \\ h_1 \downarrow & & \downarrow f_2 \\ k[u_1, \dots, u_r] & \xrightarrow{f_1} & R \end{array}$$

where S is the pullback of f_1 and f_2 .

- (1) *There is a homomorphism ∂_1 such that the sequence of abelian groups*

$$0 \rightarrow \text{tors}(\text{Pic}(R)) \xrightarrow{\partial_1} B(S) \xrightarrow{(h_1, h_2)} B(k[u_1, \dots, u_r]) \oplus B(k[X])$$

is exact.

- (2) *If k has characteristic $p = 0$, then there is a split exact sequence*

$$0 \rightarrow \text{tors}(\text{Pic}(R)) \xrightarrow{\partial_1} B(S) \rightarrow B(k) \rightarrow 0.$$

If k has characteristic $p > 0$, then the sequence is split exact modulo p -torsion subgroups.

PROOF. (1): It follows from Step 2 of the proof of Theorem 14.2.12 and Corollary 14.2.15 of [7] that there is a one-to-one homomorphism $\partial_1 : \text{tors}(\text{Pic}(R)) \rightarrow B(S)$ of abelian groups. We sketch the key ingredients of the proof. To define ∂_1 , let L be an R -progenerator module of constant rank one such that the class of L in the Picard group has finite order. An application of a theorem of Bass (see [7, Corollary 14.2.2]) implies there is an R -module isomorphism $\tau : L \otimes_R R^{(t)} \cong R^{(t)}$ for some $t > 0$. Then τ induces an R -algebra automorphism on the ring of matrices

$\bar{\tau} : M_t(R) \cong M_t(R)$. In fact, $\bar{\tau}$ is conjugation by τ . There is a cartesian square of rings

$$\begin{array}{ccc} A & \xrightarrow{\quad\quad\quad} & M_t(k[X]) \\ \downarrow & & \downarrow \\ M_t(k[u_1, \dots, u_r]) & \xrightarrow{\quad\quad\quad} & M_t(R) \xrightarrow{\bar{\tau}} M_t(R) \end{array}$$

where A is the pullback. Then A is an Azumaya S -algebra, and $\partial_1(L)$ is the Brauer class containing A . The proof that ∂_1 is one-to-one follows from the identities $\text{Pic}(k[u_1, \dots, u_r]) = \text{Pic}(k[X]) = \text{Pic}(k) = (0)$. Now we show exactness at $B(S)$. For notational simplicity, we write $R_1 = k[u_1, \dots, u_r]$ and $R_2 = k[X]$. By [7, Theorem 14.2.11] every Azumaya S -algebra Λ is defined by a pullback diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\quad\quad\quad} & \Lambda_2 \\ \downarrow & & \downarrow \phi_2 \\ \Lambda_1 & \xrightarrow{\phi_1} R \otimes_{R_1} \Lambda_1 \xrightarrow{\sigma} R \otimes_{R_2} \Lambda_2 \end{array}$$

where each Λ_i is an Azumaya R_i -algebra and σ is an isomorphism of R -algebras. We have $R_i \otimes_S \Lambda \cong \Lambda_i$. Let Λ be an Azumaya S -algebra of rank t^2 and assume Λ is split by $R_1 \oplus R_2$. Then $\Lambda_i = R_i \otimes_S \Lambda$ is the endomorphism ring of an R_i -progenerator module. By Serre’s Theorem (see [15, Theorem 4.62] or [12]), a finitely generated projective module over R_1 or R_2 is free. Therefore, Λ_i is isomorphic to the ring of matrices $M_t(R_i)$. Reversing the above steps shows that in this case, Λ is in the image of ∂_1 .

(2): Since h_2 is onto, we know S has a maximal ideal \mathfrak{m} such that $S/\mathfrak{m} \cong k$. Because S is a k -algebra, the natural homomorphism $B(k) \rightarrow B(S)$ splits. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B(k) & \longrightarrow & B(S) & \longrightarrow & \frac{B(S)}{B(k)} \longrightarrow 0 \\ & & \downarrow & & \downarrow (h_1, h_2) & & \downarrow \\ 0 & \longrightarrow & B(k) \oplus B(k) & \longrightarrow & B(R_1) \oplus B(R_2) & \longrightarrow & \frac{B(R_1) \oplus B(R_2)}{B(k) \oplus B(k)} \longrightarrow 0 \end{array}$$

where the rows are split exact sequences. By a theorem of Auslander and Goldman (see [7, Theorem 13.6.3]), the natural maps $B(k) \rightarrow B(R_1)$ and $B(k) \rightarrow B(R_2)$ are isomorphisms if $p = 0$. In this case we see that the kernel of (h_1, h_2) is isomorphic to $B(S)/B(k)$. If $p > 0$, then by a theorem of R. Hoobler (see [7, Theorem 13.6.4]), the natural maps $B(k) \rightarrow B(R_1)$ and $B(k) \rightarrow B(R_2)$ are isomorphisms modulo p -torsion subgroups. □

REMARK 3.2. Let R and S be as in Theorem 3.1. Since h_1 is an embedding of S into the polynomial ring $k[u_1, \dots, u_r]$, S is an integral domain. As we see in Example 4.4 below, in general S is not a finitely generated k -algebra. Assume R is of Krull dimension m . By E. Noether’s Normalization Lemma, there is a one-to-one homomorphism $f_2 : k[v_1, \dots, v_m] \rightarrow R$ such that R is a finitely generated module over the ring $k[v_1, \dots, v_m]$. Propositions 2.1 and 2.2 show that in this case the homomorphism h_1 makes $k[u_1, \dots, u_r]$ into a finitely generated S -module and S is a finitely generated k -algebra.

Using information obtained in the proof of Theorem 3.1 together with the Mayer-Vietoris sequence of Milnor ([7, Theorem 14.2.10]), we compute the group of units of S and the Picard group of S .

PROPOSITION 3.3. *Let k , R and S be as in Theorem 3.1. Then*

- (1) $S^* = k^*$, and
- (2) *there is an exact sequence of abelian groups*

$$1 \rightarrow k^* \rightarrow R^* \xrightarrow{\partial_0} \text{Pic}(S) \rightarrow 0.$$

PROOF. In our context, the Mayer-Vietoris sequence of Milnor is

$$(3.1) \quad 1 \rightarrow S^* \rightarrow (k[v_1, \dots, v_m])^* \times (k[X])^* \xrightarrow{(f_1, 1/f_2)} R^* \xrightarrow{\partial_0} \text{Pic}(S) \rightarrow \text{Pic}(k[v_1, \dots, v_m]) \oplus \text{Pic}(k[X]) \rightarrow \text{Pic}(R).$$

We have $(k[v_1, \dots, v_m])^* = (k[X])^* = k^*$ and $\text{Pic}(k[v_1, \dots, v_m]) = \text{Pic}(k[X]) = \text{Pic}(k) = (0)$. The map $(f_1, 1/f_2) : k^* \times k^* \rightarrow R^*$ is the difference map which is defined by $(a, b) \mapsto ab^{-1}$. The kernel and image of the difference map are both isomorphic to k^* . The sequence (3.1) breaks up into the isomorphism of Part (1) and the short exact sequence of Part (2). □

4. Examples and applications

We include in this section some examples and applications of Theorem 3.1. We begin by generalizing Example 2.3 to curves that are not necessarily hyperelliptic.

EXAMPLE 4.1. Let k be a field of characteristic p . Let f be a nonconstant polynomial in $k[x, y]$ defining the affine plane curve $C = Z(f)$. Let $\phi_1 : k[x, y] \rightarrow k[x, y]/(f)$ be the natural map. By Noether’s Normalization Lemma, there is a one-to-one homomorphism of k -algebras $\phi_2 : k[z] \rightarrow k[x, y]/(f)$ such that $k[x, y]/(f)$ is finitely generated as a $k[z]$ -module. Let

$$S = \{(u, v) \in k[x, y] \times k[z] \mid \phi_1(u) = \phi_2(v)\}$$

be the pullback. Then the cartesian square

$$\begin{array}{ccc} S & \xrightarrow{h_2} & k[z] \\ h_1 \downarrow & & \downarrow \phi_2 \\ k[x, y] & \xrightarrow{\phi_1} & \mathcal{O}(C) = k[x, y]/(f) \end{array}$$

commutes, where ϕ_1 and h_2 are onto, and ϕ_2 and h_1 are one-to-one. By Theorem 3.1, there is a sequence of abelian groups

$$0 \rightarrow \text{tors}(\text{Pic}(C)) \xrightarrow{\partial_1} \text{B}(S) \rightarrow \text{B}(k) \rightarrow 0$$

which is split exact modulo p -torsion subgroups. By Propositions 2.1 and 2.2, h_1 makes $k[x, y]$ into a finitely generated S -module and S is a finitely generated k -algebra.

REMARK 4.2. Suppose k has characteristic zero and S is a ring constructed using a plane curve $C \subseteq \mathbb{A}^2$ as in Example 4.1 such that S is a finitely generated k -subalgebra of $k[x, y]$ and $k[x, y]$ is a finitely generated module over S . Then the Brauer group functor can be used to show that the noetherian integral domain S is in general nonnormal. For instance, in [13], H. Li proves that if k is one of the

fields \mathbb{R} or \mathbb{C} , and $k \subseteq S \subseteq k[x, y]$ is a tower of subrings such that S is normal and the inclusion $S \subseteq k[x, y]$ is integral, then the natural maps $B(k) \rightarrow B(S) \rightarrow B(k[x, y])$ are isomorphisms. By Example 4.1, we know that $B(k) \rightarrow B(S)$ is not an isomorphism when $\text{tors}(\text{Pic}(C)) \neq (0)$. If $k = \mathbb{C}$, then a general plane curve in \mathbb{P}^2 is nonsingular (see for example [9, Exercise I.5.15]) and the dimension of its jacobian is greater than or equal to one. Therefore, for a sufficiently general C , the Brauer group of S will be nontrivial.

EXAMPLE 4.3. In the context of Example 2.3, let k be a field in which 2 is invertible and $g(x) = (x^2 - 1)^2 \in k[x]$. The hyperelliptic curve C defined by the equation $y^2 = (x^2 - 1)^2$ is the union of two conics. Following [7, Exercise 14.2.20], one shows that $\text{Pic}(C)$ is isomorphic to k^* , the group of invertible elements in k . If $S = k[x, y^2, y(y^2 - g(x))]$, then by Eq. (2.4), $B(S)$ contains a subgroup isomorphic to $\mu(k)$, where $\mu(k)$ denotes the group of all roots of unity in k .

The next example shows that the ring S in Theorem 3.1 can be a k -algebra that is not finitely generated.

EXAMPLE 4.4. Let k be a field. Consider the cartesian square of commutative k -algebras

$$\begin{array}{ccc} S & \xrightarrow{h_2} & k \\ h_1 \downarrow & & \downarrow f_2 \\ k[x, y] & \xrightarrow{f_1} & k[x, y]/(y) \end{array}$$

where f_1 and f_2 are the natural maps and S is the pullback. Then S can be identified with the subring of $k[x, y]$ consisting of all polynomials $p(x, y)$ such that $p(x, 0)$ is in k . Hence S can be identified with $k[y, xy, x^2y, x^3y, \dots]$. Notice that S is not a finitely generated k -algebra.

In Example 4.5 we show that any finite cyclic group is the Brauer group of an affine variety.

EXAMPLE 4.5. Let k be a field of characteristic p , where $p = 0$ is allowed. Let $m \geq 2$ and $S = k[x_0, \dots, x_m]$. Let $f \in S$ be a homogeneous irreducible polynomial of degree $n \geq 2$ and R the subring of homogeneous elements in $S[f^{-1}]$ of degree 0. Then R is a finitely generated k -algebra, R is a regular noetherian integral domain of Krull dimension m , R is birational to $k[x_1, \dots, x_m]$, $\text{Pic}(R) = \text{Cl}(R) \cong \mathbb{Z}/n$, and $R^* = k^*$. For proofs, see [7, Example 6.5.12].

Now assume n , the degree of f , is relatively prime to p . Since R is a finitely generated k -algebra, for some $r > 0$, there is a finite presentation of R

$$0 \rightarrow I \rightarrow k[u_1, \dots, u_r] \xrightarrow{f_1} R \rightarrow 0$$

where I is the kernel of f_1 .

Let $V = \{v_1, \dots, v_m\}$ be a set of indeterminates and X any subset of V . Since $\dim(R) = m$, there is a one-to-one homomorphism $f_2 : k[X] \rightarrow R$ of k -algebras. Consider the cartesian square of commutative k -algebras

$$\begin{array}{ccc} S & \xrightarrow{h_2} & k[X] \\ h_1 \downarrow & & \downarrow f_2 \\ k[u_1, \dots, u_r] & \xrightarrow{f_1} & R \end{array}$$

where S is the pullback. By Theorem 3.1,

$$0 \rightarrow \text{Pic}(R) \rightarrow \text{B}(S) \rightarrow \text{B}(k) \rightarrow 0$$

is split exact modulo subgroups of p -torsion. If we assume $p = 0$ and k is algebraically closed, then $\text{B}(S) \cong \mathbb{Z}/n$. If we pick $f_2 : k[V] \rightarrow R$ such that R is a finitely generated module over $k[V]$, then S is a finitely generated k -algebra (Remark 3.2).

Our final example, Example 4.6, shows that any finite abelian group is the Brauer group of an affine variety.

EXAMPLE 4.6. Let k be an algebraically closed field of characteristic zero. Let n_1, \dots, n_s be integers, each of which is greater than or equal to 2. Following the construction in Example 4.5, for each i let R_i be a finitely generated k -algebra of dimension $\dim(R_i) = m_i \geq 2$ such that $\text{Pic}(R_i) \cong \mathbb{Z}/n_i$. For each i let X_i be a set of m_i indeterminates, and $f_{2,i} : k[X_i] \rightarrow R_i$ a one-to-one homomorphism of k -algebras. For some $r > 0$, there is a finite presentation

$$0 \rightarrow I \rightarrow k[u_1, \dots, u_r] \xrightarrow{f_1} R_1 \oplus \dots \oplus R_s \rightarrow 0$$

of the k -algebra $R_1 \oplus \dots \oplus R_s$, where I is the kernel of f_1 . Consider the cartesian square of commutative k -algebras

$$\begin{array}{ccc} S & \xrightarrow{h_2} & k[X_1] \oplus \dots \oplus k[X_s] \\ h_1 \downarrow & & \downarrow (f_{2,1}, \dots, f_{2,s}) \\ k[u_1, \dots, u_r] & \xrightarrow{f_1} & R_1 \oplus \dots \oplus R_s \end{array}$$

where S is the pullback. Then S is an integral domain with the property that $\text{B}(S) \cong \text{Pic}(R_1 \oplus \dots \oplus R_s) \cong \mathbb{Z}/n_1 \oplus \dots \oplus \mathbb{Z}/n_s$. By Noether's Normalization Lemma, for each i we can pick $f_{2,i}$ so that R_i is finitely generated as a module over $k[X_i]$. In this case S is a finitely generated k -algebra (Remark 3.2).

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