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Curves in isomonodromy and isospectral deformations: Painlevé VI as a case study

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Abstract. Certain integrable dynamical problems can be described by isospectral deformations; under certain restrictions, the 'spectrum' is an algebraic curve. On the other hand, certain isomonodromy deformations may have an associated algebraic curve, under certain restrictions on the monodromy group. We explore the relationship between these two occurrences of algebraic curves using Painlevé VI as a "case study".

1. Introduction

The Painlevé equations have a rich history and a vast amount of applications, we refer to [**IKSY**] for information and connections with special function and to [**SHC**, Sec. 1] for the farthest-reaching range of applications. In the original approach, the six Painlevé equations, PI to PVI, were found to be (up to holomorphic change of the independent, and linear fractional transformation of the dependent variable) the only second-order non-linear ODEs with solutions whose only movable singularities are poles ("Painlevé property", which defines equations of "P-type", for short), to exclude branch points or essential singularities. The problem was posed by E. Picard [**Pi**]: we found three slightly different versions and adopt [**IKSY**, III.1.1], where the differential equation is of the form

$$F(t, y, \frac{dy}{dt}, ..., \frac{d^n y}{dt^n}) = 0,$$

defined in a domain D of the complex plane, with coefficients meromorphic in $t \in D$ and polynomial in $(y, \frac{dy}{dt}, ..., \frac{d^n y}{dt^n})$ (in [**C2**], F is assumed to be analytic in t and rational in $(y, \frac{dy}{dt}, ..., \frac{d^n y}{dt^n})$; in [**BBT**, 8.11], F is a rational function of all arguments). Painlevé and his students found the only such ODOs of order $n = 2$ that could not be integrated in terms of previously known functions (namely, elementary or elliptic); they are now known as PI–PVI. It is also possible to produce them as (time-dependent) Hamiltonian flows, and in that context the question of complete integrability, e.g, is still to some extent open, and was settled only recently for certain classes of functions (playing the role of independent variables) [**ŻF**]. There is, however, a seemingly unrelated way to come across the same equations, namely "isomonodromic deformation". The discovery of PVI in this way is originally due to

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R. Fuchs [**Fu**]. The question he posed, following investigations of his father Lazarus Fuchs, was: find conditions on a second-order linear ODE that has four regular-singular points so that the monodromy group be independent of the coefficients (namely, one parameter t, since the other three essential singularities can be moved to $0,\ 1, \infty$ by a linear-fractional transformation). By Poincaré's theory [**Po**], such an equation must have an additional apparent singularity (where he coefficients of the equation have poles but the solutions are single-valued meromorphic functions), say λ, and Fuchs' startling discovery was that isomonodromy implies that λ as a function ot t satisfies PVI. In this way, a (genus-one) curve turns out to be related to PVI. Other curves arise, both in the problem of isomonodromy, and in the theory of isospectral deformations that gives rise to "integrable hierarchies" of non-linear PDEs. The research reported in this note was prompted by the goal of identifying relationships among these curves. The finding is puzzling: to the best of our knowledge, no theoretical understanding of any such relationship is known, despite several announcements (e.g., [**FN**, **K**], detailed below) where the question was posed and a future publication was anticipated to contain the answer (we were unable to place that publication). This note therefore is devoted to creating a context for the questions, linking references that pursued similar issues unawares of each other, exploring the difference among their methods, and highlighting connections which were not made before. We then set ourselves a seemingly less ambitious task. There is another way in which P-type equations appear within the theory of integrable hierarchies, and that is "similarity reduction". We briefly summarize the emergence of that relationship.

Integrable hierarchies are sequences of non-linear PDEs that can be viewed as integrable Hamiltonian systems in a sequence of commuting variables $t_1, ..., t_n, ...$ and as such, they admit certain symmetries. Some of these symmetries produce, by reduction, the "self-similar" solutions, which typically depend only on one variable: strikingly, in that variable the solutions were found to satisfy an ODE of P-type (possibly of order higher than two). This led Ablowitz, Ramani and Segur to propose the "Painlevé conjecture": *A nonlinear PDE is solvable by an inverse scattering transform (IST) only if every nonlinear ODE obtained by exact reduction is of P-type, perhaps after a transformation of variables* [**AS2**, 3.7.b]. The conjecture prompted a great deal of work and was confirmed in special cases, but our paper is concerned instead with finding a structure behind this phenomenon. This need arises from a lack of systematic relationships between the PDEs and the ODEs: notably, "wheras the self-similar solutions of integrable evolution equations appear to give nonautonomous ordinary differential equations, there are many members of the latter class which do not result from self-similar limits of the former" [**FN**, Subsection 2B]. To begin with, "characterizing the set of PDEs that can be solved by IST" relies largely on "clever guesswork" [**AS2**, Ch. 3 Overview]. If we already have a given IST, then further analyticity assumptions, both on the kernel of the transform and on the type of solutions does make it clear, using classical theorems, that the solutions to the ODE should only have poles as singularities [**McLO**]. In this delicate setting, then, where detecting a certain qualitative behavior depends on 'clever guesses' and changes of variables, we felt that it was worthwhile to ask the simplest possible questions, and look for all available information in one "case study", which we describe next.

In Section 2 we present all six equations via isomonodromy. In that context, coalescence (or confluence) of the singularities (and also, by a different deformation procedure, introducing into variables and parameters of the equation an additional deformation parameter and taking the limit as it approaches zero) reduces PVI to all the others, according to the following diagram [**IKSY**, Prop. 1.2.1]:

$$\begin{array}{ccccccc} & & & PIV & & & \\ & & \nearrow & & \searrow & & \\ PVI & \rightarrow & PV & & & & PII \rightarrow PI. \\ & & \searrow & & \nearrow & & \\ & & & PIII & & & \end{array}$$

On the PDE side, the Kadomtsev-Petiashvili (KP) equation:

$$(u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x + \frac{3}{4}\sigma u_{yy} = 0, \quad \sigma = \pm 1,$$

and its hierarchy, reduce to the Korteweg-de Vries (KdV), Boussinesq equation (and their hierarchy), respectively, when the solution is independent of y, t, resp. Since KdV has self-similar solutions that satisfy PI, others PII, and Boussinesq, under the same symmetry reduction, solutions that satisfy PI, PII, or PIV (according to the choice of parameters), we originally asked a most naïve question: does KP have self-similar solutions that satisfy PVI? If so, a diagram with vertical arrows corresponding to similarity reduction, and horizontal arrows corresponding to coalescence, would have been commutative in some sense. But of course we were wrong: KP does not reduce to KdV or Boussinesq by any sort of coalescence of the poles; the relationship between the two types of isomonodromy that connect an integrable PDE and a P-type equation is much more elusive, and we survey one version of the connection for PVI in Subsection 2.4. It can also be safely stated that "[PVI yields all other PI-V by appropriate limiting procedure], yet the geometric content of this statement is still very obscure" [**FN**, Subsection 2C] In any case, heuristically, since the KP hierarchy is governed by an eigenfunction that has one essential singularity on the spectral curve, the isospectral deformation problem [**FN**, (2.12-13)] posed by Flaschka and Newell has only one essential singularity, so there is no issue of coalescence.

We refocused the goal of our search, and decided that it seemed useful, on the one hand, to bring together all the similarity reductions we could find for KP, on the other, all the ways that PVI was produced by a similarity reduction (Section 3).

The interplay between isomonodromic and isospectral deformations, envisioned in [**FN**, Section 1], is still completely mysterious. We focus on the case when both the spectral curve and the monodromy curve are algebraic. In Section 2 we present different constructions of the attendant isomonodromy curve, which to our knowledge have not been related either, focusing on obtaining PVI, which again, in one of the contexts at least, is a recent result [**B**].

In Section 4, we summarize the structural relationships between the aspects we covered and list other aspects of the P-type equations that are related to the theory of algebraic curves and integrable systems, with the ultimate goal of finding a common root for all of them. Generalizations are also outlined.

2. Connecting integrable PDEs and Painlevé-type. I: Isomonodromy aspects

As mentioned in the Introduction, Fuchs found that PVI is equivalent to an isomonodromy problem. This discovery was generalized to the Schlesinger isomonodromy problem to yield all equations PI-PVI. We briefly recall his method, setting out two steps; the second step does not appear to have been generalized to the several-essential-singularities case, posed by Garnier in his thesis [**G**], and proposed in [**ENP**].

2.1. Elliptic curve. Step one, in the case of order 2 and $g+1$ regular-singular points with $g=1$, amounted to the following statement:

THEOREM 2.1. *The isomonodromy problem for the equation*

$$\frac{d^2y}{dx^2} = \left[\frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t} + \frac{e}{(x-\lambda)^2} + \frac{\epsilon}{(x-\lambda)}\right] y$$

is solved, and admits four arbitrary constants $k_0,\ k_1,\ k_\infty,\ k_t$[1] *when λ as a function of t satisfies PVI:*

$$\frac{d^2\lambda}{dt^2} + \left[\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda - t}\right]\frac{d\lambda}{dt} - \frac{1}{2}\left[\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right]\left(\frac{d\lambda}{dt}\right)^2$$

$$= \frac{1}{2}\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[k_\infty - k_0\frac{t}{\lambda^2} + k_1\frac{t-1}{(\lambda-1)^2} - (k_t-1)\frac{t(t-1)}{(\lambda-t)^2}\right],$$

with k, a, b, c given (linearly) in terms of $k_0,\ k_1,\ k_t,\ k_\infty$:

$$-4(k-1) + 4a + 4b + 4c = k_\infty,$$

$$4\left(a + \frac{1}{4}\right) = k_0,\ 4\left(b + \frac{1}{4}\right) = k_1,\ 4\left(c + \frac{1}{4}\right) = k_t.$$

Step two was a remarkable observation which Fuchs possibly came across while looking into the special case $k_0 = k_1 = k_t = k_\infty = 0$; this corresponds to Gauss' hypergeometric equation, and the result is the following:

THEOREM 2.2. *The roots $y_1,\ y_2$ of Gauss' hypergeometric equation undergo a monodromy independent of t, upon circling the points $t,\ 0,\ 1,\ \infty$, if and only if the function*

$$u = \int_0^\lambda \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}}$$

satisfies Legendre's equation:

$$\frac{d^2u}{dt^2} + \frac{2t-1}{t(t-1)}\frac{du}{dt} + \frac{u}{t(t-1)} = 0.$$

[1] The right number of parameters, together with λ, for the fixed monodromy group to be arbitrary, cf. e.g. [**G**, p. 74].

Note that λ *is an elliptic function of* t*. More generally, PVI is equivalent to the equation*

$$\frac{d^2u}{dt^2}+\frac{2t-1}{t(t-1)}\frac{du}{dt}+\frac{u}{t(t-1)}$$
$$=\frac{\sqrt{\lambda(\lambda-1)(\lambda-t)}}{2t^2(t-1)^2}\times\left[k_\infty-k_0\frac{t}{\lambda^2}+k_1\frac{t-1}{(\lambda-1)^2}-k_t\frac{t(t-1)}{(\lambda-t)^2}\right]$$

for the elliptic integral.

Notably, Manin [**M1**] gave a beautiful interpretation of Fuchs' work in moduli, and connected it with mirror symmetry [**M2**].

2.2. Schlesinger flow. We recall this setting, not only for completeness, but also because its interpretation is still evolving and recent research [**B**] checks directly that PI-PVI are the compatibility condition of the Schlesinger flow with the initial Fuchsian system, an important way to obtain "spectral curves", in this case bi-spectral. In [**B**] PVI is produced rigorously by Hamitonian dynamics under symplectic projection, and notably, a counterexample is given, to the fact that PVI, conversely, implies isomonodromy, in this particular setting. We refer to [**B**] for more specific definitions, since our goal is just to highlight the conversion between the ODE problem and the Schlesinger monodromy problem.

The monodromy of an equation

$$\frac{d}{dz}\Psi=A(z,t)\Psi,\quad A\in\mathrm{gl}(N),\ \Psi\in\mathrm{GL}(N),$$

where z is a variable ranging over a suitable complex domain, the complex parameter t belongs to a complex disk, and N is a natural number, is defined by a conjugacy class (up to constant) of matrices in $\mathrm{GL}(N)$.

THEOREM 2.3. *The monodromy is constant in* t *(i.e., "isomonodromy" holds for the equation) if and only if there exists an analytic single-valued function* $B(z,t)$ *such that the form* $Adz+Bdt=\omega$ *is flat, i.e.,* $d\omega=\omega\wedge\omega$*, also expressed by the zero-curvature condition* $A_t-B_z+[A,B]=0$*, or:*

$$\begin{cases}\frac{\partial}{\partial z}\Psi=A\Psi\\ \frac{\partial}{\partial t}\Psi=B\Psi.\end{cases}$$

Specifically, Babich analyzes the Fuchsian system (with rational coefficients)

$$\frac{d}{dz}\Psi=\sum_{k=1}^{M}\frac{A^{(k)}}{z-z_k}\Psi,$$

with $\Psi\in\mathrm{GL}(N)$, $A^{(k)}\in\mathrm{gl}(N)$, $\sum_k A^{(k)}=0$ and the poles z_k viewed as deformation parameters t_k. He proves that the Schlesinger equations

$$dA^{(k)}=-\left[A^{(k)},\sum_{i\neq k}A^{(i)}d\log(z_k-z_i)\right]$$

are a sufficient condition for isomonodromy (but gives a counterexample to the converse). He also gives an equivalent Hamiltonian formulation over a space of orbits of matrices with natural symplectic structure; under Hamiltonian reduction (with respect to diagonal conjugation by constant matrices), he obtains the (multi-time) Schlesinger system for the Hamiltonian:

$$\omega - \sum_{i<j} d\mathrm{tr}(A^{(i)}A^{(j)}) \wedge d\log(z_i - z_j),$$

and this allows him to derive PVI, as the Lagrange equation corresponding to the above-mentioned Hamiltonian on $\prod_{k=1}^{4} \mathcal{O}_{J_k}//\mathrm{PSL}(2)$, with the four $A^{(k)} \in \mathrm{End}(\mathbb{C}^2)$, J_k an orbit representative, for example the Jordan form, and the deformation parameter t taken to be the cross-ratio of the four singularities z_k; the four arbitrary constants in PVI are essentially the eigenvalues of the four orbits.

However, we are unable to construct Fuchs' elliptic curve from this Hamiltonian.

We also believe it is very interesting that an elliptic curve arises from a different monodromy aspect, which we describe in the next subsection.

2.3. Algebraic monodromy curve: Hitchin's example. Hitchin's approach [**Hi**] is to interpret the Schlesinger equation as the holonomy of a flat connection for a parabolically-stable rank-two vector bundle over the Riemann sphere, with a weighted flag structure at four marked points. Under the assumption that the holonomy group $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ of the connection is finite, the corresponding solution of PVI is algebraic. Hitchin assumes the group to be the dihedral group D_k, and in that case $y(x)$ satisfies PVI with parameters $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$, which are respectively $(k_\infty, -k_0, k_1, -(k_t - 1))$ in the previous notation of Subsection 2.1. Hitchin's proof consists of a staggering set of identifications that range from twistor spaces to k-sided plane polygons inscribed in a fixed conic C and circumscribed about a conic B (a case of Poncelet porism [**BM**]), so that the elliptic curve is the incidence correspondence in $C \times B^*$, with choice of origin. Hitchin gives the explicit equation of the conics in terms of the poles of the connection, thus the elliptic curve. We do not see any way to generalize this construction to Fuchs' curve (PVI with arbitrary parameters) because in that case the curve defived by the holonomy of the connection is not algebraic. But one could ask this question for other algebraic solutions of PVI (which have been classified), particularly in view of significant relationships between reflection groups and algebraic solutions of PVI [**DM**].

2.4. Sato's tau function. On the other hand, equations of statistical mechanics were brought to isomonodromic form: the earliest instances we could find are [**BMcCW**] and [**SMJ**]). Thus, the question [**AS2**, Section 3.7]: "Is there some connection between the two-dimensional Ising model and [Inverse Scattering Transform]?" This connection is indeed our main goal, namely: an explicit relation between isospectral and isomonodromy curves.

Flaschka and Newell [**FN**] state precisely this goal, take it so far as to work out two examples and announce "results will be presented in a sequence of future papers" that never appeared. Again, the elusiveness of this goals motivated the research we present in this note.

We review the framework in [**FN**] in Subsection 3.3 below, because the way they obtain an ODE of P-type is to impose a similarity reduction; in fact, they do not obtain PVI; here, we follow the expository [**BBT**, Ch. 8], after [**SMJ**] (cf. also [**KaKa**] for the setting that links isomonodromy and isospectrality), noting that PVI is derived in a way which resembles Fuchs' yet differs in the *ad hoc* introduction of parameters to serve as new variables.

What we find most interesting in [**BBT**, Ch. 8] is the explicit relationship between the isomonodromy and the isospectral curve provided by Sato's tau function; however, what becomes more mysterious is the appearance of PI-PVI [**BBT**, Section 8.11] "which can be interpreted as isomonodromic deformations but not as isospectral deformations". This seems to negate our attempt at linking isomonodromy and isospectrality, but we don't think the authors intend this negation, since they do exhibity a wave function for an isospectral hierarchy. We found a clue to what we believe they mean in [**SHC**, Section 1], "It is worth making some comments on the use of the word "hierarchy." In the theory of soliton equations, (...) all of the equations can be derived from a sequence of Lax pairs $\mathcal{L}\phi = \lambda\phi,\ \partial\phi/\partial t_j = \mathcal{M}_j\phi$, where the Lax operator $\mathcal{L}$ is the same for each j," giving rise to a sequence of commuting flows $\partial/\partial t_j,\ j = 1, 2, ...$ whereas the corresponding flows of P-type equations that arise as similarity reductions of the PDE hierarchy may not commute even when each has a Lax pair. "To add to the confusion," (*ibid.*) there is an additional sequence of P-type equations, also called "hierarchy", obtained through a sequence of Bäcklund transformations, "[t]herefore, for each N the corresponding [P-type] equation is a scaling similarity reduction of a flow in a different hierarchy of (...) PDEs." In summary, the wave function undergoes a hierarchy of commuting isospectral flows, but as it also keeps track of isomonodromy equations, those do not organize in a sequence of ODEs whose independent variables commute. This is one of the reasons why we pose the final question of Section 4.

For the case study of PVI, we specify the general construction to 2×2 matrices in $\mathfrak{sl}(2)$ whose entries are rational functions of λ:

$$M_\lambda(\lambda) = \sum_{k=1}^{K}\left(\frac{A_1^{(k)}}{\lambda-\lambda_k} + ... + \frac{A_{n_k+1}^{(k)}}{(\lambda-\lambda_k)^{n_k+1}}\right) - A_0^{(\infty)} - ... - A_{n_\infty-1}^{(\infty)}\lambda^{n_\infty-1}$$

(set $M_\lambda(\lambda) = \sum_k M_\lambda^{(k)}(\lambda)$ where $M_\lambda^{(k)}(\lambda)$ is the polar part of M at λ_k, including ∞), with one singularity at $\lambda = \infty$ and three regular singularities at $\lambda_0 = 0, \lambda_1 = 1, \lambda_t = t$.

Goal: Study the isomonodromy-deformation problem with respect to the parameter t.

The explicit link with isospectral deformations is the following: view the solution of $\partial_\lambda\Psi = M_\lambda\Psi$ as a "wave function":

$$\Psi(\lambda) = g(\lambda)e^{\sum_i \xi_i(\lambda)t_i},\ i = (k, n, \alpha),$$

$g(\lambda)$ a regular matrix, $\xi_i(\lambda) = \sum_\alpha e_{\alpha\alpha}/(\lambda-\lambda_k)^n$, with

$$\partial_{t_i}\Psi(\lambda) = M_i(\lambda)\Psi(\lambda),\ M_i(\lambda) = \left(g(\lambda)\xi_i(\lambda)g^{-1}(\lambda)\right)_-$$

(the notation $()_-$ means taking the polar part at λ_k).

This gives a hierarchy of commuting flows, satisfying the zero-curvature equations:

$$\partial_{t_i}M_j - \partial_{t_j}M_i - [M_i, M_j] = 0.$$

Now the $\Psi(\lambda)$ have essential singularities at the $\lambda = \lambda_k$ and at $\lambda = \infty$ and in general have non-trivial monodromy around the singularities (thus, are only defined on a Riemann surface),

$$\Psi(\lambda) \sim g^{(k)}(\lambda)e^{\xi^{(k)}(\lambda)},$$

where

$$g^{(k)}(\lambda) = g_0^{(k)} + g_1^{(k)}(\lambda - \lambda_k) + ...$$

is regular at λ_k, and

$$\xi^{(k)} = B_0^{(k)} \log(\lambda - \lambda_k) + \sum_{\alpha;n=1}^{n_k} \frac{t_{(k,n,\alpha)}}{(\lambda - \lambda_k)^n} e_{\alpha\alpha}$$

$$\xi^{(\infty)} = B_0^{(\infty)} \log\left(\frac{1}{\lambda}\right) + \sum_{\alpha;n=1}^{n_\infty} t_{(\infty,n,\alpha)} \lambda^n e_{\alpha\alpha}$$

Goal: write evolution equations that describe the deformations of $M_\lambda(\lambda)$ under which the monodromy of $\Psi(\lambda)$ around a singularity of $M(\lambda)$ is fixed. The isomonodromic deformation parameters will include the $t_{(k,n,\alpha)}$, and we have an enriched family of commuting flows:

$$\partial_{\lambda_k}\Psi(\lambda) = M_{\lambda_k}\Psi(\lambda),\ M_{\lambda_k} = \left((g^{(k)}\partial_{\lambda_k}\xi_i^{(k)} g^{(k)-1}(\lambda)\right)_- .$$

In our illustration we choose the singularity data:

$$\xi^{(k)}(\lambda) = \begin{bmatrix} \theta_k & 0 \\ 0 & 0 \end{bmatrix} \log(\lambda - \lambda_k),\ k = 0, 1, t,$$

we diagonalize M at ∞ by global gauge transformation by a matrix constant in λ:

$$M(\lambda) = \frac{1}{\lambda}\begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} + O(\lambda^{-2}),\ \lambda \to \infty.$$

The equation $\partial_\lambda \Psi = M\Psi$ has a regular singularity at ∞ and the Fuchs condition reads: $\kappa_1 + \kappa_2 = \theta_0 + \theta_1 + \theta_t$. Set:

$$M(\lambda) = \begin{bmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{bmatrix} = \frac{1}{\lambda(\lambda-1)(\lambda-t)} A(\lambda),$$

with the entries of the matrix A polynomial in λ, in particular, setting: $A_{12}(\lambda) = \gamma(\lambda - y)$ introduces the parameter y which will turn out to satisfy PVI. We also parametrize $A_{11}(\lambda)$ by the values it takes at $\lambda = t$ and $\lambda = y$, in other words we interpolate as follows:

$$A_{11}(\lambda) = (\lambda - t)(\lambda - y)\kappa_1 + \frac{(\lambda - y)A_{11}(t) - (\lambda - t)A_{11}(y)}{t - y}.$$

Since $A_{21}(\lambda)$ is a linear function of λ,

$$A_{21}(t) = (1 - t)A_{21}(0) + tA_{21}(1)$$

and from the condition that the determinant of A is zero at the three singularities $\lambda = 0, 1, t$ we derive three constraints:

$$A_{21}(\lambda) = \frac{A_{11}(\lambda)A_{22}(\lambda)}{\gamma(\lambda - y)},$$

which by substitution of the interpolation formula give a quadratic condition in $A_{11}(y)$ and "unexpectedly" (*loc. cit.*) linear in $A_{11}(y)$. The dynamical variables are now $y, \gamma, A_{11}(y)$ We use the equation of motion: $\partial_t M_\lambda = \partial_\lambda M_t + [M_t, M_\lambda]$ and afterwards set $\lambda = y$:

$$\begin{bmatrix} \dot m_{11}(\lambda) & \dot m_{12}(\lambda) \\ \dot m_{21}(\lambda) & \dot m_{22}(\lambda) \end{bmatrix}_{\lambda = y} = \frac{1}{t(t-1)(y-t)^2} \cdot \left\{ A(t) - \frac{1}{y(y-1)}[A(t), A(y)] \right\}.$$

Taking the (1,2) entry of this equation and evaluating it at $\lambda = y$ gives:

$$\dot{y} = \frac{y(y-1)(y-t)}{t(t-1)}\left[2m_{11}(y) - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t - 1}{y-t}\right].$$

Taking the (1,1) entry of this equation and evaluating it at $\lambda = y$ gives:

$$\dot{m}_{11}(\lambda)|_{\lambda=y} = \frac{A_{11}(t)}{t(t-1)(y-t)^2} + \frac{\gamma}{t(t-1)}m_{21}(y).$$

Introduce the dynamical variable $z = m_{11}(y)$ and compute:

$$\dot{z} = \dot{m}_{11}(\lambda)|_{\lambda=y} + (\partial_\lambda m_{11})|_{\lambda=y} \cdot \dot{y}.$$

By eliminating z between the equations for $\dot{y}$ and $\dot{z}$ we obtain PVI for y, with appropriate values for the parameters: $\alpha = 1/2(\kappa_1 - \kappa_2 + 1)^2,\ \beta = -(1/2)\theta_0^2,\ \gamma = 1/2\theta_1^2,\ \delta = (1/2)(1-\theta_t)^2$.

In this context, the link between isomonodromy and integrable hierarchies of isospectral type arises as follows:

The τ function. The deformation equations imply that the following 1-form is closed:

$$Y = -\sum_k \mathrm{Res}_{\lambda=\lambda_k} \mathrm{Tr}(g^{(k)-1}\partial_\lambda g^{(k)} d\xi^{(k)})d\lambda,$$

summed over all the singularities including ∞.

Example. In the case of the Schlesinger equations,

$$Y = \frac{1}{2}\sum_{k\neq l} \mathrm{Tr}(A_k A_l)\frac{d\lambda_k - d\lambda_l}{\lambda_k - \lambda_l}.$$

Definition 2.4. To define Sato's tau-function, set: $Y = d\log\tau$.

Theorem 2.5. *The tau-function satisfies the Hirota equations, in the following sense.*

Define the bilinear operator

$$D_i^n f\cdot g = \left(\frac{\partial}{\partial y_i}\right)^n f(x+y)g(x-y)|_{y=0},$$

then for the polynomial operator

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau\cdot\tau = 0,$$

implies the KP equation:

$$3u_{t_2t_2} + (-4u_{t_3} - 6uu_{t_1} + u_{t_1t_1t_1})_{t_1} = 0, \quad u = \frac{\partial^2}{\partial t_1^2}\log\tau.$$

The proof consists in showing that the tau-function obeys the Schlesinger transformations.

3. Connecting integrable PDEs and Painlevé-type. II: Symmetry reduction

One feature of integrability for a dynamical system is that it is invariant under a continuous group. There are many possible definitions of invariance, even more (computational) techniques to search for such a group, and a remaining gaping question whether all possible symmetries were found [**C1**, Example]. Our concern is limited to the following issues: Bringing together results by different authors who are apparently unaware of each other; Focusing on "soliton equations", and even more narrowly, on hierarchies that can be defined as isospectral deformations by a Lax pair, in particular have a spectral curve under certain finiteness conditions (which we will specify); And attempting to find a unifying theory for these symmetries, ideally by solving an 'inverse spectral problem', namely starting with geometric data to produce the deformations, their solutions, and their groups of symmetries at the same time. Much of the geometric ground at the basis of dynamical symmetries is covered in [**O**]; [**AC**] is a rich reference for examples and interconnections, although we were unable to satisfy ourselves that certain references specifically relevant to PVI (given, as most other results, without proofs or summaries) achieved the wanted results (we will highlight this in Subsection 3.4 below).

The somewhat surprising link between integrable equations and the Painlevé condition for ODEs was pointed out in [**AS1**], where the focus was on inverse scattering as a kind of "non-linear Fourier transform". Pursuant the linearized dispersion relation for KdV

$$u_t + 6uu_x + u_{xxx},$$

solutions to KdV were rescaled, following the observation that the scattering data have a homogeneity, one power for x versus cube powers for t and "self-similar" solutions were constructed cf. [AS2, §3.4]

$$u = \frac{1}{(3t)^{2/3}} w(z), \quad z = \frac{x}{(3t)^{1/3}},$$

and $w(z)$ satisfies $w''' + 6ww' - 2w - zw = 0$; a 'Miura transformation'[2] $w(z) = V'(z) - V(z)^2$ reduces this ODE:

$$(-2V + \partial_z)(V''' - 6V^2V' - V - zV') = 0.$$

Integration of the latter factor yields PII: $V'' - 2V^3 - zV - \alpha = 0$, where α is a constant.

On the other hand, $x' = x + 3t\lambda,\ t' = t + \lambda/\alpha,\ u' = u + \lambda$ reduces KdV to PI; the technique is to make $t' = 0$, or $\lambda = -\alpha t$, $z := x' = x - 3\alpha t^2$, $U(z) := u' = u - \alpha t$, and put $u_t,\ u_x$ in KdV to obtain PI: $U''' + 6UU' + \alpha = 0$.

These discoveries led to many further results linking specific PDEs to specific Painlevé equations, and most notably to conjectures for integrability which will not be reviewed here (cf. [**AS2**, **AC**] for surveys), except for mentioning that the Non-Linear Schrödinger equation was reduced to PIV [**BoPe**] with the interesting consequence that Bäcklund transformations for PIV correspond to the addition rule for the Weierstrass $\wp$-function.

Our question is, how to give a systematic understanding of the correspondence. For example, as recalled in the Introduction, given that PVI is the 'most general' of the six Painlevé equations in the sense that it can be reduced to the previous five in suitable limits, and the fact that the KP hierarchy reduces to both KdV and Boussinesq, one might naïvely ask whether KP is connected to PVI. However, in the treatments of KP that we found, this was not the case. In [**KL**], and in [**R**] (cf. [**AS2**, Exercise 12, Section 3.7] and [**TNK**], by similarity KP is reduced to PI or PII. None of these works cites any of the others, and the method is slightly different. In

[2]Recall that under the transform $u = v_x - v^2$, KdV becomes $(-2v + \partial_x)(v_t - 6v^2v_x + v_{xxx}) = 0$, giving the mKdV equation. Solutions to mKdV therefore always transform back to solutions of KdV, but as is well known, the converse does not always hold.

both [**KL**] and [**TNK**], the general similarity method is implemented by postulating an infinitesimal deformation, and Lie-invariance is determined (see Subsection3.1 below), one reduction is found, but the invariance equations being non-linear, some simplifying assumptions lead to seemingly different results, namely only Boussinesq in [**KL**], Boussinesq or KdV in [**TNK**]. However, since both Boussinesq or KdV can be reduced once more to either PI or PII, from the point of vew of our question, the outcome is essentially the same. The method in [**R**] achieves both reductions at once, indeed Redekopp uses the linearized dispersion relation, as had Ablowitz and Segur:

$$u_{xt} + \frac{1}{4}u_{xxxx} + \frac{3}{4}u_{yy} = 0$$

for KP:

$$(u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x + \frac{3}{4}\sigma u_{yy} = 0, \quad \sigma = \pm 1,$$

hence the phase

$$kx + my + \frac{t}{4}\left(k^3 - 3\frac{m^2}{k}\right) = (lx - \imath l^2 y + l^3 t) + nx + \imath n^2 y + n^3 t)$$

in the Fourier transform, with separation: $k = l + n$, $m = -\imath(l^2 - n^2)$, sets

$$p = (3t)^{1/3} l + \imath \frac{y}{(3t)^{2/3}}, \quad s = (3t)^{1/3} n - \imath \frac{y}{(3t)^{2/3}}$$

and achieves similarity by:

$$z = (3t)^{-1/3} x + \sigma (3t)^{-4/3} y^2,$$

integrating once the resulting ODE for $u = F(z)$ and multiplying by F, F^2 is seen to satisfy PII. We are unable to find a proof, however, that one or the other method yields potentially more possibilities, namely reductions to different Painlevé equations. We believe that the problem is revealed by a phenomenon observed, apparently for the first time, in [**OR**]: there may be transformation groups that act on the (independent and dependent) variables of an equation but are only "conditional symmetries", in the sense that they do not preserve the equation but still give rise to a condition of ODE type. Thus, both methods of similarity reduction and "nonclassical-reduction" were implemented by several authors, starting from the theory in [**BC**] and up to currently active computational advances; our point here is just that there is no way to find 'all' possible P-type equations that can be obtained from a given integrable PDE; to the best of our understanding, even when reaching a point in the reduction which where no special assumptions were made, the problem, being non-linear or overdetermined, may not have a "general solution" computable by available algorithms. Almost as a footnote, another work on self-similar reductions of KP (later than [**R**, **TNK**] and referring to neither) adopts an idea of P.A. Clarkson and M.D. Kruskal which creates self-similar reductions 'directly', and depending on arbitrary functions of time [**L**]. Ultimately, we must endorse [**FN**] as a statement of our understanding: "not even the correct general definition of [the class of "multiphase similarity solution"] has been found."

3.1. Aspect I: Lie symmetry reduction. Close to the previous similarity method but somehow working 'from the top down' is the theory of (Lie) symmetries of the PDE: these give rise to self-similar solutions, though as we saw above other similarity reductions may also be relevant; the interest of this method is that it gives a complete and systematic way to study the symmetry group of a PDE, in analogy with the differential Galois group of an ODE. For our purposes, the criterion is given in [**McLO**],

THEOREM 3.1. *A vector field*

$$v = \sum_{i=1}^{m} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \phi_j(x,u)\frac{\partial}{\partial u_j}$$

is the infinitesimal generator of a symmetry group G of a system of differential equations $\Delta(x,u) = 0$, in the sense that the action of G on the space $\mathbb{R}^m \times \mathbb{R}^n \ni (x,u)$ sends solutions of the system to other solutions, if and only if the prolongation of v, acting on the spaces of partial derivatives of u with respect to x induced by the action of G on functions $u = f(x)$, is 0 whenever $\Delta = 0$.

Under technical conditions, it is then possible to obtain the G-invariant solutions by integrating a reduced system.

3.2. Aspect II: Inverse scattering. Originally, Ablowitz and Segur propose a method for solving the KdV equation, cf. [**AS2**]. The more general point of view of symmetry groups of PDEs was brought to bear in [**McLO**], where the focus is the Gel'fand-Levitan representation for an integrable equation; indeed, in [**FA**] the authors show that if φ satisfies the linear integral equation

$$\varphi(k) + \imath \exp\left[\imath(kx + k^3 t)\right] \int_C \frac{\varphi(l)}{l+k}\, d\lambda(l) = \exp\left[\imath(kx + k^3 t)\right]$$

where $d\lambda(l)$ and C are a measure and a contour, respectively, then $u = \partial_x \int_C \varphi(k)\, dk$ is a solution of the KdV equation. Furthermore, it is shown that the usual Gel'fand-Levitan equation is contained in the above as a special case. More general initial value problems are not solved. This result, however, generates a three-parameter family of solutions for Painlevé II, which is the nonlinear ordinary differential equation $d^2V/dx^2 - xV - 2V^3 - \alpha = 0$, where α is a constant. The generated solutions are only valid for $-\frac{1}{2} < \alpha < \frac{1}{2}$, as noted by D. Kaup [**FA**].

Following this program, namely a linearization of Painlevé-type problems, which can be used not only to give criteria for (non-linear PDE) integrability but also to detect special solutions of Painlevé equations and transformations between them, [**ARS**] went back from the (linear) integral equation to PIII and PIV; in this way one could aim to find PVI first, then connect it to an integrable hierarchy, and lastly 'geometrize' the connection.

The cleanest way of phrasing the appearance of the similarity in the Gel'fand-Levitan transform of inverse scattering that we found is given in [**AvdL**, Section 1], "a self-similarity reduction is closely related to the scaling behavior of [the] solutions". Let us illustrate this statement more precisely: in the example we gave at the beginning of the section, where substituting such function $u(x,t)$ into the mKdV equation yields PII, notice that the single similarity variable z ensures the scaling property;

$$u(\lambda x, \lambda^3 t) = \lambda^{-1} u(x,t)$$

"which characterizes the function $u(x,t)$ in the self-similarity condition." Now notice that this is also equivalent to the linear condition

$$(x\partial_x + 3t\partial_t)u(x,t) = -u(x,t),$$

a special case of the L_{-1} Virasoro constraint. The authors generalize to other integrable hierarchies this "equivalence of introducing a self-similarity limit by either imposing a certain scaling behavior or a certain Virasoro condition." This again begs the question whether all the self-similar reductions can be so characterized, particularly those that are "non-classical" and therefore do not come from a Lie algebra.

3.3. Aspect III: Isomonodromy. We now review the Flaschka-Newell procedure for deriving self-similar solutions of a zero-curvature deformation problem, in which they exhibit a link with P-type equations by specifically showing that the zero-curvature problem is equivalent to isomonodromy. However, they apply self-similarity first.

In [**FN**], in one of the two specific examples given, putting the scattering problem for mKdV $q_t + 6q^2q + q_{xxx} = 0$,

$$\begin{aligned} v_{1x} &= -\imath\zeta v_1 + qv_2 \\ v_{2x} &= \imath\zeta v_2 + rv_1, \end{aligned}$$

$$\begin{aligned} v_{1t} &= Av_1 + Bv_2 \\ v_{2t} &= Cv_1 + Dv_2 \end{aligned}$$

(A, B, C, D functions independent of the vector (v_1, v_2)) in self-similar form: $z = x/(3t)^{1/3}, \chi = \zeta(3t)^{1/3}, q(x,t) = (3t)^{-1/3}f(x(3t)^{-1/3})$, using the invariance of the (Baker) eigenvector $\underline{v} = (v_1, v_2)$, namely $\underline{v}(\beta x, \beta^3 t, \beta^{-1}\zeta)$ giving a solution to the same compatibility condition (note that this is the property for which Krichever [**K**] solves explicitly), then $\underline{w}(x(3t)^{-1/3}, \zeta(3t)^{1/3}) = \underline{v}(x,t,\zeta)$

$$\begin{aligned} w_{1\zeta} &= -\imath(4\zeta^2 + x + 2f^2)w_1 + (4\zeta f + 2\imath f')w_2 \\ w_{1\zeta} &= (4\zeta f - 2\imath f')w_1 + \imath(4\zeta^2 + x + 2f^2)w_2 \end{aligned}$$

(using x for $x(3t)^{-1/3}$ and ζ for $\zeta(3t)^{1/3}$) gives a system of ODE's whose Stokes multipliers near an irregular singular point are independent of z only if $w(z)$ satisfies a form of PII. This is a specific example of irregular singular point; the general theory runs along the same lines. A deformation of the system is said to be monodromy-preserving if the Stokes multipliers associated with formal solutions at $\zeta = \infty$, the monodromy matrix at $\zeta = 0$ and the matrix connecting fundamental solutions at $\zeta = 0$ and ∞ are unchanged when x is varied and q, r change as functions of x. For this, it is necessary and sufficient that $r = q_x$ and that q satisfy the second Painlevé equation $q_{xx} = 2q^3 + xq - \nu$. In order to prove this result, the authors give properties of the transform data. The functions r and q are derived via a linear singular integral equation for the columns of the fundamental solution matrix of the system. In the particular case $\nu = 0$ and $f(x) \sim \rho\mathrm{Ai}(x)$, the authors reproduce the solution found in [**AS1**]. When $\nu = n$, the integral equation yields the known rational solutions. This discovery agrees with the theme set by R. Fuchs, when he found an equivalence between PVI and an isomonodromy condition for a linear 2nd-order ODE of Fuchsian (L. Fuchs) type. In particular, the authors give comprehensive references to early examples relating Painlevé equations and isomonodromy (R. Fuchs, G.D. Birkhoff). In the spirit of the three proposals for generalization of PVI in Garnier's doctoral dissertation [**G**], they ask what is the condition on the x-deformation so that the two equations

$$y_x = Ay + y_\zeta, \quad y_{\zeta\zeta} = \left[\sum_0^m a_j \zeta^j + \sum_0^n \left(\frac{3/4}{(\zeta - \lambda_k)^2} + \frac{\rho_k}{\zeta - \lambda_k} \right) \right] y$$

form an integrable system (in the sense of Frobenius, namely compatibility), with A, B assumed rational in ζ; for $n = 0$ and $\lambda_0 = \lambda$ the answer turns out to be PII: $\lambda'' = 2\lambda^3 + x\lambda - \nu$.

What is more intriguing and presented as a speculation in [**FN**] is the idea that both problems should reduce to an algebraic condition. Through examples, [**FN**] advocate associating to an integrable PDE not only a similarity solution in one variable (and its attendant Painlevé problem) but also a multivariable P-type problem, which expresses an isomonodromic deformation.

3.4. The case study. As we were looking for similarity reduction of PDEs that yield PVI in the literature, we also found an early reference for the discovery of the link between integrability and the Painlevé property [**FY**], where the focus is not so much on 'general' similarity reduction but on the applications (Bäcklund transformations, cf. [**FA**] for a survey of applications of inverse scattering to the relevant ODEs). We also note that in [**AC**, Example 6.5.15] there are references, e.g. [**FLMS**], aiming to similarity reduction to PVI, but they appear instead to be relating the similarity ODE to PVI, short of showing equivalence; our understanding was vindicated by [**CGM**], "at the present time, various reductions of this system have been integrated with most of the six Painlevé functions [including the reference given in [**AC**]], but no explicit link with the generic sixth Painlevé equation has been found." In [**CGM**, Section 8], these authors then express their gratitude to the referee for providing reference [**KK**], which we had found earlier: we choose to report that calculation below, although PVI is likewise obtained in[**CGM**]. Lastly, we note that in [**AvdL**] the same integrable equation is reduced to PVI also, but the four PVI parameters are not arbitrary; indeed, 2α, -2β, 2γ and $1 - 2\delta$ are squares of integers.

The program that we briefly sketch [**KK**] extends (in principle) to the whole Drinfel'd-Sokolov integrable hierarchy the symmetry-group reduction leading to ODEs of Painlevé type. However, we see no way to predict which specific Painlevé equation one obtains.

The authors express the three-wave resonant system:

$$\partial_\tau u_i + c_i \partial_\chi u_i = \sqrt{-1}\gamma_i u_j^* u_k^*, \quad \{i, j, k\} = \{1, 2, 3\}$$

as the $(1, 1, 1)$-reduction of the 3-component KP hierarchy, equivalently the Drinfel'd-Sokolov hierarchy associated with the (1,1,1) homogeneous Heisenberg subalgebra of the affine Lie-algebra $\widehat{\mathfrak{gl}}_3$. The link with isomonodromy (also explored by other authors listed in the references of [**KK**]) is provided by a scaling symmetry of the $\widehat{\mathfrak{gl}}_3$ homogenoeous hierarchy, which sends solutions to solutions viewed as elements $g(z; \underline{t}) = \Psi_0(z; \underline{t}) g(z; 0)$, $g(z; 0)$ being an element of the exponentiated algebra (its Kac-Moody group) and

$$\Psi_0(z; \underline{t}) := \exp\left(\sum_{a=1}^n \left(\sum_{j>0} z^j t_j^a \right) E_{aa} \right)$$

(E_{aa} are the diagonal matrices with just one entry equal to 1). The rescaling, by $\lambda \in \mathbb{C}^*$, acting on the Gauss decomposition $g(z;\underline{t}) = g_{<0}(z;\underline{t})^{-1} g_{\geq 0}(z;\underline{t})$ (with respect to the homogeneous gradation,

$$\widehat{\mathfrak{g}} = \oplus_j \in \mathbb{Z}\widehat{\mathfrak{g}}_j,\ \widehat{\mathfrak{g}}_j = \{x \in \widehat{\mathfrak{g}} | [d,x] = jx\},$$

is:

$$\underline{t}_\lambda = (\lambda^j t_j^1, ..., \lambda^j t_j^3)_{j>0}$$
$$\bar{g}_{<0}(z,\underline{t}) = \lambda^{D(\alpha)} g_{<0}(\lambda^{-1}z;\underline{t}_\lambda)\lambda^{-D(\beta)} \bar{g}_{\geq 0}(z,\underline{t}) = \lambda^{D(\alpha)} g_{\geq 0}(\lambda^{-1}z;\underline{t}_\lambda)\lambda^{-D(\beta)}$$

where $D(\alpha) = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$, $D(\beta) = \mathrm{diag}(\beta_1, \beta_2, \beta_3)$, and a choice $\beta_3 = -1$ will be made later to obtain the Schlesinger equations, while the parameters

$$\alpha = \frac{(\beta_1 - \beta_2 - 1)^2}{2}, \beta = \frac{(\alpha_1+1)^2}{2}, \gamma = \frac{(\alpha_2+1)^2}{2}, \delta \frac{1-(\alpha_3+1)^2}{2}$$

will yield the general PVI equation. This permits reducing the equations so that the Baker functions with parameters,

$$\Psi = \Psi^{(0)}(z;\underline{t},\beta) = g_{\geq 0}(0;\underline{t}) z^{D(\beta)},\ \text{or}\ \Psi = \Psi^{(\infty)}(z;\underline{t},\alpha) = g_{<0}(0;\underline{t})\Psi_0(z;\underline{t}) z^{D(\alpha)}$$

both satisfy a (nonautonomous) ODE, compatible with the time evolutions:

$$z\frac{\partial \Psi}{\partial z} = \left(D(\alpha) + \sum_{a=1}^3 \sum_{j>0} j t_j^a B_j^a\right)\Psi$$
$$\frac{\partial \Psi}{\partial t_j^a} = B_j^a \Psi,\ 1 \leq a \leq 3,\ j > 0.$$

The key to transform this into a Schlesinger system (setting $t_i := t_1^i, t_j^i = 0$ for $j > 1$, is a Laplace transform[3]!

$$\Psi(z) \mapsto \Phi(x) = L[\Psi(z)](x) = \int_\gamma e^{-zx}\Psi(z)dz,\ \gamma \text{ such that } \int_\gamma \frac{\partial f}{\partial z} dz = 0.$$

The transformed (Schlesinger) equations have 4 regular singular points (t_1, t_2, t_3 and ∞); the constraint (rather, a WLOG choice) $\beta = -1$ makes the third column of the matrices equal zero, PVI is as customary the isomonodromy condition for the reduction to a 2×2-system on the space $(y_1, y_2, 0)$.

4. Comments and further aspects

We conclude with a discussion of what was and was not achieved, and related directions.

4.1. Our goals. We note that to some extent, in [**BePr**] we answered our question of finding a systematic way to link an integrable hierarchy, through its most general similarity reduction, and a P-type equation, or–in the opposite direction–guess from a P-type equation the attendant integrable hierarchies: we combined two unrelated constructions, sketched in short announcements in the 1980s by H. Flaschka and I.M. Krichever [**Fl**, **K**]: both authors stated that they would then provide the full technique in later papers, which never appeared. Roughly, the concept is this: use the PDE as in Subsection 3.3 to produce monodromy data [**Fl**]; use the monodromy data to build a Baker function and therefore the ODE of isomonodromy, as well as the PDE related to it, as in Subsection 3.3, by Riemann-Roch data on the curve of isospectrality [**K**]. In all this, one gets stuck because

[3]The authors refer to a 1994 paper by J. Harnad for this tool [**Ha**], which here they apply to a general monodromy problem for a 3×3 system with a simple pole at 0 and a double pole at ∞).

solving non-linear equations is not always possible, and they are produced by "clever guesswork" (cf. the Introduction), not by systematic association. The advantage is that the P-type equation is produced by differential algebra techniques, and we developed a "Sato-operator" type of isomonodromy. In fact, in [**BePr**] we were only able to go from Boussinesq to a new third-order equation of P-type: we were able to go from a P-type ODE to a integrable hierarchy which has symmetries yielding the ODE one begins with.

In conclusion, this article collects the connections we found between integrable hierarchies and P-type ODOs, hoping to facilitate further analyses of the exchange of geometric information.

4.2. Computational aspects. There are powerful computational aids that can detect the "classical symmetries" of ODEs and PDEs, namely actions of groups on the independent and dependent variables that leave the equation invariant, referred to as "Lie symmetry algebra" of the (system of) differential equations. References in [**C1**] discuss the available symbolic software, largely based on the method of differential Gröbner bases. However, computational methods for Gröbner bases over non-commutative coefficients only seem to have been proposed in [**Z**] (and published versions of various chapters of this dissertation). This would be a very important technique to study the ring of differential operators, which is non-commutative, and whose maximal-commutative subrings correspond to spectral curves/vector bundles as we illustrated above. One question, for example, is to check for the commuting Ordinary Differential Operators (ODOs) L_4 and L_{4g+2} (of order 4g+2) of [**Mi**], where $L_4 = \left(\partial_x^2 + \sum_{i=0}^3 \alpha_i x^i\right)^2 + g(g+1)\alpha_3 x$, with generic $\alpha_i \in \mathbb{C},\ i = 0, ..., 3$, whether the $\mathbb{C}$-algebra generated by L_4 and L_{4g+2} is maximal-commutative in the Weyl algebra.

In addition, Clarkson and Mansfield have focused on the "nonclassical method of symmetry reduction" [**CM**], namely additional vector fields (obtained by solving in fact fewer conditions) that leave the equations invariant, but do not in general form a Lie algebra. There exists also an algorithmic "direct method" for finding symmetry reductions, but the situation is complicated enough to say that "the precise relationships between these methods has yet to be ascertained" [**C1**].

In our vein of posing overarching questions for this theory to come together, we would rather pose the following:

After symmetry reduction of a PDE whose Bäcklund transformations are known, once an ODE of P-type is found, as well as its Bäcklund transformations, or more generally a Weyl group acting on it, is the reduction diagram equivariant?

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