

Pointed braided tensor categories

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ABSTRACT. We classify finite pointed braided tensor categories admitting a fiber functor in terms of bilinear forms on symmetric Yetter-Drinfeld modules over abelian groups. We describe the groupoid formed by braided equivalences of such categories in terms of certain metric data, generalizing the well-known result of Joyal and Street [JS93] for fusion categories. We study symmetric centers and ribbon structures of pointed braided tensor categories and examine their Drinfeld centers.

1. Introduction

In this paper, we work over an algebraically closed field k of characteristic 0. All tensor categories are assumed to be k -linear and finite. All Hopf algebras and modules over them are defined over k and are assumed to be finite dimensional.

A tensor category is called *pointed* if all its simple objects are invertible. An example of such a category is the category of finite dimensional corepresentations $\text{Corep}(H)$ of a pointed Hopf algebra H . Any pointed tensor category admitting a fiber functor is equivalent to some $\text{Corep}(H)$. The classification of pointed Hopf algebras having abelian group of group-like elements is nearing its completion, see [A14].

This paper deals with classification of *braided* pointed tensor categories. Such a classification is well known in the semisimple case, i.e., for fusion categories. It was proved by Joyal and Street [JS93] that the 1-categorical truncation of the 2-category of braided fusion categories is equivalent to the category of pre-metric groups, i.e., pairs (Γ, q) , where Γ is a finite abelian group and $q : \Gamma \rightarrow k^\times$ is a quadratic form. In particular, braidings on pointed fusion categories are in bijection with abelian 3-cocycles. In the presence of a fiber functor such cocycles are precisely bicharacters on abelian groups. Explicitly, a braided fusion category having a fiber functor is equivalent to $\mathcal{C}(\Gamma, r_0) := \text{Corep}(k[\Gamma], r_0)$, where the r -form r_0 is given by a bicharacter on Γ .

In this work we extend the above results to non-semisimple braided tensor categories. We classify co-quasitriangular pointed Hopf algebras up to tensor equivalence of their corepresentation categories, thereby obtaining a classification of braided tensor categories having a fiber functor.

The work of the second author was partially supported by the National Security Agency grant H98230-16-1-0008 and the National Science Foundation grant DMS-1801198.

THEOREM 1.1. *Let \mathcal{C} be a pointed braided tensor category having a fiber functor. Then \mathcal{C} is completely determined by a finite abelian group Γ , a bicharacter $r_0 : \Gamma \times \Gamma \rightarrow k^\times$, an object $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_-$, and a morphism $r_1 : V \otimes V \rightarrow k$. More precisely,*

$$(1) \quad \mathcal{C} \cong \mathcal{C}(\Gamma, r_0, V, r_1) := \text{Corep}(\mathfrak{B}(V)\#k[\Gamma], r),$$

where $r|_{\Gamma \times \Gamma} = r_0$ and $r|_{V \otimes V} = r_1$.

Here the symmetric center $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))$ has a canonical (possibly trivial) grading by $\mathbb{Z}/2\mathbb{Z}$:

$$\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0)) = \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_+ \oplus \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_-,$$

where $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_+$ denotes the maximal Tannakian subcategory.

Theorem 1.1 is proved in Section 4, where details of the construction of r can be found. Quasitriangular structures on $\mathfrak{B}(V)\#k[\Gamma]$ were explicitly described by Nenciu in [Ne04] in terms of generators of Γ and a basis of V . The classification of co-quasitriangular structures can be obtained by duality. Theorem 1.1 says that every pointed co-quasitriangular Hopf algebra is equivalent to the above by a 2-cocycle deformation. Also, our description of r -forms is given in invariant terms and avoids the use of bases and generators.

We describe the symmetric center of $\mathcal{C}(\Gamma, r_0, V, r_1)$ and show that a pointed braided tensor category is not factorizable unless it is semisimple. We also show that this category is always ribbon and classify its ribbon structures.

We obtain a parameterization of pointed braided tensor categories similar to the parameterization of braided fusion categories by quadratic forms [JS93]. Namely, we introduce the groupoid of *metric data* (\mathcal{P}, V, r) , where \mathcal{P} is a braided fusion category admitting a fiber functor, V is an object in $\mathcal{Z}_{sym}(\mathcal{P})_-$, and $r : V \otimes V \rightarrow k$ is an alternating morphism.

THEOREM 1.2. *The groupoid of isomorphism classes of equivalences of pointed braided tensor categories having a fiber functor is equivalent to the groupoid of metric data.*

Theorem 1.2 is proved in Section 7. The alternating form r can be canonically recovered from the restriction of the squared braiding on two-dimensional objects of a category, see Remark 7.4.

The Drinfeld center of $\mathcal{C}(\Gamma, r_0, V, r_1)$ is not pointed when $V \neq 0$. We show that when $V \cong V^*$ the trivial component of the universal grading of $\mathcal{Z}(\mathcal{C}(\Gamma, r_0, V, r_1))$ is pointed and corresponds to a certain braided vector space with a symplectic bilinear form (the Drinfeld double of V), see Section 3.2 and Theorem 8.5 for details.

The paper is organized as follows.

Section 2 contains background material about braided tensor categories, co-quasitriangular Hopf algebras and their twisting deformations by 2-cocycles.

In Section 3 we discuss quantum linear spaces of symmetric type and their bosonizations $\mathfrak{B}(V)\#k[\Gamma]$. We use Mombelli's classification of Galois objects for quantum linear spaces [Mo11] to obtain a classification of 2-cocycles on such bosonizations in terms of V (equivalently, we obtain a classification of fiber functors on corresponding corepresentation categories), see Proposition 3.9. We also compute the second invariant cohomology group of $\mathfrak{B}(V)\#k[\Gamma]$ in Proposition 3.13.

In Section 4 we prove Theorem 1.1.

In Section 5 we study the symmetric center of $\mathcal{C}(\Gamma, r_0, V, r_1)$. It turns out that this center is always non-trivial if $V \neq 0$.

In Section 6 we show that a pointed braided tensor category always has a ribbon structure and classify all such structures, improving the result of [Ne04].

In Section 7 we prove Theorem 1.2. The correspondence between pointed braided tensor categories and metric data is useful, in particular, for computing the groups of autoequivalences, see Corollary 7.5.

Finally, Section 8 contains a description of the Drinfeld center of the pointed braided tensor category $\mathcal{C}(\Gamma, r_0, V, r_1)$ when V is self-dual. This category is no longer pointed if $V \neq 0$, but it has a faithful grading with a pointed trivial component. We describe the structure of this component and show that it is the corepresentation category of the bosonization of the Drinfeld double of V .

2. Preliminaries

2.1. Finite tensor categories and Hopf algebras. We assume familiarity with basic results of the theory of finite tensor categories [EGNO15], and the theory of Hopf algebras [M93, R11].

For a Hopf algebra H we denote by Δ, ε, S the comultiplication, counit, and antipode of H , respectively. We make use of Sweedler's summation notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}$, $x \in H$. We denote by $*$ the convolution, i.e., the multiplication in the dual Hopf algebra. By H^{cop} we denote the co-opposite Hopf algebra of H , i.e., the algebra H with co-multiplication $\Delta^{\text{op}}(x) = x_{(2)} \otimes x_{(1)}$, $x \in H$. We denote $G(H)$ the group of group-like elements of H (i.e., elements $g \in H$ such that $\Delta(g) = g \otimes g$). Also, we denote $H^+ = \text{Ker}(\varepsilon)$.

Let $\text{Rep}(H)$ and $\text{Corep}(H)$ be tensor categories of left H -modules and right H -comodules, respectively, over a Hopf algebra H . Note that there is a canonical tensor equivalence between $\text{Corep}(H)$ and $\text{Rep}(H^*)$. In general, a tensor category \mathcal{C} is equivalent to the co-representation category of some Hopf algebra H if and only if there exists a fiber functor (i.e., an exact faithful tensor functor) $F : \mathcal{C} \rightarrow \text{Vec}$, where Vec is the tensor category of k -vector spaces.

A tensor category is *pointed* if all of its simple objects are invertible with respect to the tensor product. A Hopf algebra H is *pointed* if $\text{Corep}(H)$ is pointed. The classification of finite dimensional pointed Hopf algebras is still an open problem, though important progress has been made so far (see [A14] and the references therein). The best understood class is that of pointed Hopf algebras with abelian coradical [AS10]. It was shown by Angiono [An13] that such a Hopf algebra H is generated by its group-like elements and skew-primitive elements.

2.2. Braided tensor categories and co-quasitriangular Hopf algebras. A *braiding* on a finite tensor category \mathcal{C} is a natural isomorphism

$$c_{X,Y} : X \otimes Y \rightarrow Y \otimes X, \quad X, Y \in \mathcal{C},$$

satisfying the hexagon axioms. A *braided* tensor category is a pair consisting of a tensor category and a braiding on it.

A *co-quasitriangular* Hopf algebra is a pair (H, r) , where H is a Hopf algebra and $r : H \otimes H \rightarrow k$ is a convolution invertible linear map, called an *r-form*, satisfying the following conditions:

$$\begin{aligned} (2) \quad & x_{(1)}y_{(1)}r(y_{(2)}, x_{(2)}) = r(y_{(1)}, x_{(1)})y_{(2)}x_{(2)}, \\ (3) \quad & r(x, yz) = r(x_{(1)}, z)r(x_{(2)}, y), \\ (4) \quad & r(xy, z) = r(x, z_{(1)})r(y, z_{(2)}), \end{aligned}$$

for all $x, y, z \in H$.

REMARK 2.1. Let $r : H \otimes H \rightarrow k$ be a linear map and let $\varphi_r : H \rightarrow H^{*\text{cop}}$ be defined by $\varphi_r(x) = r(x, -)$, for all $x \in H$. Then r satisfies (3), respectively (4), if and only if φ_r is a coalgebra map, respectively an algebra map.

There is a bijective correspondence between the set of *r-forms* on a Hopf algebra H and the set of braidings on $\text{Corep}(H)$. The braiding corresponding to $r : H \otimes H \rightarrow k$ is given by

$$(5) \quad c_{U,V} : U \otimes V \rightarrow V \otimes U, \quad u \otimes v \mapsto \sum r(u_{(1)}, v_{(1)})v_{(0)} \otimes u_{(0)},$$

where U, V are H -comodules and $u \in U, v \in V$.

We denote by $\text{Corep}(H, r)$ the braided tensor category $\text{Corep}(H)$ with braiding given by r .

2.3. Ribbon categories and ribbon elements. A *ribbon* tensor category is a braided tensor category \mathcal{C} together with a *ribbon structure* on it, i.e., an element $\theta \in \text{Aut}(\text{id}_{\mathcal{C}})$ such that

$$\begin{aligned} (6) \quad & \theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \\ (7) \quad & (\theta_X)^* = \theta_{X^*} \end{aligned}$$

for all $X, Y \in \mathcal{C}$.

If (H, r) is a co-quasitriangular Hopf algebra then ribbon structures on $\text{Corep}(H, r)$ are in bijection with *ribbon elements* of (H, r) , i.e., convolution invertible central elements $\alpha \in H^*$ such that $\alpha \circ S = \alpha$ and

$$\alpha(xy) = \alpha(x_{(1)})\alpha(y_{(1)})(r_{21} * r)(x_{(2)}, y_{(2)})$$

for all $x, y \in H$. The ribbon structure associated to the ribbon element α is

$$\theta_V : V \rightarrow V, \quad v \mapsto \sum \alpha(v_{(1)})v_{(0)}.$$

The ribbon elements of (H, r) can be determined in the following way (see [R94, Proposition 2] where the result appears in dual form). Let $\eta : H \rightarrow k$, $\eta(h) = r(h_{(2)}, S(h_{(1)}))$, $h \in H$, be the Drinfeld element of (H, r) . Then $\eta^{-1}(h) = r(S^2(h_{(2)}), h_{(1)})$, for all $h \in H$, the element $(\eta \circ S) * \eta^{-1}$ is a group-like element of H^* , and the map

$$\gamma \mapsto \gamma * \eta$$

establishes a one-to-one correspondence between the set of group-like elements $\gamma \in H^*$ satisfying $\gamma^2 = (\eta \circ S) * \eta^{-1}$ and $S_{H^*}^2(p) = \gamma^{-1} * p * \gamma$, for all $p \in H^*$, and the set of ribbon elements of (H, r) .

2.4. Pointed braided fusion categories. Let \mathcal{C} be a pointed braided fusion category. The isomorphism classes of simple objects of \mathcal{C} form a finite abelian group Γ . The braiding determines a function $c : \Gamma \times \Gamma \rightarrow k^\times$ and the function $q : \Gamma \rightarrow k^\times$, $q(g) = c(g, g)$, $g \in \Gamma$, is a *quadratic form* on Γ , i.e., the symmetric function

$$b(g, h) = \frac{q(gh)}{q(g)q(h)}, \quad g, h \in \Gamma,$$

is a bicharacter on Γ and $q(g^{-1}) = q(g)$ for all $g \in \Gamma$. The result of [JS93] (see also [DGNO10, Appendix D]) says, in particular, that the assignment

$$\mathcal{C} \mapsto (\Gamma, q)$$

determines an equivalence between the 1-categorical truncation of the 2-category of pointed braided fusion categories and the category of pre-metric groups. The objects of the latter category are finite abelian groups equipped with a quadratic form, and morphisms are group homomorphisms preserving the quadratic forms.

A pointed braided fusion category associated to (Γ, q) is determined up to a braided tensor equivalence. This category admits a fiber functor if and only if q is the diagonal of some bilinear form $r_0 : \Gamma \times \Gamma \rightarrow k^\times$, i.e., $q(g) = r_0(g, g)$ for all $g \in \Gamma$. Such a category is the corepresentation category of a co-quasitriangular pointed semisimple Hopf algebra $k[\Gamma]$ with the r -form r_0 . We will use the following notation:

$$\mathcal{C}(\Gamma, r_0) := \text{Corep}(k[\Gamma], r_0).$$

2.5. The symmetric center. Let \mathcal{C} be a braided tensor category with braiding $\{c_{X,Y}\}_{X,Y \in \mathcal{C}}$. The *symmetric center* $\mathcal{Z}_{sym}(\mathcal{C})$ of \mathcal{C} is the full tensor subcategory of \mathcal{C} consisting of all objects Y such that $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all $X \in \mathcal{C}$.

A braided tensor category \mathcal{C} is called *symmetric* if $\mathcal{Z}_{sym}(\mathcal{C}) = \mathcal{C}$. Any symmetric fusion category \mathcal{C} has a canonical (possibly trivial) $\mathbb{Z}/2\mathbb{Z}$ -grading

$$(8) \quad \mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-,$$

where \mathcal{C}_+ is the maximal Tannakian subcategory of \mathcal{C} [De02]. In terms of the canonical ribbon element θ of \mathcal{C} , one has $\theta_X = \pm \text{id}_X$ when $X \in \mathcal{C}_\pm$.

A braided tensor category \mathcal{C} is called *factorizable* if $\mathcal{Z}_{sym}(\mathcal{C})$ is trivial.

Let $\mathcal{C} = \text{Corep}(H)$, where H is a co-quasitriangular Hopf algebra with an r -form r (so that the braiding of \mathcal{C} is given by (5)). Then $\mathcal{Z}_{sym}(\mathcal{C}) = \text{Corep}(H_{sym})$ for a Hopf subalgebra $H_{sym} \subset H$. The category \mathcal{C} is symmetric if and only if $H_{sym} = H$ and it is factorizable if and only if $H_{sym} = k1$. This Hopf subalgebra H_{sym} was described by Natale in [Na06] (note that in [Na06] quasitriangular Hopf algebras were considered while we deal with the dual situation). Below we reproduce this description using our terminology.

Consider the linear map $\Phi_r : H \rightarrow H^*$ given by

$$\Phi_r(x)(y) = r(y_{(1)}, x_{(1)}) r(x_{(2)}, y_{(2)}), \quad x, y \in H.$$

Its image $\Phi_r(H)$ is a normal coideal subalgebra of H^* . Hence, $H^* \Phi_r(H)^+$ is a Hopf ideal of H^* . We have $H_{sym} = (H^* \Phi_r(H)^+)^{\perp}$. Here for $I \subset H^*$ we denote $I^{\perp} \subset H$ its annihilator, i.e, $I^{\perp} = \{x \in H \mid f(x) = 0 \text{ for all } f \in I\}$. Explicitly,

$$(9) \quad H_{sym} = \{x \in H \mid x_{(1)} r(x_{(2)}, y_{(1)}) r(y_{(2)}, x_{(3)}) = \varepsilon(y)x, \text{ for all } y \in H\},$$

Equivalently, H_{sym} consists of all $x \in H$ such that the squared braiding $c_{H,H}^2 : H \otimes H \rightarrow H \otimes H$ fixes $x \otimes y$ for all $y \in H$.

2.6. The Drinfeld center of a tensor category and Yetter-Drinfeld modules. An important example of a braided tensor category is the *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ of a finite tensor category \mathcal{C} . The objects of $\mathcal{Z}(\mathcal{C})$ are pairs (Z, γ) consisting of an object Z of \mathcal{C} and a natural isomorphism $\gamma : - \otimes Z \rightarrow Z \otimes -$ satisfying a hexagon axiom.

The braiding of $\mathcal{Z}(\mathcal{C})$ is

$$c_{(Z,\gamma),(Z',\gamma')} = \gamma'_Z : (Z, \gamma) \otimes (Z', \gamma') \rightarrow (Z', \gamma') \otimes (Z, \gamma).$$

EXAMPLE 2.2. If H is a finite dimensional Hopf algebra then the Drinfeld center of $\text{Rep}(H)$ is braided equivalent to $\text{Rep}(D(H))$, where $D(H)$ is the *Drinfeld double* of H . As a coalgebra, $D(H) = H^{*\text{cop}} \otimes H$, where $H^{*\text{cop}}$ is the co-opposite dual of H , while the algebra structure is given by

$$(p \otimes h)(p' \otimes h') = p(h_{(1)} \rightharpoonup p' \leftarrow S^{-1}(h_{(3)})) \otimes h_{(2)}h', \quad p, p' \in H^*, \quad h, h' \in H,$$

where $(h \rightharpoonup p \leftarrow g)(x) = p(gxh)$, for all $h, g, x \in H, p \in H^*$.

It is well-known that $\text{Rep}(D(H))$ is braided equivalent to the category ${}^H_H\mathcal{YD}$ of (left) *Yetter-Drinfeld modules* over H . An object V in this category has simultaneously a structure of a left H -module, $h \otimes v \mapsto h \cdot v$, and a structure of a left H -comodule $\delta : V \rightarrow H \otimes V$, $\delta(v) = v_{(-1)} \otimes v_{(0)}$, such that the following condition is satisfied:

$$\delta(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, \quad v \in V.$$

Morphisms between such objects are linear maps preserving both the action and the co-action of H . The braiding of ${}^H_H\mathcal{YD}$ is given by:

$$c_{U,V} : U \otimes V \rightarrow V \otimes U, \quad u \otimes v \mapsto u_{(-1)} \cdot v \otimes u_{(0)}$$

for all $u \in U, v \in V$ and $U, V \in {}^H_H\mathcal{YD}$.

For a finite group Γ we denote ${}^k_{k[\Gamma]}\mathcal{YD}$ by ${}^\Gamma_\Gamma\mathcal{YD}$. By definition, this category consists of Γ -equivariant vector bundles on Γ , where Γ acts on itself by conjugation. If Γ is abelian then

$${}^\Gamma_\Gamma\mathcal{YD} \simeq \mathcal{Z}(\text{Vec}_\Gamma) \simeq \mathcal{C}(\Gamma \times \widehat{\Gamma}, h),$$

where $h : \Gamma \times \widehat{\Gamma} \rightarrow k^\times$ is the canonical quadratic form, $h(g, \chi) = \chi(g), g \in \Gamma, \chi \in \widehat{\Gamma}$.

REMARK 2.3. For a braided tensor category \mathcal{C} there exist canonical braided tensor embeddings

$$(10) \quad \mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c_{-,X}) \quad \text{and} \quad \mathcal{C}^{\text{rev}} \hookrightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c_{X,-}^{-1}).$$

The intersection of images of \mathcal{C} and \mathcal{C}^{rev} in $\mathcal{Z}(\mathcal{C})$ is equivalent to $\mathcal{Z}_{\text{sym}}(\mathcal{C})$, the symmetric center of \mathcal{C} .

2.7. 2-cocycles and deformations. A *2-cocycle* on H is a convolution invertible linear map $\sigma : H \otimes H \rightarrow k$ such that $\sigma(x, 1) = \varepsilon(x) = \sigma(1, x)$ and

$$(11) \quad \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

for all $x, y, z \in H$. Two 2-cocycles σ and σ' are *gauge equivalent* if there exists a convolution invertible map $u : H \rightarrow k$ such that

$$\sigma'(x, y) = u^{-1}(x_{(1)})u^{-1}(y_{(1)})\sigma(x_{(2)}, y_{(2)})u(x_{(3)}y_{(3)}), \quad x, y \in H.$$

for all $x, y \in H$.

The isomorphism classes of fiber functors on $\text{Corep}(H)$ are in bijection with the set of gauge equivalence classes of 2-cocycles on H .

Twisting the multiplication of H on both sides by a 2-cocycle σ , we obtain a new Hopf algebra, denoted H^σ and called a *cocycle deformation* of H . We have $H^\sigma = H$ as a coalgebra and the multiplication of H^σ is given by

$$(12) \quad x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}), \quad x, y \in H.$$

If H is a co-quasitriangular Hopf algebra with an r -form r then H^σ is also co-quasitriangular with r -form r^σ , given by

$$(13) \quad r^\sigma(x, y) = \sigma(y_{(1)}, x_{(1)})r(x_{(2)}, y_{(2)})\sigma^{-1}(x_{(3)}, y_{(3)}) \quad x, y \in H.$$

For gauge equivalent 2-cocycles σ and σ' co-quasitriangular Hopf algebras H^σ and $H^{\sigma'}$ are isomorphic.

A 2-cocycle σ on H is called *invariant* if

$$(14) \quad \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = x_{(1)}y_{(1)}\sigma(x_{(2)}, y_{(2)})$$

for all $x, y \in H$. (Invariant cocycles are called *lazy cocycles* in [BC06]). Note that σ is invariant if and only if $H^\sigma = H$ as Hopf algebras. The set of invariant 2-cocycles is a group under convolution product denoted by $Z_{\text{inv}}^2(H)$.

For a convolution invertible linear map $u : H \rightarrow k$ such that $u(1) = 1$ and $u(x_{(1)})x_{(2)} = x_{(1)}u(x_{(2)})$, for all $x \in H$, the map

$$\sigma_u : H \otimes H \rightarrow k, \quad \sigma_u(x, y) = u(x_{(1)})u(y_{(1)})u^{-1}(x_{(2)}y_{(2)}), \quad x, y \in H$$

is an invariant 2-cocycle. The set of all such 2-cocycles is a subgroup of $Z_{\text{inv}}^2(H)$ denoted by $B_{\text{inv}}^2(H)$.

The *second invariant (or lazy) cohomology group* of H [BC06] is the quotient group

$$H_{\text{inv}}^2(H) = Z_{\text{inv}}^2(H)/B_{\text{inv}}^2(H).$$

For example, if Γ is a group, then $H_{\text{inv}}^2(k[\Gamma]) = H^2(\Gamma, k^\times)$, the second cohomology group of Γ with coefficients in k^\times .

2.8. Galois objects and 2-cocycles. We recall here the connection between Galois objects and 2-cocycles. Let H be a Hopf algebra.

A *left H -Galois object* is a non-zero left H -comodule algebra A such that $A^{\text{co}H} = k$ and the linear map $A \otimes A \rightarrow H \otimes A, a \otimes b \mapsto a_{(-1)} \otimes a_{(0)}b$, for all $a, b \in A$, is bijective.

If σ is a 2-cocycle on H then H , with the comodule structure given by Δ and multiplication

$$(15) \quad x \cdot y = x_{(1)}y_{(1)}\sigma^{-1}(x_{(2)}, y_{(2)}), \quad x, y \in H,$$

is a left H -Galois object, denoted by $H_{\sigma^{-1}}$.

Conversely, if A is a left H -Galois object, then there exists a left H -colinear isomorphism $\psi : H \rightarrow A$ such that $\psi(1) = 1$. The map $\kappa : H \otimes H \rightarrow k$, defined by

$$(16) \quad \kappa(x, y) = \varepsilon\left(\psi^{-1}(\psi(x)\psi(y))\right), \quad x, y \in H$$

is convolution invertible, $\sigma := \kappa^{-1}$ is a 2-cocycle and $\psi : H_{\sigma^{-1}} \rightarrow A$ is a left H -comodule algebra isomorphism.

3. Quantum linear spaces of symmetric type

3.1. Quantum linear spaces. An important class of pointed Hopf algebras with a given abelian group Γ of group-like elements can be constructed as follows [AS98].

Let g_1, \dots, g_n be elements of Γ and let χ_1, \dots, χ_n be elements of the dual group $\widehat{\Gamma}$ such that

$$\chi_i(g_i) \neq 1 \quad \text{and} \quad \chi_j(g_i)\chi_i(g_j) = 1$$

for all $i, j = 1, \dots, n, i \neq j$.

DEFINITION 3.1. A *quantum linear space* associated to the above datum $(g_1, \dots, g_n, \chi_1, \dots, \chi_n)$ is the Yetter-Drinfeld module

$$(17) \quad V = \bigoplus_{i=1}^n kx_i \in {}_{\Gamma}\mathcal{YD},$$

with $h \cdot x_i = \chi_i(h)x_i$, for all $h \in \Gamma$, and $\rho(x_i) = g_i \otimes x_i$, for all i .

Let V_g^{χ} denote the simple object in ${}_{\Gamma}\mathcal{YD}$ corresponding to $g \in \Gamma$ and $\chi \in \widehat{\Gamma}$. Then $x_i \in V_{g_i}^{\chi_i}, i = 1, \dots, n$.

The braiding on $V \otimes V$ on the basic elements $x_i \otimes x_j, i, j = 1 \dots, n$, is given by

$$(18) \quad c_{V \otimes V}(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i.$$

DEFINITION 3.2. We will say that a quantum linear space V is of *symmetric type* if $c_{V,V}^2 = \text{id}_{V \otimes V}$ (i.e., $\chi_i(g_i) = -1$ for all $i = 1, \dots, n$).

Note that this definition does not depend on the choice of a “basis” $g_i, \chi_i, i = 1, \dots, n$.

REMARK 3.3. Let θ denote the canonical ribbon element of ${}_{\Gamma}\mathcal{YD}$. Its value on the object $V_{g_i}^{\chi_i}$ is $\chi_i(g_i)$. Thus, a quantum linear space V of symmetric type satisfies $\theta_V = -\text{id}_V$.

Note that the transposition map

$$(19) \quad \tau_{V,V} : V \otimes V \rightarrow V \otimes V : v_1 \otimes v_2 \mapsto v_2 \otimes v_1, \quad v_1, v_2 \in V,$$

is a morphism in ${}_{\Gamma}\mathcal{YD}$.

LEMMA 3.4. Let $V \in {}_{\Gamma}\mathcal{YD}$ be a quantum linear space of symmetric type and $r : V \otimes V \rightarrow k$ be a morphism in ${}_{\Gamma}\mathcal{YD}$. Then $r \circ c_{V,V} = -r \circ \tau_{V,V}$.

PROOF. It suffices to check that $c_{U,U^*} = -\tau_{U,U^*}$ for every simple object $U = V_{g_i}^{\chi_i} \subset V$ such that U^* is also a subobject of V . In this case $(g_i, \chi_i) = (g_j^{-1}, \chi_j^{-1})$ for some j . Therefore,

$$c_{U,U^*}(y \otimes y') = \chi_j(g_i)y' \otimes y = \chi_i^{-1}(g_i)y' \otimes y = -y' \otimes y$$

for all $y \in U, y' \in U^*$, as required. □

Given a quantum linear space V we associate to it the bosonization $\mathfrak{B}(V)\#k[\Gamma]$ of the Nichols algebra $\mathfrak{B}(V)$ by $k[\Gamma]^1$. This Hopf algebra is generated by the group-like elements $h \in \Gamma$ and the $(g_i, 1)$ -skew primitive elements x_i (i.e., such that

¹We will abuse the terminology and will also refer to $\mathfrak{B}(V)\#k[\Gamma]$ as a quantum linear space.

$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1$, $i = 1, \dots, n$, satisfying the following relations:

$$\begin{aligned} hx_i &= \chi_i(h)x_ih, & x_i^{h_i} &= 0, & h &\in \Gamma, \quad i = 1, \dots, n, \\ x_i x_j &= \chi_j(g_i)x_j x_i, & & & i, j &= 1, \dots, n, \end{aligned}$$

where h_i is the order of the root of unity $\chi_i(g_i)$. The set

$$\{gx_1^{i_1} \cdots x_n^{i_n} \mid g \in \Gamma, 0 \leq i_j < h_j, j = 1, \dots, n\}$$

is a basis of $\mathfrak{B}(V)\#k[\Gamma]$.

If V is a quantum linear space then the liftings of $\mathfrak{B}(V)\#k[\Gamma]$, i.e., the pointed Hopf algebras H for which there exists a Hopf algebra isomorphism

$$\text{gr } H \simeq \mathfrak{B}(V)\#k[\Gamma],$$

where $\text{gr } H$ is the graded Hopf algebra associated to the coradical filtration of H , were classified in [AS98, Theorem 5.5]. Namely, for any such a lifting H , there exist scalars $\mu_i \in \{0, 1\}$ and $\lambda_{ij} \in k$ ($1 \leq i < j \leq n$), such that

$$\begin{aligned} \mu_i &\text{ is arbitrary if } g_i^{h_i} \neq 1 \text{ or } \chi^{r_i} = 1, \text{ and } \mu_i = 0 \text{ otherwise,} \\ \lambda_{ij} &\text{ is arbitrary if } g_i g_j \neq 1 \text{ and } \chi_i \chi_j = 1, \text{ and } \lambda_{ij} = 0 \text{ otherwise.} \end{aligned}$$

The Hopf algebra H is generated by the group-like elements $g \in \Gamma$ and the $(g_i, 1)$ -skew-primitive elements a_i , $i = 1, \dots, n$, such that

$$\begin{aligned} ga_i &= \chi_i(g)a_i g, & a_i^{h_i} &= \mu_i(1 - g_i^{h_i}), & i &= 1, \dots, n, \\ a_i a_j &= \chi_j(g_i)a_j a_i + \lambda_{ij}(1 - g_i g_j), & & & 1 \leq i < j \leq n. \end{aligned}$$

It was shown in [Ma01] that these liftings are cocycle deformations of $\mathfrak{B}(V)\#k[\Gamma]$ (see Section 2.7).

REMARK 3.5. Suppose that V is a quantum linear space of symmetric type. Then $x_i^2 = 0$ for all $i = 1, \dots, n$. For a subset $P = \{i_1, i_2, \dots, i_s\} \subseteq \{1, \dots, n\}$ such that $i_1 < i_2 < \dots < i_s$ we denote the element $x_{i_1} \cdots x_{i_s}$ by x_P and use the convention that $x_\emptyset = 1$. Clearly, the set $\{gx_P \mid g \in \Gamma, P \subseteq \{1, \dots, n\}\}$ is a basis of $\mathfrak{B}(V)\#k[\Gamma]$.

Let $F \subseteq P$ be subsets of $\{1, \dots, n\}$ and let $\psi(P, F)$ be the element of k such that $x_P = \psi(P, F)x_F x_{P \setminus F}$. Thus,

$$(20) \quad \psi(P, F) = \prod_{\substack{j \in F, i \in P \setminus F \\ i < j}} \chi_j(g_i).$$

It is easy to check that the comultiplication formula for x_P is given by

$$(21) \quad \Delta(x_P) = \sum_{F \subseteq P} \psi(P, F)g_F x_{P \setminus F} \otimes x_F,$$

where $g_F = \prod_{i \in F} g_i$ and $g_\emptyset = 1$.

3.2. The double of a quantum linear space. We introduce here a construction which will appear in Section 8 where we discuss the adjoint subcategory of the center of a pointed braided finite tensor category.

Let $V \in {}^\Gamma \mathcal{YD}$ be the quantum linear space of symmetric type associated to a datum $(g_1, \dots, g_n, \chi_1, \dots, \chi_n)$. Let Σ be the subgroup of $\Gamma \times \widehat{\Gamma}$ generated by (g_i, χ_i^{-1}) , $i = 1, \dots, n$, and define characters $\varphi_i : \Sigma \rightarrow k^\times$ by

$$\varphi_i(g, \chi) = \chi_i(g), \quad \text{for all } (g, \chi) \in \Sigma, \quad i = 1, \dots, n.$$

We have

$$\varphi_i(g_i, \chi_i^{-1}) = -1 \quad \text{and} \quad \varphi_j(g_i, \chi_i^{-1})\varphi_i(g_j, \chi_j^{-1}) = 1$$

for all $i, j = 1, \dots, n$. Thus we can consider the quantum linear space of symmetric type $W \in \Sigma \mathcal{YD}$ associated to the datum $((g_1, \chi_1^{-1}), \dots, (g_n, \chi_n^{-1}), \varphi_1, \dots, \varphi_n)$.

DEFINITION 3.6. We call the quantum linear space $D(V) := W \oplus W^* \in \Sigma \mathcal{YD}$ the *Drinfeld double* of V .

Note that the quantum linear space $D(V)$ is of symmetric type.

There is a canonical bilinear form $r_{D(V)} : D(V) \otimes D(V) \rightarrow k$ given by

$$(22) \quad r_{D(V)}((w, f) \otimes (w', f')) := \text{ev}_W(f \otimes w') + \text{ev}_{W^*}c_{W, W^*}(w \otimes f')$$

for all $w, w' \in W, f, f' \in W^*$, where $\text{ev}_W : W^* \otimes W \rightarrow k$ is the evaluation morphism and $c_{W, W^*} : W \otimes W^* \rightarrow W^* \otimes W$ is the braiding in $\Sigma \mathcal{YD}$.

Note that $r_{D(V)}$ is a symplectic bilinear form on $W \oplus W^*$. Indeed, if $\{x_i\}$ is a basis of W such that $x_i \in W^{\varphi_i}_{(g_i, \chi_i^{-1})}$, $i = 1, \dots, n$, and $\{x_i^*\}$ is the dual basis of W^* , then the matrix of $r_{D(V)}$ with respect to the basis $(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ is

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the n -by- n identity matrix.

3.3. Galois objects for quantum linear spaces. Let $V \in \Gamma \mathcal{YD}$ be a quantum linear space of symmetric type and let $H = \mathfrak{B}(V) \# k[\Gamma]$. In [Mo11] Mombelli classified equivalence classes of exact indecomposable $\text{Rep}(H)$ -module categories. In particular, he classified H -Galois objects and, hence, 2-cocycles on H . Here we recall this classification, see [Mo11, Section 4] for details. Galois objects for some quantum linear spaces were also classified by Bichon in [Bi06].

A typical H -Galois object is determined by a 2-cocycle $\psi \in Z^2(\Gamma, k^\times)$ and two families of scalars $\xi = (\xi_i)_{i=1, \dots, n}$ and $\alpha = (\alpha_{ij})_{1 \leq i < j \leq n}$, satisfying:

$$(23) \quad \xi_i = 0 \quad \text{if} \quad \chi_i^2(g) \neq \frac{\psi(g, g_i^2)}{\psi(g_i^2, g)},$$

$$(24) \quad \alpha_{ij} = 0 \quad \text{if} \quad \chi_i \chi_j(g) \neq \frac{\psi(g, g_i g_j)}{\psi(g_i g_j, g)},$$

for all $g \in \Gamma$. To this datum one assigns a left H -comodule algebra $\mathcal{A}(\psi, \xi, \alpha)$ generated as an algebra by $\{e_g\}_{g \in \Gamma}$ and v_1, \dots, v_n subject to the relations:

$$\begin{aligned} e_f e_g &= \psi(f, g) e_{fg}, & f, g \in \Gamma \\ e_f v_i &= \chi_i(f) v_i e_f, & f \in \Gamma, i = 1, \dots, n \\ v_i v_j - \chi_j(g_i) v_j v_i &= \alpha_{ij} e_{g_i g_j}, & 1 \leq i < j \leq n \\ v_i^2 &= \xi_i e_{g_i^2}, & i = 1, \dots, n \end{aligned}$$

The left H -comodule structure of $\mathcal{A}(\psi, \xi, \alpha)$ is $\lambda : \mathcal{A}(\psi, \xi, \alpha) \rightarrow H \otimes \mathcal{A}(\psi, \xi, \alpha)$,

$$\lambda(v_i) = g_i \otimes v_i + x_i \otimes 1 \quad \text{and} \quad \lambda(e_f) = f \otimes e_f$$

for all $i = 1, \dots, n$ and $f \in \Gamma$. Two H -Galois objects $\mathcal{A}(\psi, \xi, \alpha)$ and $\mathcal{A}(\psi', \xi', \alpha')$ are isomorphic if and only if ψ, ψ' are cohomologous, $\xi = \xi'$, and $\alpha = \alpha'$.

REMARK 3.7. Suppose that $\psi = 1$. Then the 2-cocycle σ corresponding to the H -Galois object $\mathcal{A}(1, \xi, \alpha)$ above satisfies

$$\sigma(x_i, x_j) - \chi_j(g_i)\sigma(x_j, x_i) = \alpha_{ij}, \quad \text{and} \quad \sigma(x_i, x_i) = \xi_i, \quad 1 \leq i < j \leq n.$$

In this case the conditions (23) and (24) are equivalent to $\sigma - \sigma \circ c_{V,V} : V \otimes V \rightarrow k$ being a Γ -module map.

3.4. 2-cocycles on quantum linear spaces of symmetric type. Let $V \in {}_{\Gamma}\mathcal{YD}$ be a quantum linear space of symmetric type with braiding $c_{V,V} : V \otimes V \rightarrow V \otimes V$ and let $b : V \otimes V \rightarrow k$ be a bilinear form.

Let $\tau_{V,V} : V \otimes V \rightarrow V \otimes V$ denote the transposition map.

DEFINITION 3.8. We will say that b is *symmetric* (respectively, *alternating*) if $b \circ \tau_{V,V} = b$ (respectively, $b \circ \tau_{V,V} = -b$).

There is a canonical decomposition of b into the sum of symmetric and alternating parts:

$$(25) \quad b = b_{sym} + b_{alt},$$

where $b_{sym} = \frac{1}{2}(b + b \circ \tau_{V,V})$ and $b_{alt} = \frac{1}{2}(b - b \circ \tau_{V,V})$.

We will denote $\text{Sym}_{\Gamma}^2(V^*)$ (respectively, $\text{Alt}_{\Gamma}^2(V^*)$) the spaces of symmetric (respectively, alternating) morphisms $V \otimes V \rightarrow k$ in ${}_{\Gamma}\mathcal{YD}$. We have

$$\text{Sym}_{\Gamma}^2(V^*) \subset S^2(V^*), \quad \text{Alt}_{\Gamma}^2(V^*) \subset \wedge^2(V^*),$$

where $S^2(V^*)$ and $\wedge^2(V^*)$ are spaces of usual symmetric and alternating bilinear forms on the vector space V .

Let $H = \mathfrak{B}(V) \# k[\Gamma]$.

Let $\sigma : H \otimes H \rightarrow k$ be a 2-cocycle on H such that $\sigma|_{\Gamma \times \Gamma} = 1$ and set

$$(26) \quad \alpha = \frac{1}{2}\sigma|_{V \otimes V} \circ (\text{id}_{V \otimes V} - c_{V \otimes V}).$$

Denote

$$\alpha_{ij} := \alpha(x_i, x_j) = \frac{1}{2}(\sigma(x_i, x_j) - \chi_j(g_i)\sigma(x_j, x_i)), \quad i, j = 1, \dots, n.$$

By Remark 3.7, $\alpha : V \otimes V \rightarrow k$ is a Γ -module map and $\alpha \circ c_{V,V} = -\alpha$.

We have

$$(27) \quad \alpha_{ij} = -\chi_j(g_i)\alpha_{ji}$$

and

$$(28) \quad \alpha_{ij} = 0 \text{ if } \chi_i\chi_j \neq \varepsilon, \quad i, j = 1, \dots, n.$$

Conversely, given $\alpha : V \otimes V \rightarrow k$ such that scalars $\{\alpha_{ij} = \alpha(x_i, x_j)\}_{i,j=1}^n$ satisfy conditions (27) and (28) it follows from Remark 3.7 that there is a 2-cocycle σ on H such that

$$\sigma|_{\Gamma \times \Gamma} = 1 \quad \text{and} \quad \sigma|_{V \otimes V} \circ (\text{id}_{V \otimes V} - c_{V \otimes V}) = \alpha.$$

Such a 2-cocycle σ is unique up to a gauge equivalence.

We summarize these results in the following statement.

PROPOSITION 3.9. *The map $\sigma \mapsto \sigma|_{V \otimes V} \circ (\text{id}_{V \otimes V} - c_{V \otimes V})$ is a bijection between the set of gauge equivalence classes of 2-cocycles on H whose restriction on Γ is trivial and the set of bilinear forms r on V that are Γ -module maps satisfying $r \circ c_{V,V} = -r$.*

Next, we discuss invariant 2-cocycles on H .

PROPOSITION 3.10. *Let σ be a 2-cocycle on H whose restriction on Γ is trivial and let α be defined as in (26). Then σ is invariant if and only if $\alpha(x_i, x_j) = 0$ whenever $g_i g_j \neq 1$, i.e., if and only if α is a morphism in ${}_{\Gamma}\mathcal{YD}$.*

PROOF. Suppose that σ is invariant. Taking $x = x_i, y = g \in \Gamma$ in (14) we obtain $\sigma(x_i, g) = 1$ for all $i = 1, \dots, n$ and $g \in \Gamma$. Similarly, taking $x = g$ and $y = x_i$ we get $\sigma(g, x_i) = 1$. Next, taking $x = x_i, y = x_j$ in (14) we obtain $\sigma(x_i, x_j)(1 - g_i g_j) = 0$ for all $i, j = 1, \dots, n$. This means that σ is a Γ -comodule map. Hence, α is a Γ -comodule map. Combining this with Proposition 3.9 we see that α is a morphism in ${}_{\Gamma}\mathcal{YD}$.

Conversely, suppose that a 2-cocycle σ on H is such that $\sigma|_{\Gamma \times \Gamma} = 1$ and α is a morphism in ${}_{\Gamma}\mathcal{YD}$. Then the multiplication in the corresponding twisted Hopf algebra H^σ satisfies relations $g \cdot_\sigma x_i = \chi_i(g)x_i \cdot_\sigma g$ and

$$x_i \cdot_\sigma x_j - \chi_j(g_i)x_j \cdot_\sigma x_i = \alpha_{ij}(1 - g_i g_j), \quad i, j = 1, \dots, n.$$

But the right hand side of the last equality is equal to 0, so that $H^\sigma = H$, i.e., σ is invariant. □

COROLLARY 3.11. *The map $\sigma \mapsto (\sigma|_{V \otimes V})_{sym}$ is an isomorphism between the space of gauge equivalence classes of invariant 2-cocycles on H whose restriction on Γ is trivial and the space $\text{Sym}_{\Gamma}^2(V^*)$.*

PROOF. Note that $\text{id}_{V \otimes V} - c_{V,V}$ is an invertible morphism, so by Propositions 3.9 and 3.10 for an invariant twist σ its restriction $\sigma|_{V \otimes V}$ is a morphism in ${}_{\Gamma}\mathcal{YD}$. Using Lemma 3.4 we get

$$\sigma|_{V \otimes V} \circ \tau_{V,V} = -\sigma|_{V \otimes V} \circ c_{V,V} = \sigma,$$

i.e., $\sigma|_{V \otimes V} \circ (\text{id}_{V \otimes V} - c_{V \otimes V}) = 2(\sigma|_{V \otimes V})_{sym}$. □

We now analyze the general situation when $\sigma|_{\Gamma \times \Gamma}$ is not necessarily trivial.

Let Γ_0 denote the subgroup of Γ generated by $g_i, i = 1, \dots, n$.

PROPOSITION 3.12. *Let σ be an invariant 2-cocycle on H . There is $\rho \in Z^2(\Gamma/\Gamma_0, k^\times)$ such that $\sigma|_{\Gamma \times \Gamma}$ is cohomologous to $\rho \circ (\pi_{\Gamma_0} \times \pi_{\Gamma_0})$, where $\pi_{\Gamma_0} : \Gamma \rightarrow \Gamma/\Gamma_0$ is the quotient homomorphism.*

PROOF. It suffices to check that the alternating bilinear form $\text{alt}(\sigma) : \Gamma \times \Gamma \rightarrow k^\times$ given by

$$(29) \quad \text{alt}(\sigma)(g, h) = \frac{\sigma(g, h)}{\sigma(h, g)}, \quad g, h \in \Gamma$$

vanishes on $\Gamma \times \Gamma_0$. But this follows from invariance of σ since we must have $\sigma(g_i, g) = \sigma(g, g_i) = 1$ for all $i = 1, \dots, n$ and $g \in \Gamma$. □

PROPOSITION 3.13. $H_{inv}^2(H) \cong H^2(\Gamma/\Gamma_0, k^\times) \times \text{Alt}_{\Gamma}^2(V^*)$.

PROOF. By Corollary 3.11 the group $\text{Alt}_{\Gamma}^2(V^*)$ is identified with the normal subgroup of $H_{inv}^2(H)$ consisting of gauge equivalence classes of invariant 2-cocycles with trivial restriction on Γ .

Next, there is a surjective Hopf algebra homomorphism $p : H \rightarrow k[\Gamma/\Gamma_0]$ obtained by composing the canonical projection $H \rightarrow k[\Gamma]$ with $\pi_{\Gamma_0} : \Gamma \rightarrow \Gamma/\Gamma_0$. Thus, for any 2-cocycle $\rho \in Z^2(\Gamma/\Gamma_0, k^\times)$ its pullback $p^*(\rho)$ is a 2-cocycle on H .

Using the explicit formula (21) for the comultiplication on H we check that this 2-cocycle satisfies

$$p^*(\rho)(x_{(1)}, y_{(1)})x_{(2)} \otimes y_{(2)} = x_{(1)} \otimes y_{(1)}p^*(\rho)(x_{(2)}, y_{(2)}), \quad x, y \in H.$$

Indeed, for $x = hx_P, y = fx_Q$, where $h, f \in \Gamma$, both sides of this equality are equal to $\rho(\pi_{\Gamma_0}(h), \pi_{\Gamma_0}(f))hx_P \otimes fx_Q$.

In particular, $p^*(\rho)$ is an invariant 2-cocycle on H and belongs to the center of $H^2_{\text{inv}}(H)$. Thus, there is a central embedding $H^2(\Gamma/\Gamma_0, k^\times) \subset H^2_{\text{inv}}(H)$. The statement follows from Proposition 3.12. \square

4. Classification of co-quasitriangular structures on pointed Hopf algebras (Proof of Theorem 1.1)

Let Γ be a finite abelian group with a bicharacter $r_0 : \Gamma \times \Gamma \rightarrow k^\times$. Recall that $\mathcal{C}(\Gamma, r_0)$ denotes the fusion category of corepresentations of cosemisimple co-quasitriangular Hopf algebra $(k[\Gamma], r_0)$.

Let H be a co-quasitriangular pointed Hopf algebra with the r -form $r : H \otimes H \rightarrow k$ and let Γ be the group of its group-likes. Clearly, Γ must be abelian. By the result of Angiono [An13], the algebra H is generated by Γ and skew-primitive elements $x_i, i = 1, \dots, n$, with $\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1$ for some $g_i \in \Gamma, i = 1, \dots, n$. Furthermore, there exist characters $\chi_i \in \widehat{\Gamma}$ such that $gx_i g^{-1} = \chi_i(g)x_i$.

Let V be the span of $\{x_i\}_{i=1}^n$. Then V is an object in ${}^{\Gamma}\mathcal{YD}$ with the above conjugation action and coaction $\rho(x_i) = g_i \otimes x_i$.

It is clear that r restricts to a bicharacter r_0 on Γ . Observe that (2) for the pair $(x, y) = (x_i, g)$ yields $r_0(g, g_i) = \chi_i^{-1}(g)$, and the same condition for the pair $(x, y) = (g, x_i)$ yields $r_0(g_i, g) = \chi_i(g)$. Thus,

$$r_0(g_i, -) = \chi_i = r_0(-, g_i)^{-1}, \quad \text{for all } i = 1, \dots, n.$$

This condition is equivalent to $V \in \mathcal{Z}_{\text{sym}}(\mathcal{C}(\Gamma, r_0)) \subset {}^{\Gamma}\mathcal{YD}$. It follows that $\chi_i(g_j)\chi_j(g_i) = 1$ for all $i, j = 1, \dots, n$. Furthermore, $\chi_i(g_i) = -1$ for all $i = 1, \dots, n$ (indeed, if $\chi_i(g_i) = 1$ for some i then the Hopf subalgebra of H generated by g_i and x_i is non-semisimple commutative, and, hence, infinite dimensional).

Thus,

$$V \in \mathcal{Z}_{\text{sym}}(\mathcal{C}(\Gamma, r_0))_- \subset {}^{\Gamma}\mathcal{YD},$$

where the embedding $\mathcal{C}(\Gamma, r_0) \hookrightarrow {}^{\Gamma}\mathcal{YD}$ is given by (10). So V is a quantum linear space of symmetric type.

REMARK 4.1. It follows from the classification of symmetric fusion categories [De02] that in this situation there is a character $\chi : \Gamma \rightarrow k^\times$ such that $\chi(g_i) = -1$ for all $i = 1, \dots, n$ (in other words, $\chi = -1$ on the support of V).

It follows from results of Andruskiewitsch and Schneider [AS98] and Masuoka [Ma01] that H is a 2-cocycle twisting of $\mathfrak{B}(V)\#k[\Gamma]$. Since we are interested in the tensor category of corepresentations of H (which does not change under 2-cocycle twisting) from now on we will assume that $H = \mathfrak{B}(V)\#k[\Gamma]$.

If $g \in \Gamma$ and $i \in \{1, \dots, n\}$, then it follows from (3) that $r(g, x_i^m) = r(g, x_i)^m$, for all $m \geq 1$. Since x_i is nilpotent, we have $r(g, x_i) = 0$. Similarly, using (4), we deduce that $r(x_i, g) = 0$.

For a non-empty subset P of $\{1, \dots, n\}$ and for $g, h \in \Gamma$ one can show, using again (3) and (4), that $r(gx_P, h) = 0$ and $r(h, gx_P) = 0$ (the notation x_P was introduced in Remark 3.5).

Consider now subsets P and Q of $\{1, \dots, n\}$. Condition (2) for the pair $(x, y) = (x_P, x_Q)$ is

$$(x_P)_{(1)}(x_Q)_{(1)}r((x_Q)_{(2)}, (x_P)_{(2)}) = r((x_Q)_{(1)}, (x_P)_{(1)})(x_Q)_{(2)}(x_P)_{(2)}.$$

The terms in the coradical of H of each side of the equality are $g_P g_Q r(x_Q, x_P)$ and $r(x_Q, x_P)1$, respectively. Since these have to be equal, we have

$$(30) \quad r(x_Q, x_P)(1 - g_P g_Q) = 0.$$

This means that the restriction $r_1 := r|_{V \otimes V}$ is a morphism of Γ -comodules, i.e. a morphism in $\mathcal{C}(\Gamma, r_0) \subset {}^{\Gamma}\mathcal{YD}$.

Thus, we showed that if H admits a co-quasitriangular structure r , then V is a quantum linear space of symmetric type and the pair $(r_0, r_1) := (r|_{\Gamma \times \Gamma}, r|_{V \otimes V})$ satisfies the conditions of Theorem 1.1.

It remains to prove that, conversely, given an object $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_-$ and a pair $(r_0 : \Gamma \times \Gamma \rightarrow k^\times, r_1 : V \otimes V \rightarrow k)$ there is a unique r -form on H such that $(r|_{\Gamma \times \Gamma}, r|_{V \otimes V}) = (r_0, r_1)$.

We need the following general facts.

LEMMA 4.2. *Let $H = \mathfrak{B}(V) \# k[\Gamma]$. If $r : H \otimes H \rightarrow k$ is a convolution invertible linear map satisfying conditions (3) and (4) then r is a co-quasitriangular structure on H if and only if condition (2) holds for all pairs $(x, y) \in H_1 \times H_1$, where H_1 is the second term in the coradical filtration of H .*

PROOF. We need only prove sufficiency. By induction on m we show that condition (2) holds for all pairs (x, y) for which either x or y is in H_m , the m -th term of the coradical filtration.

Assume first that $x \in H_1$. Using induction on $k \geq 1$ we show that condition (2) holds for all pairs (x, y) and (y, x) with $y \in H_k$. If $k = 1$ there is nothing to prove. Assume that the claim is true for $k \geq 1$ and consider $z \in H_1$. Then, using the induction hypothesis and (3), we have

$$\begin{aligned} x_{(1)}(yz)_{(1)}r((yz)_{(2)}, x_{(2)}) &= x_{(1)}y_{(1)}z_{(1)}r(y_{(2)}z_{(2)}, x_{(2)}) \\ &= x_{(1)}y_{(1)}z_{(1)}r(y_{(2)}, x_{(2)})r(z_{(2)}, x_{(3)}) \\ &= y_{(2)}x_{(2)}z_{(1)}r(y_{(1)}, x_{(1)})r(z_{(2)}, x_{(3)}) \\ &= y_{(2)}z_{(2)}x_{(3)}r(y_{(1)}, x_{(1)})r(z_{(1)}, x_{(2)}) \\ &= y_{(2)}z_{(2)}x_{(2)}r(y_{(1)}z_{(1)}, x_{(1)}) \\ &= (yz)_{(2)}x_{(2)}r((yz)_{(1)}, x_{(1)}) \end{aligned}$$

and using (4) we have

$$\begin{aligned} (yz)_{(1)}x_{(1)}r(x_{(2)}, (yz)_{(2)}) &= y_{(1)}z_{(1)}x_{(1)}r(x_{(2)}, y_{(2)}z_{(2)}) \\ &= y_{(1)}z_{(1)}x_{(1)}r(x_{(2)}, z_{(2)})r(x_{(3)}, y_{(2)}) \\ &= y_{(1)}x_{(2)}z_{(2)}r(x_{(1)}, z_{(1)})r(x_{(3)}, y_{(2)}) \\ &= x_{(3)}y_{(2)}z_{(2)}r(x_{(1)}, z_{(1)})r(x_{(2)}, y_{(1)}) \\ &= x_{(2)}y_{(2)}z_{(2)}r(x_{(1)}, y_{(1)}z_{(1)}) \\ &= x_{(2)}(yz)_{(2)}r(x_{(1)}, (yz)_{(1)}). \end{aligned}$$

Since $H_{k+1} = H_k H_1$ it follows that (2) holds for all pairs (x, y) and (y, x) with $y \in H_{k+1}$. Thus, (2) holds for all pairs (x, y) with either x or y in H_1 .

Suppose now that (2) holds for all pairs (x, y) with either x or y in H_m . Then a similar argument as the one before shows that (2) holds for all pairs (x, yz) and (yz, x) with $y \in H_m, z \in H_1$ and arbitrary x . Since $H_{m+1} = H_m H_1$, it follows that (2) is satisfied for all pairs (x, y) with either x or y in H_{m+1} . This proves the induction step and concludes the proof. \square

LEMMA 4.3. *Let H be a Hopf algebra generated as an algebra by h_1, \dots, h_n and such that the vector space V spanned by h_1, \dots, h_n is a subcoalgebra. Then any co-quasitriangular structure on H is uniquely determined by its restriction to $V \otimes V$.*

PROOF. Suppose r' and r'' are two co-quasitriangular structures on H such that $r'(h_i, h_j) = r''(h_i, h_j)$, for all $i, j \in \{1, \dots, n\}$. If $i, i_1, \dots, i_t \in \{1, \dots, n\}$ then, using (4), we have

$$\begin{aligned} r'(h_{i_1} \cdots h_{i_t}, h_i) &= r'(h_{i_1}, (h_i)_{(1)}) \cdots r'(h_{i_t}, (h_i)_{(t)}) \\ &= r''(h_{i_1}, (h_i)_{(1)}) \cdots r''(h_{i_t}, (h_i)_{(t)}) \\ &= r''(h_{i_1} \cdots h_{i_t}, h_i). \end{aligned}$$

Thus, $r'(h, h_i) = r''(h, h_i)$, for all $h \in H$ and $i \in \{1, \dots, n\}$. Let $h \in H$ and i_1, \dots, i_t be in $\{1, \dots, n\}$. Then, using (3), we have

$$\begin{aligned} r'(h, h_{i_1} \cdots h_{i_t}) &= r'(h_{(1)}, h_{i_t}) \cdots r'(h_{(t)}, h_{i_1}) \\ &= r''(h_{(1)}, h_{i_t}) \cdots r''(h_{(t)}, h_{i_1}) \\ &= r''(h, h_{i_1} \cdots h_{i_t}). \end{aligned}$$

Since h_1, \dots, h_n generate H as an algebra, we conclude that $r' = r''$. \square

We now proceed to complete the proof of Theorem 1.1.

For $g \in \Gamma$ and $i = 1, \dots, n$ let $\gamma_g, \xi_i \in H^*$ be defined by

$$\gamma_g(hx_P) = \delta_{P, \emptyset} r_0(g, h), \quad \xi_i(hx_P) = \begin{cases} 0 & \text{if } |P| \neq 1 \\ r_1(x_i, x_j) & \text{if } P = \{j\} \end{cases}$$

for all $h \in \Gamma$ and $P \subseteq \{1, \dots, n\}$. We will show that $\varphi : H \rightarrow H^{*\text{cop}}$ defined by

$$\varphi(g) = \gamma_g \text{ and } \varphi(x_i) = \xi_i, \quad \text{for all } g \in \Gamma, i = 1, \dots, n,$$

is a bialgebra map. Using Remark 2.1, this will prove that $r : H \otimes H \rightarrow k$, $r(x, y) = \varphi(x)(y)$, for all $x, y \in H$, is a linear map, satisfying (3) and (4), whose restriction to $\Gamma \times \Gamma$, respectively $V \otimes V$, is r_0 , respectively r_1 .

Notice first that, since $r_1 : V \otimes V \rightarrow k$ is a morphism of Yetter-Drinfeld modules, we have $\xi_i(x_j)(1 - \chi_i \chi_j) = 0$ and $\xi_i(x_j)(1 - g_i g_j) = 0$, for all $i, j \in \{1, \dots, n\}$. Using this, it is straightforward to check that $\gamma_g \gamma_h = \gamma_{gh}$, $\gamma_g \xi_i = \chi_i(g) \xi_i \gamma_g$, $\xi_i \xi_j = \chi_j(g_i) \xi_j \xi_i$ and $\xi_i^2 = 0$, for all $g, h \in \Gamma$ and $i, j = 1, \dots, n$. For example, $\xi_i \xi_j(hx_P) = 0 = \chi_j(g_i) \xi_j \xi_i(hx_P)$ when $|P| \neq 2$. If $1 \leq u < v \leq n$, then

$$\begin{aligned} \xi_i \xi_j(hx_u x_v) &= \xi_i(hg_u x_v) \xi_j(hx_u) + \chi_u^{-1}(g_v) \xi_i(hg_v x_u) \xi_j(hx_v) \\ &= \xi_i(x_v) \xi_j(x_u) + \chi_i(g_j)^{-1} \xi_i(x_u) \xi_j(x_v) \\ &= \chi_j(g_i) (\xi_i(x_u) \xi_j(x_v) + \chi_j(g_i)^{-1} \xi_i(x_v) \xi_j(x_u)) \\ &= \chi_j(g_i) (\xi_j(hg_u x_v) \xi_i(hx_u) + \chi_u^{-1}(g_v) \xi_j(hg_v x_u) \xi_i(hx_v)) \\ &= \chi_j(g_i) \xi_j \xi_i(hx_u x_v). \end{aligned}$$

Thus, $\varphi : H \rightarrow H^{*\text{cop}}$, $\varphi(g) = \gamma_g$ and $\varphi(x_i) = \xi_i$, for all $g \in \Gamma$ and $i = 1, \dots, n$, is an algebra map. It is straightforward to check that γ_g is a group-like element and x_i a $(\gamma_{g_i}, \varepsilon)$ -skew primitive element of $H^{*\text{cop}}$ for every $g \in \Gamma$ and $i = 1, \dots, n$. Thus, φ is a bialgebra map. It follows that the linear map $r : H \otimes H \rightarrow k$, $r(x, y) = \varphi(x)(y)$, for all $x, y \in H$, satisfies (3) and (4). Moreover, $r|_{\Gamma \times \Gamma} = r_0$ and $r|_{V \otimes V} = r_1$. Since $r|_{k[\Gamma] \otimes k[\Gamma]}$ is convolution invertible, it follows that r is convolution invertible. Taking into account Lemma 4.2, in order to prove that r is a co-quasitriangular structure on H , we need only check that (2) holds for every pair of elements in the second term of the coradical filtration of H . This is straightforward, as we next show for the pair (gx_i, hx_j) :

$$\begin{aligned} r((hx_j)_{(1)}, (gx_i)_{(1)})(hx_j)_{(2)}(gx_i)_{(2)} &= r(hg_j, gg_i)hx_jgx_i + r(hx_j, gx_i)hg \\ &= r(h, g)\chi_i^{-1}(h)ghx_ix_j + r(hx_j, gx_i)gh \\ &= gx_ihx_jr(h, g) + gg_ihg_jr(hx_j, gx_i) \\ &= (gx_i)_{(1)}(hx_j)_{(1)}r((hx_j)_{(2)}, (gx_i)_{(2)}), \end{aligned}$$

where for the third equality we use the fact that $r(hx_j, gx_i)(1 - g_i g_j) = 0$. Thus, r is a co-quasitriangular structure on H which restricts on Γ to r_0 and on V to r_1 . The uniqueness of r follows from Lemma 4.3. This completes the proof of Theorem 1.1.

Let $\mathcal{C}(\Gamma, r_0, V, r_1)$ denote the braided tensor category constructed above.

EXAMPLE 4.4. Take $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and let r_0 be the unique non-trivial bicharacter of $\mathbb{Z}/2\mathbb{Z}$. Let V be a multiple of the non-identity simple object of $\text{Corep}(k[\mathbb{Z}/2\mathbb{Z}])$. Then V belongs to $\mathcal{Z}_{\text{sym}}(\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, r_0))$, so the Nichols Hopf algebra [Ni78]

$$(31) \quad E(V) := \mathfrak{B}(V) \# k[\mathbb{Z}/2\mathbb{Z}]$$

admits co-quasitriangular structures. According to Theorem 1.1 such structures are in bijection with bilinear forms $r_1 : V \otimes V \rightarrow k$ or, equivalently, with n -by- n square matrices, where $n = \dim_k(V)$. This fact was first established in [PvO99].

5. The symmetric center of $\mathcal{C}(\Gamma, r_0, V, r_1)$

The symmetric center $\mathcal{Z}_{\text{sym}}(\mathcal{C})$ of a braided tensor category \mathcal{C} was defined in Section 2.5.

Let Γ, r_0, V, r_1 be as in Theorem 1.1. Let $H = \mathfrak{B}(V) \# k[\Gamma]$ and $r : H \otimes H \rightarrow k$ be the corresponding Hopf algebra and r -form.

We have

$$\mathcal{Z}_{\text{sym}}(\mathcal{C}(\Gamma, r_0, V, r_1)) \cong \text{Corep}(H_{\text{sym}}, r|_{H_{\text{sym}} \otimes H_{\text{sym}}}),$$

where H_{sym} is the Hopf subalgebra of H defined by (9).

Let $b : \Gamma \times \Gamma \rightarrow k^\times$ be the symmetric bicharacter given by

$$b(g, h) = r(g, h)r(h, g), \quad g, h \in \Gamma.$$

Let Γ^\perp and V^\perp denote the radicals of b and $(r_1)_{\text{alt}}$, respectively, i.e.,

$$\begin{aligned} \Gamma^\perp &= \{g \in \Gamma \mid b(g, h) = 1 \text{ for all } h \in \Gamma\}, \\ V^\perp &= \{v \in V \mid (r_1)_{\text{alt}}(v, w) = 0 \text{ for all } w \in V\}. \end{aligned}$$

REMARK 5.1. If $V \neq 0$ then $\Gamma^\perp \neq 0$ since $V \in \mathcal{C}(\Gamma^\perp, r_0|_{\Gamma^\perp \times \Gamma^\perp})$.

LEMMA 5.2. H_{sym} is generated as an algebra by Γ^\perp and V^\perp .

PROOF. Since H_{sym} is a Hopf subalgebra of a pointed Hopf algebra with abelian coradical, it is pointed with abelian coradical. By the result of Angiono [An13], H_{sym} is generated by its group-like and skew-primitive elements.

It is easy to see that an element $g \in \Gamma$ is in H_{sym} if and only if $r(g, h)r(h, g) = 1$, for all $h \in \Gamma$. Thus, the set of group-like elements of H_{sym} is Γ^\perp .

Let now $g \in \Gamma^\perp$ and let x be a $(g, 1)$ -primitive element. If $g \notin \{g_i \mid i = 1, \dots, n\}$ then x is a scalar multiple of $1 - g$, so it is contained in H_{sym} . If $g = g_i$ for some $i \in \{1, \dots, n\}$ then $x = a(1 - g_i) + \sum_{j:g_j=g_i} a_j x_j$, for some $a, a_j \in k$. Let $y = \sum_{j:g_j=g_i} a_j x_j$. We claim that $x \in H_{sym}$ if and only if $y \in V^\perp$. It is clear that $x \in H_{sym}$ if and only if $y \in H_{sym}$. Next,

$$y_{(1)}r(y_{(2)}, z_{(1)})r(z_{(2)}, y_{(3)}) = \sum_{j:g_j=g_i} a_j \left(r(g_j, z_{(1)})r(z_{(2)}, x_j) + r(x_j, z) \right) g_i + \varepsilon(z)y,$$

for $z \in H$, so $y \in H_{sym}$ if and only if

$$(32) \quad \sum_{j:g_j=g_i} a_j \left(r(g_j, z_{(1)})r(z_{(2)}, x_j) + r(x_j, z) \right) = 0$$

for all $z \in \{hx_l \mid h \in \Gamma, l = 1, \dots, n\}$. For $z = hx_l$, the left-hand side of (32) becomes

$$\begin{aligned} \text{LHS}(32) &= \sum_{j:g_j=g_i} a_j \left(r(g_j, hg_l)r(hx_l, x_j) + r(x_j, hx_l) \right) \\ &= \sum_{j:g_j=g_i} a_j \left(r(g_j, g_l)r(x_l, x_j) + r(x_j, x_l) \right) \\ &= r(g_l, g_i)r(x_l, y) + r(y, x_l) \\ &= (r + r \circ c_{V,V})(y, x_l) \\ &= (r - r \circ \tau_{V,V})(y, x_l) = 2(r_1)_{alt}(y, x_l), \end{aligned}$$

where we used the fact that $r|_{V \otimes V}$ is a morphism in ${}^\Gamma \mathcal{YD}$ and Lemma 3.4. Thus, $y \in H_{sym}$ if and only if $y \in V^\perp$. It follows that non-trivial skew-primitive elements of H_{sym} generate V^\perp , so the claim follows. \square

COROLLARY 5.3. $H_{sym} = \mathfrak{B}(V^\perp) \# k[\Gamma^\perp]$ and

$$\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0, V, r_1)) \cong \mathcal{C}(\Gamma^\perp, r_0|_{\Gamma^\perp \times \Gamma^\perp}, V^\perp, r_1|_{V^\perp \otimes V^\perp}).$$

COROLLARY 5.4. (i) $\mathcal{C}(\Gamma, r_0, V, r_1)$ is symmetric if and only if $\mathcal{C}(\Gamma, r_0)$ is symmetric and r_1 is symmetric.

(ii) $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0, V, r_1))$ is semisimple if and only if $(r_1)_{alt} : V \otimes V \rightarrow k$ is non-degenerate.

(iii) $\mathcal{C}(\Gamma, r_0, V, r_1)$ is factorizable if and only if $\mathcal{C}(\Gamma, r_0)$ is factorizable and $V = 0$.

EXAMPLE 5.5. Let $E(V)$ be the Hopf algebra from Example 4.4 and let $r_1 : V \otimes V \rightarrow k$ be a bilinear form. In this case $c_{V,V} = -\tau_{V,V}$, so Corollary 5.4(i) says that the co-quasitriangular structure determined by r_1 is symmetric if and only if r_1 is symmetric (in the usual linear algebra sense). This was proved in [CC04].

6. Ribbon structures on $\mathcal{C}(\Gamma, r_0, V, r_1)$

Ribbon structures on braided tensor categories and ribbon elements of coquasitriangular Hopf algebras were defined in Section 2.3. We want to classify ribbon structures of $\mathcal{C}(\Gamma, r_0, V, r_1)$.

LEMMA 6.1. *Let A be an abelian group and let a_1, a_2, \dots, a_n be elements of A . Then there exists a group homomorphism $\gamma : A \rightarrow \{\pm 1\}$ such that $\gamma(a_i) = -1$, for all $i = 1, \dots, n$ if and only if there are no relations in A of the form $a_{i_1} a_{i_2} \cdots a_{i_k} = x^2$, with k odd.*

PROOF. Homomorphisms $A \rightarrow \{\pm 1\}$ are in bijection with homomorphisms $A/A^2 \rightarrow \{\pm 1\}$. Let $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ be images of a_1, a_2, \dots, a_n in A/A^2 . If there are no relations in A of the form $a_{i_1} a_{i_2} \cdots a_{i_k} = x^2$ with k odd then all relations in A/A^2 (viewed as a $\mathbb{Z}/2\mathbb{Z}$ -vector space) are of the form $\bar{a}_{i_1} + \bar{a}_{i_2} \cdots + \bar{a}_{i_l} = 0$ with l even. Therefore, there is a well-defined homomorphism $A/A^2 \rightarrow \{\pm 1\}$ sending each a_i to -1 . The converse implication is trivial. \square

PROPOSITION 6.2. *The set of ribbon structures on $\mathcal{C}(\Gamma, r_0, V, r_1)$ is non-empty and is in bijection with the set of group homomorphisms $\gamma : \Gamma \rightarrow \{\pm 1\}$ such that $\gamma(g_i) = -1$, for all $i = 1, \dots, n$.*

PROOF. Let $H = \mathfrak{B}(V) \# k[\Gamma]$. As explained in Section 2.3, ribbon structures on $\mathcal{C}(\Gamma, r_0, V, r_1)$ are in bijection with group-like elements $\gamma \in G(H^*)$ satisfying $\gamma^2 = (\eta \circ S) * \eta^{-1}$ and $S_{H^*}^2(p) = \gamma^{-1} * p * \gamma$, for all $p \in H^*$. Let γ be such an element. We have

$$\gamma(g)^2 = \gamma^2(g) = (\eta \circ S) * \eta^{-1}(g) = r_0(g^{-1}, g)r_0(g, g) = 1$$

for all $g \in \Gamma$, so $\gamma(\Gamma) \subseteq \{\pm 1\}$. Now $S_{H^*}^2(p) = \gamma^{-1} * p * \gamma$ for all $p \in H^*$, if and only if $S_H^2 = \gamma^{-1} * \text{id}_H * \gamma$. Since both maps are algebra maps, they are equal if and only if they agree on algebra generators. We have

$$\begin{aligned} S_H^2(g) &= g, & (\gamma^{-1} * \text{id}_H * \gamma)(g) &= g, \\ S_H^2(x_i) &= -x_i, & (\gamma^{-1} * \text{id}_H * \gamma)(x_i) &= \gamma^{-1}(g_i)x_i \end{aligned}$$

for all $g \in \Gamma$ and $i = 1, \dots, n$. Thus, $S_H^2 = \gamma^{-1} * \text{id}_H * \gamma$ if and only if $\gamma(g_i) = -1$ for all i .

It remains to show that there always exists a homomorphism $\gamma : \Gamma \rightarrow \{\pm 1\}$ such that $\gamma(g_i) = -1$, for all $i = 1, \dots, n$. Since k^\times is an injective \mathbb{Z} -module, it is enough to show that there is such a homomorphism on the subgroup $\Gamma_0 = \langle g_1, \dots, g_n \rangle \subset \Gamma$. Using Lemma 6.1, we have to show that there are no relations in Γ_0 of the form $g_{i_1} g_{i_2} \cdots g_{i_k} = x^2$ with k odd. If $x = g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_t}^{e_t}$ is an element of Γ_0 then

$$r_0(x, x) = \prod_{r=1}^t r_0(g_{i_r}, g_{i_r})^{e_r^2} \prod_{1 \leq r < s \leq t} (r_0(g_{i_r}, g_{i_s})r_0(g_{i_s}, g_{i_r}))^{e_r e_s} = (-1)^{\sum_{r=1}^t e_r^2}.$$

In particular, $r_0(x, x)^2 = 1$ for all $x \in \Gamma_0$. On the other hand, if $g_{i_1} g_{i_2} \cdots g_{i_k} = x^2$ with k odd, then $r_0(x, x)^4 = r_0(x^2, x^2) = (-1)^k = -1$, which is a contradiction. \square

7. Metric data (Proof of Theorem 1.2)

It is well known that the 1-categorical truncation of the 2-category of pointed braided fusion categories is equivalent to the 2-category of *pre-metric groups*.

The objects of the former are pointed braided fusion categories and morphisms are natural isomorphism classes of braided tensor functors. The objects of the latter are pairs (Γ, q) , where Γ is an abelian group and $q : \Gamma \rightarrow k^\times$ is a quadratic form, and morphisms are orthogonal homomorphisms (see [JS93] or [EGNO15, Section 8.4]).

The goal of this section is to extend this result to tensor categories that are not necessarily semisimple.

7.1. The category of metric data. Recall that a *groupoid* is a category in which all morphisms are isomorphisms.

Let \mathcal{B} denote a groupoid whose objects are pointed braided tensor categories admitting a fiber functor and morphisms are natural isomorphism classes of equivalences of braided tensor categories. Thus, objects of \mathcal{B} can be identified with the co-representation categories of co-quasitriangular pointed Hopf algebras.

Define a groupoid \mathcal{M} as follows. The objects of \mathcal{M} are data (\mathcal{P}, V, r) , where \mathcal{P} is pointed braided fusion category admitting a fiber functor, V is an object in $\mathcal{Z}_{\text{sym}}(\mathcal{P})_- \subset {}_{\Gamma}^{\Gamma}\mathcal{YD}$, and $r : V \otimes V \rightarrow k$ is an alternating bilinear form in \mathcal{P} . The set of morphisms

$$\text{Hom}_{\mathcal{M}}((\mathcal{P}, V, r), (\mathcal{P}', V', r'))$$

is the set of pairs (α, f) , where $\alpha : \mathcal{P} \rightarrow \mathcal{P}'$ is a braided equivalence and $f : \alpha(V) \rightarrow V'$ is an isomorphism in ${}_{\Gamma'}^{\Gamma'}\mathcal{YD}$ such that $r' \circ (f \otimes f) = \alpha(r)$, modulo the equivalence relation identifying pairs (α, f) and $(\alpha, -f)$.

DEFINITION 7.1. We will call \mathcal{M} the groupoid of *metric data*.

Define a functor

$$(33) \quad F : \mathcal{M} \rightarrow \mathcal{B}$$

as follows. Given a datum (\mathcal{P}, V, r) let Γ be the underlying Abelian group of \mathcal{P} and let $r_0 : \Gamma \times \Gamma \rightarrow k^\times$ be the corresponding r -form. Set

$$(34) \quad F(\mathcal{P}, V, r) = \mathcal{C}(\Gamma, r_0, V, r).$$

A morphism (α, f) between (\mathcal{P}, V, r) and (\mathcal{P}', V', r') (with $\mathcal{P} = \mathcal{C}(\Gamma, r_0)$ and $\mathcal{P}' = \mathcal{C}(\Gamma', r'_0)$) gives rise to an isomorphism of Hopf algebras $\varphi_{(\alpha, f)} : \mathfrak{B}(V) \# k[\Gamma] \xrightarrow{\sim} \mathfrak{B}(V') \# k[\Gamma']$ given by

$$(35) \quad \varphi_{(\alpha, f)}(g) = \alpha(g), \quad \varphi_{(\alpha, f)}(x) = f(x)$$

for all $g \in \Gamma$ and $x \in V$. Here we slightly abuse notation and use α to denote the isomorphism between Γ and Γ' corresponding to the braided equivalence $\alpha : \mathcal{P} \xrightarrow{\sim} \mathcal{P}'$. The bicharacter $r'_0 \circ (\alpha \times \alpha) / r_0$ is alternating and so is equal to $\text{alt}(\mu)$ for some $\mu \in Z^2(\Gamma/\Gamma_0, k^\times)$, see (29). This means that the r -form $r'_0 \circ (\alpha \times \alpha)$ is a twisting deformation of r_0 by means of the above 2-cocycle $\mu \in H^2(\Gamma/\Gamma_0, k^\times)$. But such μ defines an invariant 2-cocycle on $\mathfrak{B}(V) \# k[\Gamma]$ by Proposition 3.13. Thus, $\varphi_{(\alpha, f)}$ gives rise to a well defined braided tensor equivalence $F(\alpha, f)$ between $\mathcal{C}(\Gamma, r_0, V, r)$ and $\mathcal{C}(\Gamma', r'_0, V', r')$.

7.2. Proof of Theorem 1.2. Our goal is to show that the functor F defined in the previous Section is an equivalence.

Let $H := \mathfrak{B}(V) \# k[\Gamma]$ and $H' := \mathfrak{B}(V') \# k[\Gamma']$ with co-quasitriangular structures on H and H' defined by (r_0, r_1) and (r'_0, r'_1) .

LEMMA 7.2. *Co-quasitriangular Hopf algebra isomorphisms $H \rightarrow H'$ are in bijection with pairs (α, f) , where $\alpha : \Gamma \rightarrow \Gamma'$ is a group isomorphism, $f : V^\alpha \rightarrow V'$ is an isomorphism such that $r'_0 \circ (\alpha \times \alpha) = r_0$, and $r'_1 \circ (f \otimes f) = r_1^\alpha$. Here V^α is the object of ${}_{\Gamma'}^{\Gamma'}\mathcal{YD}$ induced from V and r_1^α is the bilinear form induced from r_1 by means of α .*

PROOF. Let $\phi : H \rightarrow H'$ be an isomorphism of co-quasitriangular Hopf algebras. Since f takes group-like elements to group-like elements, it restricts to a group isomorphism $\alpha : \Gamma \rightarrow \Gamma'$. Let us show that ϕ induces an isomorphism $V^\alpha \rightarrow V'$.

Notice first that $\alpha(g_i) \in \{g'_j\}$. Indeed, if this is not the case, then $\phi(x_i) = a(1 - \phi(g_i))$, for some $a \in k$. But $\phi(x_i)$ anti-commutes with $\phi(g_i)$, while $a(1 - \phi(g_i))$ commutes with $\phi(g_i)$, so $\phi(x_i) = 0$. This, however, contradicts the injectivity of ϕ . Thus, $\alpha(g_i) \in \{g'_j\}$, for all i . It follows that there exist scalars $a_i, b_{ij} \in k$ such that $b_{ji}(g'_j - \phi(g_i)) = 0$ and

$$\phi(x_i) = a_i(1 - \phi(g_i)) + \sum_j b_{ji}x'_j, \quad i = 1, \dots, n.$$

We must have $\phi(g)\phi(x_i) = \chi_i(g)\phi(x_i)\phi(g)$, for all $g \in \Gamma$. Now

$$\begin{aligned} \phi(g)\phi(x_i) &= a_i(\phi(g) - \phi(gg_i)) + \sum_j b_{ji}\phi(g)x'_j \\ &= a_i(\phi(g) - \phi(gg_i)) + \sum_j b_{ji}\chi'_j(\phi(g))x'_j\phi(g) \end{aligned}$$

and

$$\chi_i(g)\phi(x_i)\phi(g) = a_i\chi_i(g)(\phi(g) - \phi(gg_i)) + \sum_j b_{ji}\chi_i(g)x'_j\phi(g).$$

Thus, $\phi(g)\phi(x_i) = \chi_i(g)\phi(x_i)\phi(g)$ if and only if $a_i = a_i\chi_i(g)$ and $\chi'_j(\phi(g))b_{ji} = \chi_i(g)b_{ji}$, for all j and $g \in \Gamma$. Taking $g = g_i$ in the first condition, we obtain $a_i = 0$. The second condition is equivalent to $b_{ji}(\chi_i - \chi'_j \circ \phi) = 0$. It follows that

$$\phi(x_i) = \sum_j b_{ji}x'_j$$

where $b_{ji}(g'_j - \alpha(g_i)) = 0$ and $b_{ji}(\chi_i\alpha^{-1} - \chi'_j) = 0$. This means precisely that the restriction f of ϕ to V is a morphism in $\mathcal{C}(\Gamma', r'_0)$ from V^α to V' . Since ϕ is an isomorphism, the restriction is also an isomorphism.

We have proven that if $\phi : H \rightarrow H'$ is a Hopf algebra isomorphism then it induces, by restriction, isomorphisms $\alpha : \Gamma \rightarrow \Gamma'$ and $f : V^\alpha \rightarrow V'$. We have $r' \circ (\phi \otimes \phi) = r$ if and only if $r'_0 \circ (\alpha \times \alpha) = r_0$ and $r'_1 \circ (f \otimes f) = r_1^\alpha$. It is easy to

check that every such data comes from an isomorphism of co-quasitriangular Hopf algebras $H \rightarrow H'$. This completes the proof. \square

LEMMA 7.3. *Let σ be a 2-cocycle on $H = \mathfrak{B}(V)\#k[\Gamma]$ such that $\sigma|_{\Gamma \times \Gamma} = 1$. Then the twisted r -form r^σ (see (13)) on H^σ satisfies*

$$(36) \quad r^\sigma|_{V \otimes V} = r|_{V \otimes V} + 2(\sigma|_{V \otimes V})_{sym}.$$

PROOF. Using formula (13) we compute

$$\begin{aligned} r^\sigma(x_i, x_j) &= r(g_i, g_j) \sigma^{-1}(x_i, x_j) + r(x_i, x_j) + \sigma(x_j, x_i) \\ &= r(x_i, x_j) + (\sigma(x_j, x_i) - \chi_i(g_j)\sigma(x_i, x_j)) \\ &= r(x_i, x_j) + (\sigma \circ \tau_{V,V} - \sigma \circ c_{V,V} \circ \tau_{V,V})(x_i, x_j). \end{aligned}$$

Since restrictions of r and r^σ on $V \otimes V$ are morphism in ${}_{\Gamma}^{\Gamma}\mathcal{YD}$ we conclude, using Lemma 3.4, that

$$\sigma|_{V \otimes V} \circ \tau_{V,V} - \sigma|_{V \otimes V} \circ c_{V,V} \circ \tau_{V,V} = 2(\sigma|_{V \otimes V})_{sym},$$

which implies the statement. \square

We now proceed with the proof of Theorem 1.2. We need to show that the functor F (33) is surjective and fully faithful.

(1) *F is surjective.*

For this end it suffices to check that the co-quasitriangular structure on H defined using $r : V \otimes V \rightarrow k$ is gauge equivalent, by means of an invariant 2-cocycle on H , to the one defined using an alternating morphism $V \otimes V \rightarrow k$ in $\mathcal{C}(\Gamma, r_0)$. Let σ be an invariant 2-cocycle on H such that $\sigma|_{\Gamma \times \Gamma} = 1$ (such 2-cocycles are classified in Corollary 3.11) and let $r^\sigma = \sigma_{21} * r * \sigma^{-1}$. By Lemma 7.3 we have

$$r^\sigma|_{V \otimes V} = r|_{V \otimes V} + 2(\sigma|_{V \otimes V})_{sym}.$$

Thus, we can take σ such that $(\sigma|_{V \otimes V})_{sym} = -\frac{1}{2}(r \circ \tau)_{sym}$, so that $r^\sigma|_{V \otimes V} = (r|_{V \otimes V})_{alt}$, i.e., $r^\sigma|_{V \otimes V}$ is alternating.

Since $\mathcal{C}(\Gamma, r_0, V, r)$ and $\mathcal{C}(\Gamma, r_0, V, r^\sigma)$ are equivalent braided tensor categories, the surjectivity of F follows.

(2) *F is faithful.* We need to check that F is injective on morphisms. It is clear from definitions that $F(\alpha, f) = F(\alpha', f')$ implies $\alpha = \alpha'$. Therefore, it remains to check that for an automorphism (id_Γ, f) of $(\mathcal{P}, V, r) \in \mathcal{M}$ one has $F(\text{id}_\Gamma, f) \cong \text{id}_{\mathcal{C}(\mathcal{P}, V, r)}$ as a tensor functor if and only if $f = \pm \text{id}_V$. One implication is clear since $-\text{id}_V$ preserves any bilinear form.

Note that the Hopf algebra automorphism $\varphi_{(\text{id}_\Gamma, f)}$ of H defined in (35) gives rise to a trivial tensor autoequivalence of $\text{Corep}(H)$ if and only if it is given by

$$h \mapsto \chi \rightharpoonup h \leftarrow \chi^{-1}, \quad h \in H$$

for some character $\chi \in H^*$. The condition that it preserves r is equivalent to $\chi(g_i)\chi(g_j) = 1$ for all $i, j = 1, \dots, n$, i.e., to χ being identically equal to 1 or -1 on the support of V . By the Remark 4.1 there exists χ such that this value is -1 , so that $f = \pm \text{id}_V$.

(3) *F is full.* We need to check that F is surjective on morphisms. We claim that any braided tensor equivalence Φ between $\text{Corep}(\mathfrak{B}(V)\#k[\Gamma], r)$ and $\text{Corep}(\mathfrak{B}(V')\#k[\Gamma'], r')$ is isomorphic to one coming from a co-quasitriangular Hopf algebra isomorphism $\mathfrak{B}(V)\#k[\Gamma] \rightarrow \mathfrak{B}(V')\#k[\Gamma']$. By the result of Davydov [Da10]

Φ corresponds to a pair (f, σ) , where σ is a 2-cocycle on $\mathfrak{B}(V)\#k[\Gamma]$ and $f : (\mathfrak{B}(V)\#k[\Gamma])^\sigma \rightarrow \mathfrak{B}(V')\#k[\Gamma']$ is a Hopf algebra isomorphism such that

$$r \circ (f \otimes f) = r^\sigma.$$

The last condition corresponds to the braided property of the equivalence.

We must have $\sigma|_{\Gamma \times \Gamma} = 1$ since a non-trivial twisting changes the braided equivalence class of $\text{Corep}(k[\Gamma], r_0)$. By Lemma 7.3 we have

$$(r \circ (f \otimes f) - r)|_{V \otimes V} = 2(\sigma|_{V \otimes V})_{sym}.$$

The left hand side of the above equality is alternating, while the right hand side is symmetric. Hence, both sides are equal to 0 and so σ is gauge equivalent to the trivial 2-cocycle by Proposition 3.9. This means that Φ is isomorphic to the equivalence induced by a co-quasitriangular Hopf algebra isomorphism, so the result follows from Lemma 7.2.

REMARK 7.4. We can give a conceptual explanation of the reason why $(r_1)_{alt}$ is an invariant of the braided tensor category $\mathcal{C} := \mathcal{C}(\Gamma, r_0, V, r_1)$.

Let $g \in \Gamma$. We will also use g to denote the corresponding invertible H -comodule. Recall that $\text{Ext}_{\mathcal{C}}(g, 1) \cong P_{1,g}(H)/k(1-g)$, where $P_{1,g}(H)$ denotes the space of $(1, g)$ -skew primitive elements of H . Explicitly, elements of $\text{Ext}_{\mathcal{C}}(g, 1)$ are in bijection with equivalence classes of short exact sequences

$$0 \rightarrow 1 \xrightarrow{\iota} V_x \xrightarrow{p} g \rightarrow 0,$$

where 1 denotes the trivial comodule k . The 2-dimensional comodule V_x is a vector space with a basis v_0, v_1 and H -coaction given by

$$(37) \quad \rho(v_0) = v_0 \otimes 1, \quad \rho(v_1) = v_0 \otimes x + v_1 \otimes g,$$

where $x \in P_{1,g}(H)$.

Let $x' \in P_{1,g'}(H)$, $g' \in \Gamma$, be another skew-primitive element of H , let

$$0 \rightarrow 1 \xrightarrow{\iota'} V_{x'} \xrightarrow{p'} g' \rightarrow 0$$

be the corresponding extension, and let v'_0, v'_1 be a basis of $V_{x'}$ defined analogously to (37).

Let $\beta_{x,x'} = c_{V_{x'}, V_x} \circ c_{V_x, V_{x'}}$ denote the square of the braiding on $V_x \otimes V_{x'}$. Using formula (5) one computes

$$\begin{aligned} \beta_{x,x'}(v_0 \otimes v'_0) &= v_0 \otimes v'_0, \\ \beta_{x,x'}(v_0 \otimes v'_1) &= v_0 \otimes v'_1, \\ \beta_{x,x'}(v_1 \otimes v'_0) &= v_1 \otimes v'_0, \\ \beta_{x,x'}(v_1 \otimes v'_1) &= v_1 \otimes v'_1 + (r_1(x, x') + r_0(g, g')r_1(x', x))v_0 \otimes v'_0. \end{aligned}$$

Let $s := r_1 - r_1 \circ \tau$. Combining Lemma 3.4 with above computation we see that

$$(38) \quad \beta_{x,x'} = \text{id}_{V_x \otimes V_{x'}} + s(x, x')(p \otimes p') \circ (\iota \otimes \iota')$$

for all $x, x' \in V = \text{Ext}_{\mathcal{C}}(\Gamma, 1)$ (note that $(p \otimes p') \circ (\iota \otimes \iota') \in \text{End}_{\mathcal{C}}(V_x \otimes V_{x'})$ whenever $s(x, x') \neq 0$).

It follows from (38) that $s = (r_1)_{alt}$ is an invariant of the braided equivalence class of \mathcal{C} (a computation establishing this fact is straightforward and can be found in the proof of [BN15, Proposition 6.7]).

Let $\mathcal{C} = \mathcal{C}(\mathcal{P}, V, r)$. Theorem 1.2 allows to compute the group $\text{Aut}^{br}(\mathcal{C})$ of isomorphism classes of braided autoequivalences of \mathcal{C} . Namely, let

$$\begin{aligned} \text{Aut}(V, r) &:= \{f \in \text{Aut}_{\mathcal{P}}(V) \mid r \circ (f \otimes f) = r\} / \{\pm \text{id}_V\}, \\ O(\mathcal{P}, r) &:= \{\alpha \in \text{Aut}(\Gamma) \mid q \circ \alpha = q \text{ and } r^\alpha, r : V \otimes V \rightarrow k \text{ are congruent in } \mathcal{P}\}. \end{aligned}$$

COROLLARY 7.5. *There is a short exact sequence*

$$(39) \quad 1 \rightarrow \text{Aut}(V, r) \rightarrow \text{Aut}^{br}(\mathcal{C}) \rightarrow O(\mathcal{P}, r) \rightarrow 1.$$

EXAMPLE 7.6. Let us consider $\mathcal{C} = \text{Corep}(E(V), r)$, where $E(V)$ is the Hopf algebra from Example 4.4 with the co-quasitriangular structure given by the zero bilinear form on V (this structure is symmetric). In this case $\Gamma = \mathbb{Z}/2\mathbb{Z}$, so $O(\mathcal{P}, r) = 1$ and Corollary 7.5 implies that $\text{Aut}^{br}(\mathcal{C}) = GL_n(V) / \{\pm \text{id}_V\}$, cf. [BN15].

8. The Drinfeld center of a pointed braided tensor category

It is well known that the Drinfeld center of a pointed braided fusion category is pointed (see, e.g., [DN13, Proposition 5.8]). This is no longer true in the non-semisimple case. Indeed, the Drinfeld center is always factorizable, cf. Corollary 5.4(iii).

Let $\mathcal{C} = \mathcal{C}(\Gamma, r_0, V, r_1)$ be a pointed braided tensor category corresponding to a non-zero *self-dual* Yetter-Drinfeld module V . In this Section we show that the trivial component of the universal grading of $\mathcal{Z}(\mathcal{C})$ is pointed, i.e., $\mathcal{Z}(\mathcal{C})$ is nilpotent of nilpotency class 2 ([GN08]).

8.1. The group-like elements of $D(H)^*$. It is well known [R93, Proposition 10] that for a finite dimensional Hopf algebra H the group of central group-like elements of $D(H)$ is

$$G(D(H)^*) \cong G(D(H)) \cap Z(D(H)).$$

More explicitly,

$$G(D(H)^*) = \{g \otimes \gamma \mid (g, \gamma) \in G(H) \times G(H^*), \sum \gamma(h_{(1)})h_{(2)}g = \sum \gamma(h_{(2)})gh_{(1)}, \forall h \in H\}$$

Let V be a quantum linear space of symmetric type and let $H = \mathfrak{B}(V) \# k[\Gamma]$. It is not hard to check that

$$G(D(H)^*) = \{g \otimes \gamma \mid (g, \gamma) \in \Gamma \times \widehat{\Gamma}, \gamma(g_i) = \chi_i(g), \text{ for all } i = 1, \dots, n\}.$$

Let $b : (\Gamma \times \widehat{\Gamma}) \times (\Gamma \times \widehat{\Gamma}) \rightarrow k^\times$ be the canonical non-degenerate bicharacter defined by

$$b((g, \chi), (g', \chi')) = \chi(g')\chi'(g), \quad (g, \chi), (g', \chi') \in \Gamma \times \widehat{\Gamma}.$$

Consider the subgroup

$$(40) \quad \Sigma := \langle (g_i, \chi_i^{-1}) \mid i = 1, \dots, n \rangle \subset \Gamma \times \widehat{\Gamma}.$$

REMARK 8.1. The above Σ is an isotropic subgroup of $\Gamma \times \widehat{\Gamma}$, i.e., $\Sigma \subset \Sigma^\perp$, where Σ^\perp is the orthogonal complement of Σ with respect to b .

It follows that

$$(41) \quad G(D(H)^*) \cong G(D(H)) \cap Z(D(H)) \cong \Sigma^\perp.$$

Let H be a Hopf algebra. It was shown in [GN08] that for any Hopf subalgebra K of H contained in the center of H tensor category $\text{Rep}(H)$ is graded by $G(K^*)$. The trivial component of this grading is $\text{Rep}(H/HK^+)$. The maximal central Hopf subalgebra of H provides the universal grading of $\text{Rep}(H)$. In this case, the trivial component is $\text{Rep}(H)_{\text{ad}}$, the adjoint subcategory of $\text{Rep}(H)$.

If H is quasitriangular then the maximal central Hopf subalgebra of H is the group algebra of $G(H) \cap Z(H)$ and the universal grading group of $\text{Rep}(H)$ is the group of characters of $G(H) \cap Z(H)$.

We will be interested in the adjoint subcategory of $\mathcal{Z}(\text{Corep}(H))$, where $H = \mathfrak{B}(V) \# k[\Gamma]$ for a quantum linear space $V \in {}_\Gamma \mathcal{YD}$ of symmetric type. We have

$$(42) \quad \mathcal{Z}(\text{Corep}(H)) \cong \text{Rep}(D(H)^{\text{cop}}) \cong \text{Corep}(D(H)^{\text{op}}).$$

Since $D(H)^{\text{cop}}$ is quasitriangular, the universal grading group of $\mathcal{Z}(\text{Corep}(H))$ is isomorphic to $\widehat{\Sigma}^\perp$, according to (41).

REMARK 8.2. The Drinfeld double $D(\mathfrak{B}(V) \# k[\Gamma])$ was studied by Beattie in [Be03].

8.2. The adjoint subcategory of the center of \mathcal{C} . If (H, r) is a co-quasitriangular Hopf algebra then

$$\iota_r : H \rightarrow D(H)^{\text{op}}, \quad \iota_r(x) = x_{(1)} \otimes r(-, x_{(2)})$$

is a Hopf algebra homomorphism. This corresponds to the embedding $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$ from Remark 2.3.

Let $V \in {}_\Gamma \mathcal{YD}$ be a quantum linear space of symmetric type associated to the datum $(g_1, \dots, g_n, \chi_1, \dots, \chi_n)$ and let $H = \mathfrak{B}(V) \# k[\Gamma]$. Suppose H admits co-quasitriangular structures. Then, according to Theorem 1.1, r -forms on H are in bijection with pairs (r_0, r_1) , where r_0 is a bicharacter of Γ such that $V \in \mathcal{Z}_{\text{sym}}(\mathcal{C}(\Gamma, r_0))_-$ and $r_1 : V \otimes V \rightarrow k$ is a morphism in $\mathcal{C}(\Gamma, r_0)$.

Consider the r -form r on H corresponding to the pair $(r_0, 0)$. Then $\iota_r(g_i) = g_i \otimes \chi_i^{-1}$ and $\iota_r(x_i) = x_i \otimes \varepsilon$. Thus, $D(H)^*$ contains group-like elements $g_i \otimes \chi_i^{-1}$ and $(g_i \otimes \chi_i^{-1}, 1)$ -skew primitive elements $x_i \otimes \varepsilon$, $i = 1, \dots, n$.

Assume now that V is a self-dual object of ${}_\Gamma \mathcal{YD}$. In this case the set $\{(g_i, \chi_i) \mid i = 1, \dots, n\}$ is closed under taking inverses. Let $r_1 : V \otimes V \rightarrow k$ denote a non-degenerate evaluation morphism in ${}_\Gamma \mathcal{YD}$. Let r' be the r -form on H corresponding to the pair (r_0, r_1) . Then $\iota_{r'}(g_i) = g_i \otimes \chi_i^{-1}$ and $\iota_{r'}(x_i) = g_i \otimes r'(-, x_i) + x_i \otimes \varepsilon$. We see that in this case $D(H)^*$ contains, in addition to the above, $(g_i \otimes \chi_i^{-1}, 1)$ -skew primitive elements $g_i \otimes r'(-, x_i)$.

Let A be the Hopf subalgebra of $D(H)^*$ generated by group-like elements $g_i \otimes \chi_i^{-1}$ and skew-primitive elements $x_i \otimes \varepsilon$ and $g_i \otimes r'(-, x_i)$, $i = 1, \dots, n$.

REMARK 8.3. By definition (40), the group of group-likes of A is Σ . The above skew-primitive elements $x_i \otimes \varepsilon$ and $g_i \otimes r'(-, x_i)$, $i = 1, \dots, n$ constructed above are linearly independent and form a $2n$ -dimensional quantum linear space of symmetric type in $\Sigma \mathcal{YD}$. Therefore,

$$\dim_k(A) = |\Sigma| 2^{2n}.$$

PROPOSITION 8.4. *Let $K = k[G(D(H^*))]$. We have*

$$(43) \quad A^* \cong D(H)/D(H)K^+.$$

PROOF. Let $\pi : D(H) \rightarrow D(H)/D(H)K^+$ be the canonical projection. We claim that A is the image of the dual Hopf algebra homomorphism

$$\pi^* : (D(H)/D(H)K^+)^* \rightarrow D(H)^*$$

According to (41), K is the group algebra of $\{\gamma \otimes g \mid (g, \gamma) \in \Sigma^\perp\}$. Let

$$e = \frac{1}{|\Sigma^\perp|} \sum_{(g, \gamma) \in \Sigma^\perp} \gamma \otimes g.$$

Then e is a central idempotent of $D(H)$, $ze = \varepsilon(z)e$, for all $z \in K$, and $K^+ = (1 - e)K$. Thus, $D(H)K^+ = D(H)K(1 - e) = (1 - e)D(H)$. The image of π^* is

$$\text{Im}(\pi^*) = \{f \in (D(H))^* \mid f(z) = f(ez), \text{ for all } z \in D(H)\}.$$

It is easy to check that any $f \in \{g_i \otimes \chi_i^{-1}, x_i \otimes \varepsilon, g_i \otimes r'(-, x_i)\}$ satisfies $f((\gamma \otimes g)z) = f(z)$, for all $(g, \gamma) \in \Sigma^\perp$ and $z \in D(H)$. For example, if $(g, \gamma) \in \Sigma^\perp$, then

$$\begin{aligned} (x_i \otimes \varepsilon)((\gamma \otimes g)z) &= (g_i \otimes \chi_i^{-1})(\gamma \otimes g)(x_i \otimes \varepsilon)(z) + (x_i \otimes \varepsilon)(\gamma \otimes g)(1 \otimes \varepsilon)(z) \\ &= \gamma(g_i)\chi_i^{-1}(g)(x_i \otimes \varepsilon)(z) \\ &= (x_i \otimes \varepsilon)(z). \end{aligned}$$

It follows that the generators of A are contained in the image of π^* , so $A \subseteq \text{Im}(\pi^*)$. Using Remark 8.3 we compute

$$\dim \text{Im}(\pi^*) = \frac{\dim D(H)}{\dim K} = \frac{|\Gamma|^2 2^{2n}}{|\Sigma^\perp|} = |\Sigma| 2^{2n} = \dim A$$

and so $A = \text{Im}(\pi^*)$. □

Recall the notion of the Drinfeld double of V from Section 3.2. We have $D(V) = W \oplus W^* \in \mathcal{Z}_{\Sigma} \mathcal{YD}$, where W is the quantum linear space associated to the datum

$$((g_1, \chi_1^{-1}), \dots, (g_n, \chi_n^{-1}), \varphi_1, \dots, \varphi_n), \quad \text{with } \varphi_i \in \widehat{\Sigma}, \varphi_i(g, \chi) = \chi_i(g), i = 1, \dots, n.$$

Define a bicharacter $r_\Sigma : \Sigma \times \Sigma \rightarrow k^\times$ by

$$r_\Sigma((g, \chi), (g', \chi')) = \chi'(g).$$

The diagonal of this bicharacter is a quadratic form $q_\Sigma : \Sigma \rightarrow k^\times$,

$$q_\Sigma(g, \chi) = \chi(g), \quad (g, \gamma) \in \Sigma.$$

Then $D(V) \in \mathcal{Z}_{\text{sym}}(\mathcal{C}(\Sigma, r_\Sigma))_-$.

THEOREM 8.5. *Let (\mathcal{P}, V, r) be a metric datum such that $V \in \mathcal{P}$ is self-dual. There is an equivalence of braided tensor categories:*

$$\mathcal{Z}(\mathcal{C}(\mathcal{P}, V, r))_{\text{ad}} \cong \mathcal{C}(\mathcal{C}(\Sigma, r_\Sigma), D(V), r_{D(V)}),$$

where $r_{D(V)}$ is the canonical symplectic form on $D(V)$ defined in (22).

PROOF. Let $H = \mathfrak{B}(V)\#k[\Gamma]$ and let A be the Hopf subalgebra of $D(H)^*$ generated by the group-like elements $g_i \otimes \chi_i^{-1}$ and by the skew-primitive elements $x_i \otimes \varepsilon$ and $g_i \otimes r'(-, x_i)$, $i = 1, \dots, n$, where r' is an r -form on H whose restriction to $V \otimes V$ is non-degenerate. Using Proposition 8.4, we have

$$\begin{aligned} \mathcal{Z}(\text{Corep}(H))_{\text{ad}} &= (\text{Rep}(D(H)^{\text{cop}}))_{\text{ad}} \\ &= \text{Rep}(A^{*\text{cop}}) = \text{Rep}(A^*)^{\text{op}} \simeq \text{Rep}(A^*) = \text{Corep}(A). \end{aligned}$$

where the equivalence between $\text{Rep}(A^*)$ and its opposite follows from the fact that $\text{Rep}(A^*)$ is braided.

We claim that $A \cong \mathfrak{B}(D(V))\#k[\Sigma]$. Indeed, it is easy to check that for each $(g, \gamma) \in \Sigma$ we have

$$\begin{aligned} (g \otimes \gamma)(x_i \otimes \varepsilon) &= \chi_i^{-1}(g)(x_i \otimes \varepsilon)(g \otimes \gamma), \\ (g \otimes \gamma)(g_i \otimes r(-, x_i)) &= \chi_i^{-1}(g)(g_i \otimes r(-, x_i))(g \otimes \gamma), \\ (g_i \otimes r(-, x_i))(x_j \otimes \varepsilon) &= \chi_j^{-1}(g_i)(x_j \otimes \varepsilon)(g_i \otimes r(-, x_i)), \end{aligned}$$

for all $i, j = 1, \dots, n$. Note that W is self-dual because V is self-dual. Thus, there exists a basis $\{y_i\}_{i=1}^{2n}$ of $D(V)$ such that $y_i, y_{n+i} \in D(V)_{(g_i, \chi_i^{-1})}^{\varphi_i}$. It follows from the above, that the map $A \rightarrow \mathfrak{B}(D(V))\#k[\Sigma]$, given by

$$g \otimes \gamma \mapsto (g, \gamma), \quad x_i \otimes \varepsilon \mapsto y_i, \quad g_i \otimes r(-, x_i) \mapsto y_{n+i}$$

for all $(g, \gamma) \in \Sigma$ and $i = 1, \dots, n$, is a Hopf algebra isomorphism.

The braiding on $\mathcal{Z}(\mathcal{C}(\Gamma, q, V, r))_{\text{ad}}$ is obtained by restriction of the braiding of $\mathcal{Z}(\mathcal{C}(\Gamma, q, V, r))$. It corresponds to the braiding on $\text{Corep}(A)$ coming from the restriction to A of the canonical r -form $r_{D(H)^*} : D(H)^* \otimes D(H)^* \rightarrow k$ on $D(H)^*$. The latter is given by

$$r_{D(H)^*}(\alpha, \beta) = \sum_{h \in \Gamma, P \subseteq \{1, \dots, n\}} \alpha(\varepsilon \otimes hx_P) \beta((hx_P)^* \otimes 1), \quad \alpha, \beta \in D(H)^*.$$

We have

$$\begin{aligned} r_{D(H)^*}(g \otimes \gamma, g' \otimes \gamma') &= \sum_{h, P} \varepsilon(g) \gamma(hx_P) (hx_P)^*(g') \gamma'(1) = \gamma(g'), \\ r_{D(H)^*}(x_i \otimes \varepsilon, x_j \otimes \varepsilon) &= \sum_{h, P} \varepsilon(x_i) \varepsilon(hx_P) (hx_P)^*(x_j \varepsilon(1)) = 0, \\ r_{D(H)^*}(x_i \otimes \varepsilon, g_j \otimes r(-, x_j)) &= \sum_{h, P} \varepsilon(x_i) \varepsilon(hx_P) (hx_P)^*(g_j) r(1, x_j) = 0, \\ r_{D(H)^*}(g_j \otimes r(-, x_j), x_i \otimes \varepsilon) &= \sum_{h, P} \varepsilon(g_j) r(hx_P, x_j) (hx_P)^*(x_i) \varepsilon(1) = r(x_i, x_j), \\ r_{D(H)^*}(g_i \otimes r(-, x_i), g_j \otimes r(-, x_j)) &= \sum_{h, P} \varepsilon(g_i) r(hx_P, x_i) (hx_P)^*(g_j) r(1, x_j) = 0. \end{aligned}$$

Thus, $\text{Corep}(A, r_{D(H)^*}|_{A \otimes A}) \simeq \mathcal{C}(\Sigma, q_\Sigma, D(V), s)$, where the quadratic form $q_\Sigma : \Sigma \rightarrow k^\times$ is given by $q_\Sigma(g, \gamma) = \gamma(g)$ for all $(g, \gamma) \in \Sigma$ and the matrix of $s : D(V) \otimes D(V) \rightarrow k$ with respect to the basis $\{y_i\}$ is the block matrix

$$\begin{pmatrix} 0 & 0 \\ X^t & 0 \end{pmatrix}.$$

Here X^t is the transpose of the matrix $X = (s(x_i, x_j))_{i,j}$. Changing s by a cocycle deformation s^σ will not change the braided equivalence class of $\mathcal{C}(\Sigma, q_\Sigma, D(V), s)$. As explained in Section 7.2, we can choose invariant σ such that s^σ is alternating. The matrix of s^σ with respect to the basis $\{y_i\}$ is then

$$\frac{1}{2} \begin{pmatrix} 0 & -X \\ X^t & 0 \end{pmatrix}.$$

This matrix is easily seen to be congruent to $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. So after a change of basis the matrices of s^σ and $r_{D(V)}$ coincide. Therefore, $\text{Corep}(A, r_{D(H)^*}|_{A \otimes A}) \simeq \mathcal{C}(\Sigma, q_\Sigma, D(V), r_{D(V)})$. \square

Let G be a finite group. Recall that a tensor category \mathcal{C} is called a G -extension of a tensor category \mathcal{A} if there is a faithful grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that $\mathcal{C}_e \cong \mathcal{A}$.

COROLLARY 8.6. $\mathcal{Z}(\mathcal{C}(\mathcal{P}, V, r))$ is a $\widehat{\Sigma}^{-1}$ -extension of $\mathcal{C}(\mathcal{C}(\Sigma, r_\Sigma), D(V), r_{D(V)})$.

Acknowledgments.

The authors are grateful to Adriana Nenciu for helpful discussions.

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