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The α-family in the $K(2)$-local sphere at the prime 2

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Abstract. In this note, we compute the image of the α-family in the homotopy of the $K(2)$-local sphere at the prime $p = 2$ by locating its image in the algebraic duality spectral sequence. This is a steppingstone for the computation of the homotopy groups of the $K(2)$-local sphere at the prime 2 using the duality spectral sequences.

Contents

Acknowledgements. This note was born in conversations which happened during the writing of [**BGH17**] and the author is indebted to her collaborators, Paul Goerss and Hans-Werner Henn. Many of the methods and ideas used here are borrowed from that rich collaboration. She also thanks Zhouli Xu for useful conversations related to this topic.

Mark Mahowald knew how the α-family would be detected in the duality spectral sequences and this paper makes his sketches precise. Some of the computations used in the proof of this theorem are also closely related to results of Mahowald and Rezk in [**MR09**].

1. Introduction

The first periodic family in the homotopy groups of spheres was constructed by Adams in his study of the image of the J homomorphism, which culminated in what is now one of the must-read articles in algebraic topology, *On the Groups $J(X)$ – IV* [**Ada66**]. In the last section of this paper, Adams uses self-maps of Moore spaces to construct elements of the homotopy groups of spheres that he denotes by α. These elements are intimately related to K-theory and are part of what is now called the "α-family".

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The α-family is one of the few computable families of elements in the stable homotopy groups of spheres. It is the first of its kind and, with its successors the β and γ-families, it now belongs to a collection of classes known as the Greek-letter elements. In their cornerstone paper on periodicity in the Adams-Novikov Spectral Sequence, Miller, Ravenel and Wilson [**MRW77**] give an intimate connection between the Greek-letter elements and the chromatic spectral sequence, and establish the importance of the chromatic point of view for computations of the homotopy groups of spheres.

Chromatic homotopy as it is known today comes from Morava's insight that there should be higher analogs of p-completed K-theory. They should carry higher Adams operations, and detect periodic families which are generalizations of the image of J. These cohomology theories are called the Morava E-theories E_n and the associated mod p theories are called the Morava K-theories $K(n)$. The higher Adams operations form a group called the (extended) Morava stabilizer group, denoted $\mathbb{G}_n$. The theories E_n and $K(n)$ are complex oriented ring spectra whose construction is based on the deformation theory of height n formal groups.

The Morava E and K-theories detect periodic families of elements in the homotopy groups of spheres. There are various ways to make this precise. One is through the eyes of Bousfield localization. The Chromatic Convergence Theorem of Hopkins and Ravenel states that the p-local sphere spectrum $S_{(p)}$ is the (homotopy) inverse limit of the Bousfield localizations S_{E_n} of the sphere at the Morava E-theories. One then studies $S_{(p)}$ through its images under the natural maps $S_{(p)} \to S_{E_n}$. Further, the S_{E_n} can be inductively reassembled from the localizations at the Morava K-theories via a homotopy pull-back

$$\begin{array}{ccc} S_{E_n} & \longrightarrow & S_{K(n)} \\ \downarrow & & \downarrow \\ S_{E_{n-1}} & \longrightarrow & (S_{K(n)})_{E_{n-1}}. \end{array}$$

These facts highlight the importance of computing both $\pi_* S_{E_n}$ and $\pi_* S_{K(n)}$. The standard tools for computing these homotopy groups are two closely related spectral sequences. Note that the E_n-local sphere is equivalent to $S_{E(n)}$, where $E(n)$ is the Johnson-Wilson spectrum, a "thinner" version of E_n. The $E(n)$-Adams-Novikov Spectral Sequence computes the homotopy groups of $S_{E(n)} \simeq S_{E_n}$:

$$\mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*) \Longrightarrow \pi_* S_{E(n)}.$$

The second spectral sequence is the $K(n)$-local E_n-Adams-Novikov Spectral Sequence, which computes the homotopy groups of $S_{K(n)} \simeq E_n^{h\mathbb{G}_n}$. Its E_2-term can be identified with continuous cohomology groups:

$$H^*(\mathbb{G}_n, (E_n)_*) \Longrightarrow \pi_* S_{K(n)}.$$

We give an overview of what is known. First $S_{K(0)}$ and S_{E_0} are both the rational sphere $S_{\mathbb{Q}}$. The computation of $\pi_* S_{K(1)}$ and $\pi_* S_{E_1}$ can be obtained from the classical computations of Adams, Atiyah and others on the image of J and the action of the Adams operations. The computation of $\pi_* S_{E_2}$ and $\pi_* S_{K(2)}$ are entirely different beasts. Shimomura, Wang and Yabe have done extensive work on computing these homotopy groups at various primes. The case $p \geq 5$ is treated in [**SY95**] and is also nicely presented in [**Beh12**]. The case $p = 3$ is treated in [**SW02b**, **Shi00**] and the case $p = 2$ is partially treated in [**SW02a**, **Shi99**].

The height two computations are extremely difficult and the answers contain an enormous amount of information that is hard to interpret and analyze. Having multiple points of view seems to have become an imperative for our understanding of chromatic height two phenomena.

In [**GHMR05**], Goerss, Henn, Mahowald and Rezk establish a different approach to height two computations. It relies on resolutions of the $K(2)$-local sphere called the *duality resolutions*, from which one obtains various spectral sequences. For certain subgroups G of $\mathbb{G}_2$, the *topological duality spectral sequences* converge to $\pi_* E_2^{hG}$ and the *algebraic duality spectral sequences* converge to $H^*(G, (E_2)_*)$.

The advantage of the duality spectral sequences is that they organize the computations and the answers in a systematic way. For $p \geq 5$, these methods are used in [**Lad13**], for $p = 3$, in [**HKM13**] and for $p = 2$, in [**Bea17b**] to perform computations for the $K(2)$-local Moore spectrum. The homotopy of $\pi_* S_{K(2)}$ at $p = 3$ has been analyzed by Goerss, Henn, Karamanov, Mahowald using duality methods, but has not been fully recorded yet.

Duality spectral sequence techniques are also being used to solve other central problems in chromatic homotopy theory. They have been crucial in the study of the Chromatic Splitting Conjecture [**GHM14**, **Bea17a**, **BGH17**] at $p = 2$ and $p = 3$. In particular, they play a central role in the disproof of the strongest form of the conjecture at $p = 2$ [**Bea17a**, **BGH17**]. The computations of the $K(2)$-local Picard groups and of the Gross-Hopkins dual of the sphere at the prime $p = 3$ rely on the duality spectral sequences [**GHMR15**, **GH16**]. These are currently being adapted by the author and her collaborators to solve the same problems at $p = 2$. Finally, Bhattacharya and Egger use the duality techniques to compute the homotopy groups of the first example of a type 2 complex with a v_2^1 self-map [**BE17**].

The current paper is concerned with computations of $\pi_* S_{K(2)}$ at $p = 2$ using duality spectral sequence techniques and we finish the introduction by stating our result. When computing $\pi_* S_{K(2)}$, a first and essential step is to locate the α-family in the computation. The goal of this paper is to do this at $p = 2$, using the duality techniques. The results in this paper are a steppingstone for a full computation of $\pi_* S_{K(2)}$ using the duality spectral sequences. We will recall the precise definition of the α-family in Section 2. We will define the algebraic duality spectral sequence and the subgroup $\mathbb{S}_2^1 \subseteq \mathbb{G}_2$ in Section 3. Our main results (in Section 4) are summarized in the following statement.

THEOREM. *Let $p = 2$. The elements $\alpha_{i/j} \in \mathrm{Ext}^{1,2i}_{BP_*BP}(BP_*, BP_*)$ map nontrivially to $H^1(\mathbb{S}_2^1, (E_2)_{2i})$. In the algebraic duality spectral sequence*

$$E_1^{p,q,t} = H^q(F_p, (E_2)_t) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, (E_2)_t)$$

the αs are detected as follows:

(a) $\alpha_{2/2} \in E_1^{0,1,4}$
(b) $\alpha_{i/1} \in E_1^{0,1,2i}$ *if $i \geq 1$ is odd.*
(c) $\alpha_{i/j} \in E_1^{1,0,2i}$ *if i is even.*

The maps

$$H^1(\mathbb{G}_2, (E_2)_t) \to H^1(\mathbb{S}_2^1, (E_2)_t)$$

in degrees $t \neq 0$ are injective so that the image of the $\alpha_{i/j}$ have unique lifts in $H^1(\mathbb{G}_2, (E_2)_{2i})$.

In the spectral sequence

$$H^*(\mathbb{G}_2, (E_2)_*) \Longrightarrow \pi_* S_{K(2)}$$

the α-family supports the standard pattern of differentials and the family of elements detected by the αs in $\pi_* S$ maps non-trivially to $\pi_* S_{K(2)}$. The same holds in $\pi_* E_2^{h\mathbb{S}_2^1}$ and the associated homotopy fixed point spectral sequence.

2. The α-family in the Adams-Novikov Spectral Sequence

In this section, we review the construction of the α-family and fix notation. There are many references for these results: See, for example, Section 4 of [**MRW77**] and Section 4 of [**Rav78**]. We let $S = S_{(2)}$ and BP be the 2-local Brown-Peterson spectrum. For a spectrum X, the Adams-Novikov Spectral Sequence (ANSS) is given by

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X) \Longrightarrow \pi_{t-s}X_{(2)}.$$

The α-family is a collection of elements $\alpha_{i/j} \in \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*)$ which we construct below.

Remark 2.1. We also call the collection of non-trivial elements of π_*S detected by the αs the α-family, or the *topological* α-family when we wish to make the distinction clear.

To define the α-family, one first shows that there is an isomorphism

$$\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/2) \cong \mathbb{F}_2[v_1].$$

See for example Theorem 4.3.2 of [**Rav86**]. The α-family is defined by taking the image in $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*)$ of the powers of v_1 under various Bockstein homomorphisms. Define $x_{1,n} \in v_1^{-1}BP_{2^{n+1}}$ by

$$x_{1,n} = \begin{cases} v_1 & n = 0 \\ v_1^2 - 4v_1^{-1}v_2 & n = 1 \\ v_1^4 - 8v_1v_2 & n = 2 \\ x_{1,n-1}^2 & n \geq 3. \end{cases}$$

Let $s \geq 1$ be an odd integer. The reduction of $x_{1,n}^s$ modulo 2 is an element of $BP_{2^{n+1}s}/2$ congruent to $v_1^{2^n s}$. Furthermore,

$$x_{1,n}^s \in \begin{cases} BP_2/2 & n = 0 \text{ and } s \geq 1 \\ BP_4/4 & n = 1 \text{ and } s = 1 \\ BP_{2^{n+1}s}/2^{n+2} & n \geq 2, \text{ or } n = 1 \text{ and } s \geq 3 \end{cases}$$

are comodule primitives. See, for example, Lemma 4.12 of [**MRW77**].

Let

$$\delta^{(n)} \colon \mathrm{Ext}^{0,t}_{BP_*BP}(BP_*, BP_*/2^n) \to \mathrm{Ext}^{1,t}_{BP_*BP}(BP_*, BP_*)$$

be the connecting Bockstein homomorphism associated to the short exact sequence

$$0 \longrightarrow BP_* \xrightarrow{\times 2^n} BP_* \longrightarrow BP_*/2^n \longrightarrow 0\ .$$

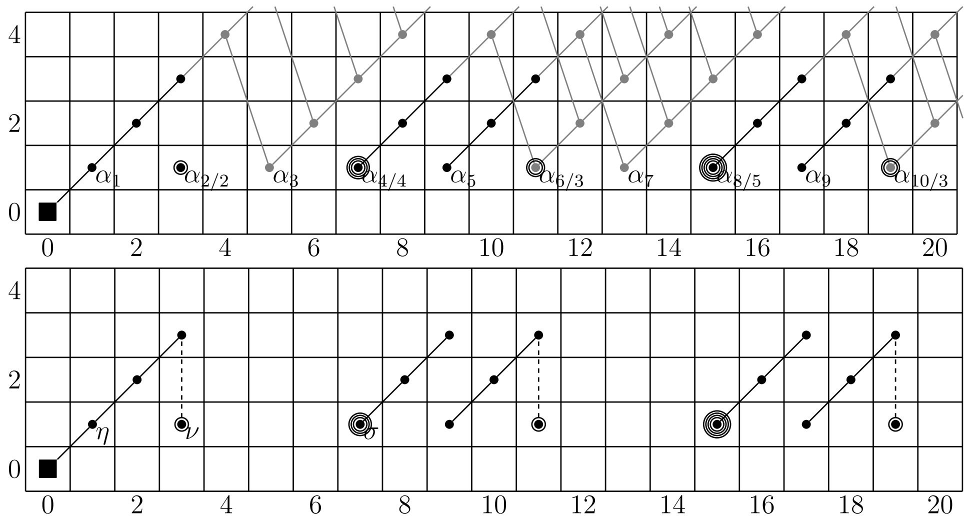

FIGURE 1. The α-family in the E_2 (top) and E_∞ (bottom) pages of the Adams-Novikov Spectral Sequence. Here, a ■ denotes a copy of $\mathbb{Z}_2$, a • denotes a copy of $\mathbb{Z}/2$, a ◉ a copy of $\mathbb{Z}/4$ and so on. Dashed lines denote exotic multiplications by 2.

Keeping the convention that $s \geq 1$ is an odd integer, there are classes

$$\alpha_{i/j} \in \mathrm{Ext}^{1,2i}_{BP_*BP}(BP_*, BP_*)$$

of order 2^j defined by

$$\alpha_{s/1} = \delta^{(1)}(x^s_{1,0}),$$
$$\alpha_{2/2} = \delta^{(2)}(x_{1,1}),$$

and

$$\alpha_{2^n s/(n+2)} = \delta^{(n+2)}(x^s_{1,n})$$

for $n \geq 2$ and $s \geq 1$, or for $n = 1$ and $s \geq 3$. We usually abbreviate $\alpha_i = \alpha_{i/1}$.

Note that $\alpha_1 \alpha_{2/2} = 0$ and otherwise

$$\alpha_1^k \alpha_{i/j} \neq 0$$

for all $k \geq 0$.

The $\alpha_{i/j}$s are classes in the E_2-term of the ANSS for the sphere spectrum. This spectral sequence has no d_2 differentials for degree reasons, so the E_2 and E_3-terms are equal. Further, there are d_3 differentials

$$d_3(\alpha_{i/j}) = \begin{cases} \alpha_1^4 & i = j = 1 \\ \alpha_1^3 \alpha_{4k+1} & i = 4k+3 \text{ and } j = 1 \\ \alpha_1^3 \alpha_{2^n s/n+2} & i = 2^n s + 2 \text{ and } j = 3. \end{cases}$$

We obtain the pattern in Figure 1, which is also in Table 2 of [**Rav78**].

3. Subgroups of $\mathbb{G}_2$ and the algebraic duality spectral sequence

Before turning to the computation of the α-family in the $K(2)$-local sphere, we recall some of the tools used in the computation. This will be brief, but we refer the reader to [**Bea15**, **Bea17b**, **BGH17**] where these techniques were explained in great detail.

We let $K(2)$ refer to the 2-periodic Morava K-theory spectrum whose formal group law is that of the super-singular elliptic curve defined over $\mathbb{F}_4$ with Weierstrass equation

$$C_0 : y^2 + y = x^3.$$

The homotopy groups of $K(2)$ are given by

$$K(2)_* = \mathbb{F}_4[u^{\pm 1}]$$

for u in degree -2. We let $E = E_2$ be the associated Morava E-theory constructed in [**Bea17b**, Section 2], chosen so that the formal group law of E is that of the universal deformation of C_0 with Weierstrass equation

$$C : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3.$$

Its homotopy groups are

$$E_* = \mathbb{W}[\![u_1]\!][u^{\pm 1}]$$

where u_1 is in E_0 and u is in E_{-2}. Here $\mathbb{W} = W(\mathbb{F}_4)$ is the ring of Witt vectors on $\mathbb{F}_4$. We choose a primitive third root of unity ω and note that $\mathbb{W} \cong \mathbb{Z}_2[\omega]/(1+\omega+\omega^2)$. This is a complete local ring with residue field $\mathbb{F}_4$. In fact, it is the ring of integers in an unramified extension of degree 2 of $\mathbb{Q}_2$. The Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, whose generator we denote by σ, acts on $\mathbb{W}$ by the $\mathbb{Z}_2$-linear map determined by $\omega^\sigma = \omega^2$. Further, the Teichmüller lifts give a natural embedding of $\mathbb{F}_4^\times \subseteq \mathbb{W}^\times$.

We let $\mathbb{S}_2$ be the group of automorphisms of the formal group law of $K(2)$. The group $\mathbb{S}_2$ is isomorphic to the units in a maximal order $\mathcal{O}$ of a division algebra of dimension 4 over $\mathbb{Q}_2$ and Hasse invariant $1/2$. A presentation for $\mathcal{O}$ is given by

$$\mathcal{O} \cong \mathbb{W}\langle T\rangle/(T^2 = -2, aT = Ta^\sigma), \quad a \in \mathbb{W}.$$

It follows that an element of $\gamma \in \mathbb{S}_2$ can be written as power series

$$\gamma = \sum_{i\geq 0} a_i(\gamma)T^i$$

where the elements $a_i(\gamma) \in \mathbb{W}$ satisfy $a_i(\gamma)^4 - a_i(\gamma) = 0$ and $a_0(\gamma) \neq 0$. The group Gal acts on $\mathcal{O}$ via its action on $\mathbb{W}$, fixing T. We let $\mathbb{G}_2$ be the extension of $\mathbb{S}_2$ by Gal, so that

$$\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathrm{Gal}\,.$$

The right action of $\mathbb{S}_2$ on $\mathcal{O}$ gives rise to a representation $\mathbb{S}_2 \to GL_2(\mathbb{W})$ whose determinant restricts to a homomorphism

$$\det\colon \mathbb{S}_2 \to \mathbb{Z}_2^\times.$$

We can extend the determinant to $\mathbb{G}_2$ by $\det(x,\sigma) = \det(x)$. The determinant composed with the projection to $\mathbb{Z}_2^\times/(\pm 1) \cong \mathbb{Z}_2$ defines a homomorphism of $\mathbb{G}_2$ onto $\mathbb{Z}_2$. For any subgroup $G \subseteq \mathbb{G}_2$, we let G^1 be the kernel of this composite. If G is $\mathbb{S}_2$ or $\mathbb{G}_2$, this is a split surjection so that

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2 \qquad \mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2 \tag{3.1}$$

We will use the map $\mathbb{Z}_2 \to \mathbb{G}_2$ which sends a chosen generator of $\mathbb{Z}_2$ to $\pi = 1+2\omega \in \mathbb{W}^\times$ as a preferred splitting.

The group $\mathbb{S}_2$ has the following important subgroups. First, it has a unique conjugacy class of maximal finite subgroups. A representative can be chosen to be the image of the automorphisms of the super-singular curve $\mathrm{Aut}(C_0)$, which we will denote by G_{24}. It is the semi-direct product of a quaternion group with the natural copy of $\mathbb{F}_4^\times$ in $\mathbb{S}_2$. The group $C_6 = (\pm 1) \times \mathbb{F}_4^\times$ is a subgroup of G_{24}. Note that the torsion is contained in $\mathbb{S}_2^1$ as $\mathbb{Z}_2$ is torsion free. So these are in fact subgroups of $\mathbb{S}_2^1$. However, we note that in $\mathbb{S}_2^1$ the groups G_{24} and $G_{24}' = \pi G_{24} \pi^{-1}$ are not conjugate ($\pi \notin \mathbb{S}_2^1$). Finally, the Galois group acts on these finite subgroups and they can all be extended to corresponding subgroups of $\mathbb{G}_2$. The maximal finite subgroup of $\mathbb{G}_2$ is denoted by

$$G_{48} \cong G_{24} \rtimes \mathrm{Gal}\,.$$

Next, we turn to the computational tools. For finite spectra X,

$$X_{K(2)} \simeq E^{h\mathbb{G}_2} \wedge X \simeq (E \wedge X)^{h\mathbb{G}_2}$$

and there is a spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_2, E_t X) \Longrightarrow \pi_{t-s} X_{K(2)} \tag{3.2}$$

where, here and everywhere, we mean the continuous cohomology groups. Analyzing the E_2-term of this spectral sequence is difficult, so we often start by studying the cohomology of the subgroup $\mathbb{S}_2^1$. We have an extremely concrete tool to compute the group cohomology of $\mathbb{S}_2^1$, a spectral sequence called the *algebraic duality spectral sequence* (ADSS), which we describe here.

For a graded profinite $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$-module M (a typical example is $M = E_*X$), the algebraic duality spectral sequence for M is a first quadrant spectral sequence:

$$E_1^{p,q,t} = E_1^{p,q,t}(M) \cong H^q(F_p, M_t) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M_t) \tag{3.3}$$

with differentials $d_r \colon E_r^{p,q,t} \to E_r^{p+r,q-r+1,t}$, where $F_0 = G_{24}$, $F_1 = F_2 = C_6$ and $F_3 = G_{24}'$. We may omit the internal grading t from the notation.

The spectral sequence has an edge homomorphism

$$H^p(H^0(F_\bullet, M_t), d_1) \to H^p(\mathbb{S}_2^1, M_t),$$

where $H^p(H^0(F_\bullet, M_t), d_1)$ is the cohomology of the complex

$$0 \longrightarrow H^0(F_0, M_t) \xrightarrow{d_1} H^0(F_1, M_t) \xrightarrow{d_1} H^0(F_2, M_t) \xrightarrow{d_1} H^0(F_3, M_t) \longrightarrow 0, \tag{3.4}$$

and $H^0(F_p, M_t) = E_1^{p,0,t} \cong M_t^{F_p}$.

Central to the computations of this paper is the differential

$$d_1 \colon E_1^{0,0,t} \to E_1^{1,0,t},$$

which we describe here. There is an element $\alpha \in \mathbb{W}^\times \subseteq \mathbb{S}_2$ which is defined so that $\alpha = 1 + 2\omega \mod (4)$ and $\det(\alpha) = -1$.[1] It is shown in Theorem 1.1.1 of [**Bea17b**] that the differential is given by the action of $1 - \alpha$:

$$d_1 = 1 - \alpha \colon H^0(F_0, E_*) \to H^0(F_1, E_*).$$

[1] At this point, we run into a conflict of notation. In the current trend of $K(2)$-local computations at $p = 2$, the element named α plays a crucial and well-established role. We will keep the name, as any element of the α-family has a subscript and this should make it easy to avoid confusion.

To compute with this spectral sequence, we will also need information about the cohomology $H^*(G_{24}, E_*)$ (which is isomorphic to $H^*(G'_{24}, E_*)$) and of $H^*(C_6, E_*)$. This is all well-known, but nicely presented in Section 2 of [**BG18**]. So we refer to that paper for the information we need.

Finally, for any closed subgroup G of $\mathbb{G}_2$ and finite 2-local spectrum X, the complex orientation of Morava E-theory and the fact that the homotopy fixed point spectral sequence (3.2) is isomorphic to the $K(2)$-local E-based Adams Spectral Sequence (see Appendix A of [**DH04**]) gives a comparison diagram

$$\begin{array}{ccc}
\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*X) & \Longrightarrow & \pi_*X \\
\cong\uparrow & & \cong\uparrow \\
\mathrm{Ext}^{*,*}_{MU_*MU}(MU_*, MU_*X) & \Longrightarrow & \pi_*X \\
\downarrow & & \downarrow \\
H^*(\mathbb{G}_2, E_*X) & \Longrightarrow & \pi_*X_{K(2)} \\
\downarrow & & \downarrow \\
H^*(G, E_*X) & \Longrightarrow & \pi_*(E^{hG} \wedge X).
\end{array}$$

To detect the α-family in π_*E^{hG}, one studies the fate of the α-family in $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ under the vertical maps when $X = S_{(2)}$.

4. The α-family in the $K(2)$-local sphere

We finally turn to the computation of the α-family in the $K(2)$-local sphere. The approach is as follows. We will identify the image of the α-family under the map

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^*(\mathbb{G}_2, E_*).$$

In particular, we will show that all of the non-trivial classes $\alpha_1^k\alpha_{i/j}$ map non-trivially. For filtration reasons, this will imply that any class from the topological α-family in π_*S maps non-trivially to $\pi_*E^{h\mathbb{G}_2}$.

We will need the following generalization of [**BGH17**, Proposition 3.2.2], which allows us to identify classes detecting the α-family in the cohomology of certain closed subgroups of $\mathbb{G}_n$. Its proof is completely analogous and is omitted here.

Proposition 4.1. *Let $E = E_n$ and $H \subseteq \mathbb{G}_n$ be a closed subgroup. Let $R = H^0(H, E_0)$. Fix $i > 0$ and suppose that*

(1) *$H^0(H, E_{2i}/2)$ is a cyclic R-module generated by v_1^i.*

Let $y_{i/j} \in E_{2i}/2^j$ be a class so that

(2) *$y_{i/j} \equiv v_1^i$ modulo 2, and*
(3) *$y_{i/j}$ is invariant under the action of H.*

Then, up to multiplication by a unit in R, the image of $\alpha_{i/j} \in \pi_{2i-1}E^{hH}$ is detected in the spectral sequence

$$H^s(H, E_t) \Longrightarrow \pi_{t-s}E^{hH}$$

by the class $\delta^{(j)}(y_{i/j}) \in H^1(H, E_{2i})$.

One of the consequences of Theorem 1.2.2 of [**Bea17b**] is the following lemma.

LEMMA 4.2. *Let H be a closed subgroup of $\mathbb{G}_2$ which contains $\mathbb{S}_2^1$. Then*

$$H^0(H, E_*/2) \cong \mathbb{F}_4[v_1].$$

In particular, any closed subgroup H of $\mathbb{G}_2$ which contains $\mathbb{S}_2^1$ satisfies condition (1) of Proposition 4.1 for any $i > 0$. So, to apply Proposition 4.1, we must identify candidates for the classes $y_{i/j}$.

To construct these classes, recall that there are classical G_{48}-invariants in E_0 associated to the curve C, which play a key role in computations at $n = p = 2$. Specifically, letting $v_1 = u_1 u^{-1}$ and $v_2 = u^{-3}$ the following are invariant for the action of G_{48}:

$$\begin{aligned}
\Delta &= 27v_2(v_1^3 - v_2)^3 \\
c_4 &= 9(v_1^4 + 8v_1v_2) \\
c_6 &= 27\left(v_1^6 - 20v_1^3v_2 - 8v_2^2\right) \\
j &= c_4^3\Delta^{-1}.
\end{aligned}$$

A few elements in the higher cohomology $H^*(G_{48}, E_*)$ will also appear in the computation. Namely, there are elements

$$\eta \in H^1(G_{48}, E_2) \qquad \nu \in H^1(G_{48}, E_4) \qquad \mu \in H^1(G_{48}, E_6).$$

The classes η and ν are chosen to be the images of α_1 and $\alpha_{2/2}$ under the map

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^*(G_{48}, E_*).$$

We choose the class μ to be the image of α_3. This will be discussed in the proof of Theorem 4.12. It has the property that $\mu = \eta v_1^2$ modulo (2) and $\eta\Delta^{-1}c_6c_4^2$ is a unit multiple of $j\mu$.

Note that $H^*(G_{24}, E_*) \cong H^*(G_{48}, E_*) \otimes_{\mathbb{Z}_p} \mathbb{W}$. The restriction

$$H^*(G_{48}, E_*) \to H^*(G_{24}, E_*)$$

is the inclusion of fixed point under the action of the Galois group on the right factor of $\mathbb{W}$. For any element in the cohomology of G_{48}, we denote its restriction in the cohomology of G_{24} by the same name.

We will prove the following result.

PROPOSITION 4.3. *Let $s \geq 1$ be odd. Then, for the action of $\mathbb{G}_2$,*

(a) *$v_1^s \in E_2$ is an invariant modulo 2,*
(b) *$v_1^2 \in E_4$ is an invariant modulo 4,*
(c) *$c_4^n \in E_{8n}$ is an invariant modulo 2^{k+4} for $n = 2^k s$ where $k \geq 0$, and*
(d) *$c_6c_4^n \in E_{8n+12}$ is invariant modulo 8 for $n \geq 0$.*

This motivates the following definition, where $s \geq 1$ is odd,

$$y_{i/j} = \begin{cases} v_1^s & i = s, j = 1, \\ v_1^2 & i = j = 2, \\ c_4^{2^k s} & i = 2^{k+2}s, j = k+4, \\ c_6c_4^{(s-3)/2} & i = 2s, j = 3,\ s \neq 1. \end{cases} \tag{4.1}$$

Note that in the last two cases of (4.1) (for $i = 2^{k+2}s$ and $j = k+4$, or $i = 2s$, $j = 3$, and $s \neq 1$), the element $y_{i/j} \in H^0(F_0, E_{2i})$ since $F_0 = G_{24} \subseteq G_{48}$ and both c_4 and c_6 are invariant for the action of G_{48}.

Following the outline of Proposition 4.1, we must compute $\delta^{(j)}(y_{i/j})$. We get specific and do this for the group $\mathbb{S}_2^1$ defined in (3.1) by using the algebraic duality spectral sequence (ADSS) of (3.3). The part of the ADSS relevant for our computations is depicted in Figure 2. Lemma 7.1.2 of [**BGH17**] gives a method for computing the Bockstein $\delta^{(n)}$ of certain elements for the spectral sequence of a double complex which is particularly suited to the ADSS. Combined with Proposition 4.1, it has the following immediate consequence.

THEOREM 4.4. *Let $(H^0(F_\bullet, E_t), d_1)$ be the complex of* (3.4). *Let $s \geq 1$ be an odd integer. Let*

(a) $i = 2^{k+2}s$ *and* $j = k+4$, *or*
(b) $i = 2s$, $j = 3$, *and* $s \neq 1$.

Then, up to multiplication by a unit in $\mathbb{W}$, $\alpha_{i/j} \in H^1(\mathbb{S}_2^1, E_{2i})$ is detected by the image of the class

$$\left[\frac{d_1(y_{i/j})}{2^j}\right] \in H^1(H^0(F_1, E_{2i}), d_1)$$

under the edge homomorphism

$$H^0(F_p, E_{2i}) \longrightarrow E_\infty^{p,0,2i} \subseteq H^p(\mathbb{S}_2^1, E_{2i}).$$

To prove Proposition 4.3 and thus apply Proposition 4.4, we will need some information about the action of $\mathbb{S}_2$ on c_4 and c_6 which we record now.

PROPOSITION 4.5. *Let $\gamma = 1 + a_2(\gamma)T^2 \mod T^3$ in $\mathbb{S}_2$. Then*

$$\gamma_*(c_4) \equiv c_4 + 16(a_2(\gamma) + a_2(\gamma)^2)v_1v_2 \mod (32, 16u_1^2)$$

and

$$\gamma_*(c_6) \equiv c_6 + 8(a_2(\gamma) + a_2(\gamma)^2)v_1^3v_2 \mod (16, 8u_1^4).$$

PROOF. The first claim is Lemma 5.2.2 of [**Bea17b**]. To prove the second claim, we proceed as in the proof of this lemma. From (3.3.1) of [**Bea17b**], we have that

$$\gamma_*(u) = t_0(\gamma)u \qquad \gamma_*(u_1) = t_0(\gamma)u_1 + \frac{2}{3}\frac{t_1(\gamma)}{t_0(\gamma)} \tag{4.2}$$

where

$$t_0(\gamma) \equiv 1 + 2a_2(\gamma) \mod (2, u_1)^2, \qquad t_1 \equiv a_2(\gamma)^2u_1 \mod (2, u_1^2). \tag{4.3}$$

We abbreviate by letting $t_i = t_i(\gamma)$ for $i = 0, 1$ and $a_2 = a_2(\gamma)$.

From (4.2), we deduce that, modulo (16)

$$\begin{aligned} &c_6 - \gamma_*(c_6) \\ &\quad \equiv 4u^{-6}t_0^{-6}\left(u_1^3t_0^6 + u_1^3t_0^2(3t_0 + 3u_1t_1^2 + u_1^2t_1t_0^2) + 2(u_1^2t_1t_0 + t_0^6 + 1)\right). \end{aligned}$$

By Proposition 6.3.3 of [**Bea17b**],

$$t_0^4 \equiv t_0 + u_1t_1^2 + u_1^2t_1t_0^2 \mod (2)$$

so that

$$c_6 - \gamma_*(c_6) \equiv 0 \mod (8).$$

To compute the leading term, we consider $c_6 - \gamma_*(c_6)$ modulo $(16, u_1^4)$. Using (4.3), we have that, modulo $(16, u_1^4)$

$$\begin{aligned} c_6 - \gamma_*(c_6) &\equiv 4u^{-6}t_0^{-6}\left(u_1^3(t_0^6+3t_0^3)+2u_1^2t_1t_0+2(t_0^6+1)\right) \\ &\equiv 8u_1^3u^{-6}\left(a_2+a_2^2\right). \end{aligned}$$

In the last line, we used the fact that $t_0^6 \equiv 1 \mod (2, u_1^4)$ and also modulo $(4, u_1)$. □

Proof of Proposition 4.3. Since v_1 is invariant modulo 2, (a) and (b) are immediate. Proposition 4.5 shows that c_4 and c_6 are invariant under the action of α and π modulo 16 and 8 respectively. Since c_4 and c_6 are already invariant under the action of G_{48} and $\mathbb{G}_2$ is topologically generated by G_{48}, α and π, parts (c) and (d) follow by taking appropriate powers. □

To apply Proposition 4.4, we will prove something slightly more general: We will completely compute the differential

$$d_1 \colon E_1^{0,0,*} \to E_1^{1,0,*}.$$

We first identify $E_1^{0,0,*} \cong E_*^{G_{24}} \cong H^0(G_{24}, E_*)$ and $E_1^{1,0,*} \cong E_*^{C_6} \cong H^0(C_6, E_*)$ more explicitly than we have done so far.

For example, from Section 2 and 3 of [**BG18**], we have isomorphisms

$$H^0(G_{24}, E_*) \cong \mathbb{W}[\![j]\!][c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = (12)^3\Delta, c_4^3 = \Delta j).$$

It follows that the elements

$$\{c_6^\epsilon c_4^m \Delta^n \mid m \geq 0,\ \epsilon = 0,1,\ n \in \mathbb{Z}\}$$

form a set of topological $\mathbb{W}$-module generators, so that, in the category of profinite graded $\mathbb{W}$-modules,

$$H^0(G_{24}, E_*) \cong \prod_{\substack{n,m\in\mathbb{Z}, m\geq 0 \\ \epsilon=0,1}} \mathbb{W}\{c_6^\epsilon c_4^m \Delta^n\}.$$

There is also an isomorphim

$$H^0(C_6, E_*) \cong \mathbb{W}[\![u_1^3]\!][v_1^2, v_1v_2, v_2^{\pm 2}]/\sim$$

where $\sim$ is the ideal

$$((v_1^2)^3 - (v_2^2)(u_1^3)^2, (v_1v_2)^2 - (v_1^2)(v_2^2), (v_1^2)(v_1v_2) - (u_1^3)(v_2^2)).$$

Therefore, a basis of topological $\mathbb{W}$-module generators for $H^0(C_6, E_*)$ is given by

$$\{(v_1v_2)^\epsilon (v_1^2)^m (v_2^2)^n \mid m \geq 0,\ \epsilon = 0,1,\ n \in \mathbb{Z}\}$$

and, in the category of profinite graded $\mathbb{W}$-modules,

$$H^0(C_6, E_*) \cong \prod_{\substack{n,m\in\mathbb{Z}, m\geq 0 \\ \epsilon=0,1}} \mathbb{W}\{(v_1v_2)^\epsilon (v_1^2)^m (v_2^2)^n\}.$$

We are now ready to compute d_1 explicitly. We note that this result is intimately related to Propositions 8.1 and 8.2 [**MR09**].

PROPOSITION 4.6. *The differential* $d_1\colon E_1^{0,0} \to E_1^{1,0}$ *is determined by the following information:*

(a) *For* $n, m \in \mathbb{Z}$ *of the form* $n = 2^k(2t+1)$, $m \geq 0$ *and for* $\epsilon = 0, 1$,

$$d_1(c_6^{\epsilon} c_4^m \Delta^n) \equiv (v_1^2)^{3\cdot 2^k + 2m + 3\epsilon} (v_2^2)^{2^k(1+4t)} \mod (2, v_1^{9\cdot 2^k + 4m + 6\epsilon})$$

and $d_1(\Delta^0) = 0$.

(b) *For* $n \in \mathbb{Z}$ *of the form* $n = 2^k(2t+1)$, $n \geq 1$,

$$d_1(c_4^n) \equiv 2^{k+4}(v_1 v_2)(v_1^2)^{2(n-1)} \mod (2^{k+5}, 2^{k+4} v_1^{4(n-1)+2}).$$

(c) *For* $n \in \mathbb{Z}$, $n \geq 1$ *of the form* $n = 2^k(2t+1)$ *or for* $n = 0$,

$$d_1(c_6 c_4^n) \equiv 8(v_1 v_2)(v_1^2)^{2n+1} \mod (16, 8v_1^{4n+4}).$$

PROOF. This differential is given by the action of $1-\alpha$. Further, $\alpha \equiv 1 + \omega T^2$ modulo T^4 for ω a primitive third root of unity. Hence, $a_2(\alpha) + a_2(\alpha)^2 = -1$.

The claim (a) is an immediate consequence of Proposition 5.1.1 of [**Bea17b**], which states that

$$\alpha_*(\Delta^n) \equiv \Delta^n + v_1^{6\cdot 2^k} v_2^{2^{k+1}(4t+1)} \mod (2, u_1^{9\cdot 2^k}),$$

using the fact that $c_4 \equiv v_1^4$ and $c_6 \equiv v_1^6$ modulo 2.

To prove (b), from Proposition 4.5, using the fact that $c_4 \equiv v_1^4$ modulo 2, we deduce that

$$\alpha_*(c_4^{2^k}) \equiv (c_4^{2^k} + 2^{k+4} v_1^{4(2^k-1)} v_1 v_2) \mod (2^{k+5}, 2^{k+4} u_1^{4(2^k-1)+2}).$$

Hence,

$$\alpha_*(c_4^n) \equiv c_4^n + 2^{k+4} v_1^{4(n-1)} v_1 v_2 \mod (2^{k+5}, 2^{k+4} u_1^{4(n-1)+2}).$$

Similarly, to prove (c), using that $\alpha_*(c_4) \equiv c_4$ modulo (16) we have

$$\begin{aligned} \alpha_*(c_6 c_4^n) &\equiv \alpha_*(c_6) c_4^n \mod (16) \\ &\equiv c_6 c_4^n + 8 v_1^{4n+3} v_2 \mod (16, 8v_1^{4n+4}). \end{aligned}$$

□

REMARK 4.7. For (ϵ, a, b) such that $\epsilon = 0, 1$, $a \geq 0$, and $b \in \mathbb{Z}$, we define elements $b_{\epsilon,a,b}$ in $E_1^{1,0,t} \cong H^0(C_6, E_t)$ for $t = 8\epsilon + 4a + 12b$ that satisfy

$$b_{\epsilon,a,b} = (v_1 v_2)^{\epsilon} (v_1^2)^a (v_2^2)^b + \dots$$

as follows:

(a) For $n, m \in \mathbb{Z}$ of the form $n = 2^k(2t+1)$, $m \geq 0$ and for $\epsilon = 0, 1$,

$$b_{0, 3\cdot 2^k + 2m + 3\epsilon, 2^k(1+4t)} = d_1(c_6^{\epsilon} c_4^m \Delta^n)$$

(b) For $n \in \mathbb{Z}$ of the form $n = 2^k(2t+1)$, $n \geq 1$,

$$b_{1,2(n-1),0} = \frac{d_1(c_4^n)}{2^{k+4}}.$$

(c) For $n \in \mathbb{Z}$, $n \geq 1$ of the form $n = 2^k(2t+1)$ or for $n = 0$,

$$b_{1,2n+1,0} = \frac{d_1(c_6 c_4^n)}{8}.$$

(d) In all other cases,

$$b_{\epsilon,a,b} = (v_1 v_2)^{\epsilon} (v_1^2)^a (v_2^2)^b.$$

Although we will not refer to all of the elements $b_{\epsilon,a,b}$ defined above, it will be useful to have a fixed name for them in future computations.

We now give some consequences of Proposition 4.6. We start with an immediate corollary:

THEOREM 4.8. *In the ADSS*

$$E_1^{p,q,t} \cong H^q(F_p, E_t) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, E_t)$$

there is an isomorphism

$$E_2^{0,0,0} \cong E_\infty^{0,0,0} \cong \mathbb{W}\{\Delta_0\}$$

where Δ_0 *is the unit in* $E_1^{0,0,0} \cong H^0(G_{24}, E_0)$. *Further,* $E_2^{0,0,t} = 0$ *if* $t \neq 0$. *For* $r \geq 0$, *the classes* $b_{1,r,0}$ *are in the kernel of* $d_1\colon E_1^{1,0,t} \to E_1^{2,0,t}$ *and detect classes in* $E_\infty^{1,0,t}$ *of degree* $t = 8 + 4r$. *These classes have order* 8 *if* $r = 2n+1$ *and* $n \geq 0$. *They have order* 2^{k+4} *if* $r = 2n$ *for* $n = 2^k s - 1$, $s \geq 1$ *odd, and* $k \geq 0$.

REMARK 4.9. Since the edge homomorphism of the ADSS has the form

$$H^0(F_p, E_*) \longrightarrow E_\infty^{p,0,*} \subseteq H^p(\mathbb{S}_2^1, E_*),$$

even if the generators $b_{1,n,0}$ are strictly speaking elements of $E_\infty^{p,0,*}$, they represent unique elements in the cohomology of $\mathbb{S}_2^1$, and hence, we can write $b_{1,n,0} \in H^*(\mathbb{S}_2^1, E_*)$ without any ambiguity.

As an immediate consequence of Theorem 4.8, we have the following result, which was already proved in [**BGH17**]:

COROLLARY 4.10. *The inclusion* $\mathbb{Z}_2 \to E_0$ *induces an isomorphism*

$$\begin{aligned} H^0(\mathbb{S}_2^1, E_*) &\cong H^0(\mathbb{S}_2^1, E_0) \cong \mathbb{W} \\ H^0(\mathbb{G}_2^1, E_*) &\cong H^0(\mathbb{G}_2^1, E_0) \cong \mathbb{Z}_2 \\ H^0(\mathbb{G}_2, E_*) &\cong H^0(\mathbb{G}_2, E_0) \cong \mathbb{Z}_2. \end{aligned}$$

PROOF. Theorem 4.8 implies that $H^0(\mathbb{S}_2^1, E_*) \cong \mathbb{W}$. The result follows for $\mathbb{G}_2^1$ since $H^0(\mathbb{S}_2^1, E_*) \cong H^0(\mathbb{G}_2^1, E_*) \otimes_{\mathbb{Z}_2} \mathbb{W}$ with the natural action of Gal on $\mathbb{W}$ (see [**BG18**, Lemma 1.24]). The fixed points for $\mathbb{G}_2$ include in those for $\mathbb{G}_2^1$ and contain the image of $\mathbb{Z}_2 \subseteq E_0$. □

The next three results are depicted in Figure 3.

COROLLARY 4.11. *Up to multiplication by a unit in* $\mathbb{W}$,

(a) $b_{1,2n+1,0}$ *for* $n \geq 0$ *detects* $\alpha_{(4n+6)/3}$, *and*
(b) $b_{1,2n,0}$ *for* $n = 2^k s - 1$, $s \geq 1$ *odd,* $k \geq 0$ *detects* $\alpha_{2^{k+2}s/(k+4)}$.

PROOF. This follows from Proposition 4.4, using Theorem 4.8 and Corollary 4.10. □

We turn to the elements $\alpha_s = \alpha_{s/1}$ where $s \geq 1$ is odd.

THEOREM 4.12. *Let* $s \geq 1$ *be an odd integer. In the ADSS*

$$E_1^{p,q,t} \cong H^q(F_p, E_t) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, E_t)$$

there are isomorphisms

$$E_2^{0,1,2s} \cong E_\infty^{0,1,2s} \cong \begin{cases} \mathbb{F}_4\{\eta c_6^\epsilon c_4^m\} & s \neq 3,\ s = 1+6\epsilon+4m \\ \mathbb{F}_4\{\mu\} & s = 3. \end{cases}$$

Further $\alpha_s = \alpha_{s/1} \in H^1(\mathbb{S}_2^1, E_{2s})$ *is non-trivial. The edge homomorphisms*

$$H^1(\mathbb{S}_2^1, E_{2s}) \to E_\infty^{0,1,2s} \subseteq H^1(F_0, E_{2s})$$

are isomorphisms and α_s *can be identified with its image in* $H^1(F_0, E_{2s})$*. The element* α_1 *is detected by* η*,* α_3 *is detected by* μ*. If* $s \geq 5$*, then* $s = 1+6\epsilon+4m$ *for some* $\epsilon = 0, 1$ *and* $m \geq 0$*. In this case,* α_s *is detected by* $\eta c_6^\epsilon c_4^m$*.*

Proof. The associated graded of the ADSS for $H^1(\mathbb{S}^1, E_{2s})$ consists of $E_\infty^{0,1,2s}$ and $E_\infty^{1,0,2s}$. The latter is a subquotient of $E_2^{1,0,2s} \cong E_{2s}^{C_6}$, which is trivial when s is odd. Therefore, $H^1(\mathbb{S}_2^1, E_{2s}) \cong E_\infty^{0,1,2s}$ and the edge homomorphism is an isomorphism.

Now, note that the reduction modulo 2 induces isomorphisms

$$H^1(F_p, E_{2s}) \cong H^1(F_p, E_{2s}/2)$$

for $p = 0, 1$. Further, this isomorphism maps $\eta c_6^\epsilon c_4^m$ to $\eta v_1^{6\epsilon+4m}$. So, to compute $E_\infty^{0,1,2s}$ we can use the commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & H^1(F_0, E_{2s}) & \xrightarrow{d_1} & H^1(F_1, E_{2s}) \\ & \cong\downarrow & & \cong\downarrow \\ 0 \longrightarrow & H^1(F_0, E_{2s}/2) & \xrightarrow{d_1} & H^1(F_1, E_{2s}/2) \end{array}$$

The kernel of d_1 for the top row is isomorphic to the kernel of d_1 for the bottom row, which was computed in [**Bea17b**] to be generated by $\eta v_1^{s-1} = \eta v_1^{6\epsilon+4m}$ if $s = 1+6\epsilon+4m$. Therefore, $E_2^{0,1,2s} \cong \mathbb{F}_4\{\eta c_6^\epsilon c_4^m\}$ as desired.

Also implied by the extensive computations in [**Bea17b**] is the fact that $H^0(\mathbb{S}_2^1, E_*) \cong \mathbb{F}_4[v_1]$ with the edge homomorphism

$$H^0(\mathbb{S}_2^1, E_{2s}/2) \xrightarrow{\cong} E_\infty^{0,0,2s} \subseteq H^0(F_0, E_{2s}/2)$$

an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{S}_2^1, E_{2s}/2) & \xrightarrow{\delta_{\mathbb{S}_2^1}} & H^1(\mathbb{S}_2^1, E_{2s}) \\ \downarrow\cong & & \downarrow\cong \\ E_\infty^{0,0,2s} & & E_\infty^{0,1,2s} \\ \downarrow\subseteq & & \downarrow\subseteq \\ H^0(F_0, E_{2s}/2) & \xrightarrow{\delta_{F_0}} & H^1(F_0, E_{2s}) \end{array}$$

where δ_G is the connecting homomorphism for the exact sequence

$$0 \longrightarrow E_* \xrightarrow{\times 2} E_* \longrightarrow E_*/2 \longrightarrow 0.$$

Since

$$\delta_{F_0}(v_1^s) = \begin{cases} \eta c_6^{\epsilon} c_4^m & s \geq 1,\ s \neq 3,\ s = 1 + 6\epsilon + 4m \\ \mu & s = 3, \end{cases}$$

the image of α_s is $\eta c_6^{\epsilon} c_4^m$ if $s \neq 3$ and μ if $s = 3$. So, the corresponding elements of $E_2^{0,1,2s}$ are permanent cycles in the ADSS. So $E_\infty^{0,1,2s} \cong E_2^{0,1,2s}$, generated by the image of α_s for s odd. □

It remains to understand the image of $\alpha_{2/2}$.

THEOREM 4.13. *In the ADSS, there is an isomorphism*

$$E_2^{0,1,4} \cong E_\infty^{0,1,4} \cong \mathbb{W}/4\{\nu\}.$$

The edge homomorphism

$$H^1(\mathbb{S}_2^1, E_4) \to E_\infty^{0,1,4} \subseteq H^1(F_0, E_4)$$

is an isomorphism and the element $\alpha_{2/2}$ can be identified with its image in $H^1(F_0, E_4)$, where it is detected by ν.

PROOF. The contributions to $H^1(\mathbb{S}_2^1, E_4)$ in the ADSS consist of $E_\infty^{0,1,4}$ and $E_\infty^{1,0,4}$. There is an isomorphism

$$E_1^{0,1,4} \cong H^1(F_0, E_4) \cong H^1(G_{24}, E_4) \cong \mathbb{W}/4\{\nu\}$$

and ν is so named because it is the image of $\alpha_{2/2}$ under the homomorphism from the ANSS E_2-term $\mathrm{Ext}^{1,4}_{BP_*BP}(BP_*, BP_*)$. This map factors through $H^*(\mathbb{S}_2^1, E_*)$, so ν must be a permanent cycle in the ADSS. So, all elements of $E_1^{0,1,4}$ persist to E_∞.

We turn our attention to $E_\infty^{1,0,4}$ and show that

$$E_\infty^{1,0,4}(E_*) = E_2^{1,0,4}(E_*) \cong H^1(H^0(F_\bullet, E_4)) = 0.$$

This implies that the edge homomorphism is an isomorphism, and that ν lifts uniquely to an element of $H^1(\mathbb{S}_2^1, E_4)$ where it corresponds to the image of $\alpha_{2/2}$.

We begin with a computation modulo (2). There is a commutative diagram:

$$\begin{array}{ccccccc}
H^0(F_0, E_4/2) & \xrightarrow{d_1} & H^0(F_1, E_4/2) & \xrightarrow{d_1} & H^0(F_2, E_4/2) & \xrightarrow{d_1} & H^0(F_3, E_4/2) \\
{\scriptstyle v_1^2}\uparrow{\scriptstyle \cong} & & {\scriptstyle v_1^2}\uparrow{\scriptstyle \cong} & & {\scriptstyle v_1^2}\uparrow{\scriptstyle \cong} & & {\scriptstyle v_1^2}\uparrow{\scriptstyle \cong} \\
H^0(F_0, E_0/2) & \xrightarrow{d_1} & H^0(F_1, E_0/2) & \xrightarrow{d_1} & H^0(F_2, E_0/2) & \xrightarrow{d_1} & H^0(F_3, E_0/2) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^0(F_0, \mathbb{W}/2) & \xrightarrow{d_1} & H^0(F_1, \mathbb{W}/2) & \xrightarrow{d_1} & H^0(F_2, \mathbb{W}/2) & \xrightarrow{d_1} & H^0(F_3, \mathbb{W}/2)
\end{array}$$

By Theorem 5.4.1 of [**BGH17**], the vertical map from the bottom to the middle row induces an isomorphism upon taking cohomology with respect to d_1. The cohomology of the bottom row gives a copy of $\mathbb{F}_4$ in each degree, whose generators were called Δ_0, b_0, $\overline{b}_0$, $\overline{\Delta}_0$ for $p = 0, 1, 2, 3$ respectively. It follows that

$$E_2^{p,0,4}(E_*/2) \cong \begin{cases} \mathbb{F}_4\{v_1^2 \Delta_0\} & p = 0 \\ \mathbb{F}_4\{v_1^2 b_0\} & p = 1. \end{cases}$$

Let $\ker(d_1)$ be the kernel of $H^0(F_1, E_4) \xrightarrow{d_1} H^0(F_2, E_4)$ and $\widetilde{\ker}(d_1)$ that of $H^0(F_1, E_4/2) \xrightarrow{d_1} H^0(F_2, E_4/2)$. The diagram

$$\begin{array}{ccc} H^0(F_0, E_4/2) & \xrightarrow{\cong} & \mathbb{F}_4\{v_1^2\Delta_0\} \oplus H^0(F_0, E_4)/2 \\ \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle 0\oplus d_1} \\ \widetilde{\ker}(d_1) & \xrightarrow{\cong} & \mathbb{F}_4\{v_1^2 b_0\} \oplus \ker(d_1)/(2) \end{array}$$

commutes. Given the cohomology of the left vertical map, it must be the case that d_1 induces an isomorphism

$$d_1 \colon H^0(F_0, E_4)/2 \xrightarrow{\cong} \ker(d_1)/(2).$$

So, we have a commutative diagram

$$\begin{array}{ccccc} H^0(F_0, E_4) & \xrightarrow{d_1} & \ker(d_1) & \longrightarrow & H^1(H^0(F_\bullet, E_4)) \\ \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} \\ H^0(F_0, E_4) & \xrightarrow{d_1} & \ker(d_1) & \longrightarrow & H^1(H^0(F_\bullet, E_4)) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(F_0, E_4)/2 & \xrightarrow[\cong]{d_1} & \ker(d_1)/(2) & \longrightarrow & 0. \end{array} \tag{4.4}$$

The left two columns of (4.4) are short exact by definition. By Theorem 4.8 $E_2^{0,0,4} = 0$, so the map $d_1 \colon H^0(F_0, E_4) \to H^0(F_1, E_4)$ is injective. So the rows of (4.4) are short exact. It follows that the third column is short exact. So multiplication by 2 is an isomorphism on $H^1(H^0(F_\bullet, E_4))$. Since this is a complete $\mathbb{Z}_2$-module, it must be trivial. $\square$

We now identify the α-family in $H^*(\mathbb{G}_2, E_*)$. We begin with an observation.

Remark 4.14. Recall once more that $H^*(\mathbb{S}_2^1, E_*) \cong H^*(\mathbb{G}_2^1, E_*) \otimes_{\mathbb{Z}_2} \mathbb{W}$ and that the restriction

$$H^*(\mathbb{G}_2^1, E_*) \cong H^*(\mathbb{S}_2^1, E_*)^{\mathrm{Gal}} \xrightarrow{\subseteq} H^*(\mathbb{S}_2^1, E_*)$$

is an inclusion. Further, the map from the E_2-term of the ANSS to $H^*(\mathbb{S}_2^1, E_*)$ factors through this inclusion. We have identified the $\mathbb{W}$-submodule $\mathbb{W}/2^j\{\alpha_{i/j}\} \subseteq H^1(\mathbb{S}_2^1, E_{2i})$. Choose a Galois invariant $\mathbb{W}$-module generator of $\mathbb{W}/2^j\{\alpha_{i/j}\}$ and call it $\alpha_{i/j}$. Then $\mathbb{Z}/2^j\{\alpha_{i/j}\} \subseteq H^*(\mathbb{G}_2^1, E_*)$.

On the other hand, the restriction

$$\mathrm{res} \colon H^*(\mathbb{G}_2, E_*) \to H^*(\mathbb{G}_2^1, E_*)$$

is not injective. So one must proceed with care. Recall that there is a split exact sequence

$$1 \longrightarrow \mathbb{G}_2^1 \longrightarrow \mathbb{G}_2 \longrightarrow \mathbb{Z}_2^\times/(\pm 1) \longrightarrow 1.$$

Using the fact that $\mathbb{Z}_2^\times/(\pm 1) \cong \mathbb{Z}_2$, we have $\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2$. We choose π to be a topological generator for $\mathbb{Z}_2 \cong \mathbb{G}_2/\mathbb{G}_2^1$, which acts on $H^*(\mathbb{G}_2^1, E_*)$. From the

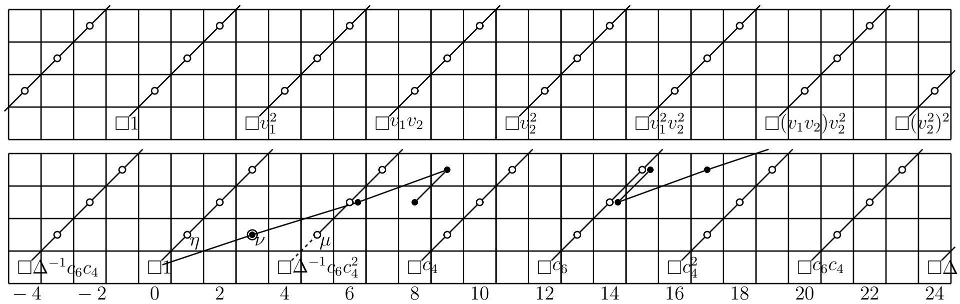

FIGURE 2. A part of the E_1-term of the algebraic duality spectral sequence, $E_1^{p,q,t} = H^q(F_p, E_t)$. The top is $E_1^{1,q,t}$ for $0 \leq q \leq 3$, drawn in the $(t-q-1, q)$-plane. The bottom is $E_1^{0,q,t}$ in the same range, drawn in the $(t-q, q)$-plane. A $\square$ denotes a copy of $\mathbb{W}[\![j]\!]$ if $p = 0$ and $\mathbb{W}[\![u_1^3]\!]$ if $p = 1$. A $\circ$ denotes a copy of $\mathbb{F}_4[\![j]\!]$ if $p = 0$ and $\mathbb{F}_4[\![u_1^3]\!]$ if $p = 1$. A $\bullet$ is a copy of $\mathbb{F}_4$ and $\odot$ a copy of $\mathbb{W}/4$. The labels denote the generators as $\mathbb{W}[\![j]\!]$-modules on the $p = 0$-line and as $\mathbb{W}[\![u_1^3]\!]$-modules on the $p = 1$-line. The lines denote multiplication by η and ν. The dashed line indicates that $\eta\Delta^{-1}c_6c_4^2 = j\mu$.

Lyndon-Hochschild-Serre Spectral Sequence for the group extension, one obtains a long exact sequence

$$\ldots \longrightarrow H^*(\mathbb{G}_2^1, E_*) \xrightarrow{\pi-1} H^*(\mathbb{G}_2^1, E_*) \xrightarrow{\delta} \tag{4.5}$$

$$H^{*+1}(\mathbb{G}_2, E_*) \xrightarrow{\text{res}} H^{*+1}(\mathbb{G}_2^1, E_*) \xrightarrow{\pi-1} \ldots$$

To fully analyze the long exact sequence (4.5), we would need a full computation of $H^*(\mathbb{G}_2^1, E_*)$, and control over the action of π on the cohomology groups $H^*(\mathbb{G}_2^1, E_*)$. Neither is available to us at this point. However, in the range of interest for computing the α-family, we get lucky.

PROPOSITION 4.15. *The restriction $H^1(\mathbb{G}_2, E_t) \to H^1(\mathbb{G}_2^1, E_t)$ is injective if $t \neq 0$.*

PROOF. By Corollary 4.10, $H^0(\mathbb{G}_2^1, E_*) \cong \mathbb{Z}_2$ and the restriction $H^0(\mathbb{G}_2, E_*) \to H^0(\mathbb{G}_2^1, E_*)$ is an isomorphism. This is the first map in (4.5), so we get an exact sequence

$$0 \longrightarrow H^0(\mathbb{G}_2^1, E_*) \xrightarrow{\delta} H^1(\mathbb{G}_2, E_*) \longrightarrow H^1(\mathbb{G}_2^1, E_*) \xrightarrow{\pi-1} H^1(\mathbb{G}_2^1, E_*)$$

The claim follows from the fact that $H^0(\mathbb{G}_2^1, E_t) = 0$ if $t \neq 0$. $\square$

COROLLARY 4.16. *There are unique classes $\alpha_{i/j} \in H^1(\mathbb{G}_2, E_{2i})$ which map to the same named classes in $H^1(\mathbb{S}_2^1, E_{2i})$ as described in Remark 4.14. These are the images of the α-family elements under the map from the E_2-term of the BP-based Adams-Novikov Spectral Sequence.*

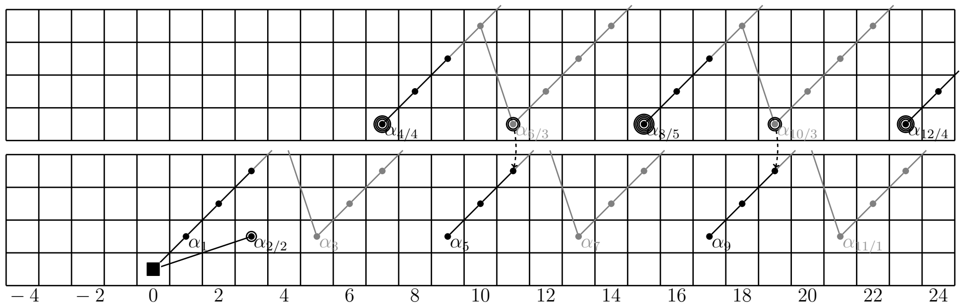

FIGURE 3. The contribution to the α-family in the algebraic duality spectral sequence. The differentials indicate differentials that occur in the homotopy fixed points spectral sequence $H^*(\mathbb{S}_2^1, E_*) \Longrightarrow \pi_* E^{h\mathbb{S}_2^1}$ and the dashed arrows indicate exotic extensions on the E_∞-term of that spectral sequence.

COROLLARY 4.17. *The topological α-family maps non-trivially in $\pi_* S_{K(2)}$, and in $\pi_* E^{h\mathbb{S}_2^1}$. The α-family is detected in $H^*(\mathbb{G}_2, E_*)$ by the classes $\alpha_1^k \alpha_{i/j}$ which support the standard pattern of differentials in the spectral sequence*

$$H^s(\mathbb{G}_2, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{G}_2}. \tag{4.6}$$

PROOF. The only thing to justify is that there are no differentials killing non-trivial elements of the image of the topological α-family. However, the ANSS filtration of the α-elements detecting non-trivial elements in homotopy is at most 3. The first differential in (4.6) and the analogue for $\mathbb{S}_2^1$ is a d_3, and the zero line of (4.6) consists of the permanent cycles $H^0(\mathbb{G}_2, E_*) \cong \mathbb{Z}_2$, respectively $H^0(\mathbb{S}_2^1, E_*) \cong \mathbb{W}$. That the latter are all permanent cycles follows from Section 1.2 of [**BG18**]. □

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