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Complex orientations for THH of some perfectoid fields

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Abstract. This sketch argues that work of Hesselholt [11] on the topological Hochschild homology of $\mathbb{C}_p$ extends, using work of Scholze and others [27], to define complex orientations for a version of topological Hochschild homology for rings of integers in a natural class of generalized cyclotomic perfectoid fields; and that the resulting spectra provide geometrically interesting targets for analogs of the Chern character, defined for certain integral lifts [21] of the extraordinary K-functors of chromatic homotopy theory.

§I Introduction and Preliminaries

1.1 About fifty years ago SP Novikov called attention to the relevance of one-dimensional formal groups in algebraic topology, and since then understanding the resulting link between homotopy theory and arithmetic geometry (following Quillen) has become one of the deepest topics in mathematics.

Recent powerful new ideas [4,5,6] in p-adic Hodge theory, applied to the (now classical) Lubin-Tate generalized cyclotomic closure $L^{\hat{\infty}}$ [15] of a local number field L, define complex orientations (*i.e.* ring homomorphisms

$$\mathrm{MU}^* \to \mathrm{THH}^*(\mathcal{O}_{L^{\hat{\infty}}}, \mathbb{Z}_p) \cong \mathsf{S}^*_{\mathcal{O}_{L^{\hat{\infty}}}}(T\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{L^{\hat{\infty}}}))$$

from the Lazard-Quillen complex cobordism ring) for the p-adic topological Hochschild homology spectra for the valuation rings of such fields. These orientations have interesting connections on one hand to chromatic homotopy theory, and on the other to the p-adic Fourier theory [26] of Schneider and Teitelbaum.

To present an accessible account of these connections requires considerable background, reviewed in this section after a sketch of the organization of this paper. The interests of workers in modern stable homotopy theory have a great deal of overlap with current work in higher local number theory, but the languages of these fields have diverged since the days of Cartan's seminars. The review below summarizes material useful in both areas; technical terms used informally in the outline immediately below will be defined more precisely in that review.

1.2 Organization This paper is organized as follows. §2 interprets the group $\boldsymbol{\mu}$ of p-power roots of unity (for example in a p-adic field big enough to contain all such), as an analog in p-adic algebraic K-theory of the classical Hopf line bundle

$\eta \to \mathbb{C}P^\infty$ in complex topological K-theory. Building on Hesselholt's pioneering 2006 work on the Dennis trace

$$\tau_{\mathbb{C}_p} : k_*^{\rm alg}(\mathbb{C}_p, \mathbb{Z}_p) \ \to \ {\rm THH}_*(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p)$$

(for the **non**-periodic p-adic algebraic K-theory of the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$), we use $k_{\rm alg}(\mathbb{C}_p)^*(B\boldsymbol{\mu}, \mathbb{Z}_p)$ to construct orientation classes $\boldsymbol{\gamma}_P$ for p-adic THH of the valuation ring $\mathcal{O}_P$ of a perfectoid subfield P of $\mathbb{C}_p$ (containing the completion $\mathbb{Q}_p^\infty$ of the field of p-power roots of unity over $\mathbb{Q}_p$)[1].

The main concern of this paper is the Galois structure of Hesselholt and Madsen's p-adic THH [10] of the valuation ring $\mathcal{O}_{L^\infty}$ of the (perfectoid) completion of a maximal **totally ramified** Abelian extension L^∞ of L, discussed in §3. Such questions are now accessible through recent extensions [6, Thm. 6.1] of Bökstedt periodicity, which reduce this problem to that of the structure of (the p-adic completion of) the module of Kähler differentials of $\mathcal{O}_{L^\infty}$ over $\mathbb{Z}_p$.

In §3.3, work of Fontaine from 1982 is extended to identify the Tate module

$$T\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{L^\infty}) \cong (\pi_0 \mathcal{D}_L)^{-1} \mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_L} T_L$$

of such objects; for example

$$T\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\mathbb{Q}_p^\infty}) \cong p_0^{-1} \mathcal{O}_{\mathbb{Q}_p^\infty} \otimes_{\mathcal{O}_{\mathbb{Q}_p}} T_{\mathbb{Q}_p}$$

(T_L being the Tate module [§3.1] of a Lubin-Tate group for L). Here $\mathcal{D}_L$ is Dedekind's different ideal of L over $\mathbb{Q}_p$, and π_0 is a certain torsion point of a Lubin-Tate group. A Jacobson-Zariski exact sequence

$$0 \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\mathbb{Q}_p^\infty}) \otimes_{\mathcal{O}_{\mathbb{Q}_p^\infty}} \mathcal{O}_{L^\infty} \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{L^\infty}) \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{L^\infty}) = 0$$

p-completes to an isomorphism

$$0 \to p_0^{-1} \mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_{\mathbb{Q}_p}} T_{\mathbb{Q}_p} \to (\pi_0 \mathcal{D}_L)^{-1} \mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_L} T_L \to 0$$

defined by multiplication by an element $p_0(\pi_0 \mathcal{D}_L)^{-1} \Omega_\partial(L) \in \mathcal{O}_{L^\infty}$ closely related to the Schneider-Teitelbaum period of the Lubin-Tate group of L. The Galois behavior of the unit $\Omega_\partial(L)$ is key to relations between the Lubin-Tate group of L and the complex orientation constructed for ${\rm THH}(\mathcal{O}_{L^\infty}, \mathbb{Z}_p)$.

The final section dicusses possible applications of this construction in chromatic homotopy theory, and an appendix summarizes some properties of lifts of the vector-space valued functors $K(n)$ to cohomology theories valued in modules over local number rings.

1.3 Background

1.3.1 Conventions from local classfield theory

To avoid overuse of the letter K, in this paper $L \supset \mathbb{Q}_p$ will denote a locally compact topological field of degree $[L : \mathbb{Q}_p] = n < \infty$, with a topology defined by a discrete valuation associated to a homomorphism

$${\rm ord}_p : L^\times \to e^{-1}\mathbb{Z} \subset \mathbb{Q}$$

[1] Unless otherwise noted, we take the prime p to be odd. Some notation may be abbreviated, as specified below, to simplify iterated subscripts and similar decorations.

(normalized by $\mathrm{ord}_p(p) = 1$). We have

$$L \supset \mathcal{O}_L \supset \mathfrak{m}_L = (\pi_L)$$

with $\mathcal{O}_L/\mathfrak{m}_L := k_L \cong \mathbb{F}_q$ with $q = p^f$; this should not lead to confusion with K-theory notation. The 'uniformizing element' π_L generating the maximal ideal $\mathfrak{m}_L$ of the (local) valuation ring $\mathcal{O}_L$ will be fixed once and for all; it satisfies an Eisenstein equation

$$\pi_L^e + \cdots + u \cdot p = 0$$

with coefficients in the Witt ring $W(k_L) \subset \mathcal{O}_L$, with u a unit, so $\mathrm{ord}_p(\pi_L) = e^{-1}$ with $n = ef$. When L is unramified over $\mathbb{Q}_p$, *i.e.* when $e = 1$, we will take $\pi_L = p$, and may omit the subscript on π_L when L is clear from context.

We will also need **non**-discretely valued topological fields, such as the completion $\mathbb{C}_p$ of an algebraic closure $\overline{\mathbb{Q}}_p \supset \mathbb{Q}_p$. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts continuously on $\mathbb{C}_p$ and its order homomorphism is surjective. Similarly, $\mathbb{Q}_p^\infty$ will denote the smallest subfield of $\mathbb{C}_p$ containing all p-power roots of unity, and $\mathbb{Q}_p^{\hat{\infty}}$ will be its completion; these completions are important examples of **perfectoid** number fields [27], *i.e.* p-adic fields P complete with respect to a non-discrete valuation, such that the Frobenius endomorphism $x \mapsto x^p$ is surjective on $\mathcal{O}_P/p\mathcal{O}_P$. We will use $\boldsymbol{\mu}$ to denote the group of p-power roots of unity in some field of interest, fixing an isomorphism

$$\epsilon : \mathbb{Q}_p/\mathbb{Z}_p \cong \boldsymbol{\mu}(\mathbb{Q}_p^\infty) \ .$$

This identifies the Tate module

$$T\boldsymbol{\mu} := \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \boldsymbol{\mu})$$

with the free rank one module $\mathbb{Z}_p(1)$ over the p-adic integers, with Galois action defined by the cyclotomic character

$$\chi : \mathrm{Gal}(\mathbb{Q}_p^\infty/\mathbb{Q}_p) \cong \mathrm{Aut}(\boldsymbol{\mu}) \cong {\mathbb{Z}_p}^\times$$

by the p-adic analog of the Kronecker-Weber theorem.

Example The field $\mathbb{Q}_p(\zeta_p) \subset \mathbb{Q}_p^\infty$ generated over $\mathbb{Q}_p$ by adjoining a primitive pth root ζ_p of unity has $f = 1$ and $\zeta_p - 1$ as uniformizing element, satisfying

$$[p]_{\hat{\mathbb{G}}_m}(x) = x^{-1}[(1+x)^p - 1] = 0 \ ,$$

so $e = p-1$. By Lubin-Tate theory [§3.1.1] this is the same as the field $\mathbb{Q}_p(p_0) =: \widetilde{\mathbb{Q}_p}$ generated over $\mathbb{Q}_p$ by adjoining a root p_0 of

$$[p]_{\hat{\mathbb{G}_0}}(x) = x^{p-1} + p = 0 \ :$$

the formal groups associated to these two series are isomorphic, via a series $\phi(x) = x + \cdots \in \mathbb{Z}_p[[x]]$ which sends $\zeta_p - 1$ to p_0. The Galois group of this field over $\mathbb{Q}_p$ is cyclic, of order $p-1$, and the ratio $p_0^{-1}(\zeta_p - 1)$ is a unit.

1.3.2 Algebraic K-theory and THH

1.3.2.1 In 2003 Hesselholt and Madsen [10, §1.5.6] constructed, for a non-Archimedean topologized field K with valuation ring A and perfect residue field, a cyclotomic spectrum $\mathrm{T}(A|K)$ and a trace morphism

$$k_{\mathrm{alg}}(K) \to \mathrm{T}(A|K) \ ,$$

which refines to a map to a pro-system $\{\mathrm{TR}^{\mathrm{n}},\ \mathrm{n} \geq 1\}$ of fixed-point spectra. These notes will unfortunately be concerned only with the first ($n = 1$) stage of this system, which is accessible in principle by methods [*e.g.* Mac Lane homology [16, §1]] of classical homological algebra, and $\mathrm{T}(A|K)$ is, for such rings, a generalized Eilenberg - Mac Lane spectrum [1, §4]. Recent work [22, §III.5] of Scholze and Nikolaus has clarified, among other things, the $\mathcal{E}_\infty$ properties of this construction, and we will be interested in the p-adic completion

$$k_{\mathrm{alg}}(K,\mathbb{Z}_p) := (k_{\mathrm{alg}}(K))_{\hat{p}} \to (\mathrm{T}(A|K))_{\hat{p}} =: \mathrm{THH}(A,\mathbb{Z}_p)$$

of this trace map, defined by the functor

$$X \mapsto X_{\hat{p}} := [M(-1,\mathbb{Q}_p/\mathbb{Z}_p), X]$$

of maps from a suitable Moore spectrum [11, §2.3].

Example The stable p-adic homotopy group $\pi^S_*(B\mathbb{Q}_p/\mathbb{Z}_p,\mathbb{Z}_p)$, *i.e.*

$$[M(-1,\mathbb{Q}_p/\mathbb{Z}_p),[S^0,B\mathbb{Q}_p/\mathbb{Z}_p]]_* = [M(*-1,\mathbb{Q}_p/\mathbb{Z}_p),B\mathbb{Q}_p/\mathbb{Z}_p] =$$
$$H^1(M(*-1,\mathbb{Q}_p/\mathbb{Z}_p),\mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,\mathbb{Q}_p/\mathbb{Z}_p) = T(\mathbb{Q}_p/\mathbb{Z}_p)$$

of $B\mathbb{Q}_p/\mathbb{Z}_p$ vanishes unless $* = 2$, when it is the Tate module $\mathbb{Z}_p(1)$. [In fact the p-adic completions [32, Ex. 2 p 43] of $B\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{C}P^\infty$ are homotopy equivalent as spaces.]

In 2006 Hesselholt [11], greatly extending the early periodicity theorem of Bökstedt [7, 12 (Intro)], calculated the p-adic completion

$$\begin{array}{ccc} k^{\mathrm{alg}}_*(\mathbb{C}_p,\mathbb{Z}_p) & \xrightarrow{\tau_{\mathbb{C}_p}} & \mathrm{THH}_*(\mathcal{O}_{\mathbb{C}_p},\mathbb{Z}_p) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}_p[\beta_{\mathbb{C}_p}] & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}[\gamma_p] \end{array}$$

of this trace on the homotopy groups of these spectra, for $K = \mathbb{C}_p$. He identified $\mathrm{THH}_*(\mathcal{O}_{\mathbb{C}_p},\mathbb{Z}_p)$ as the symmetric algebra over $\mathcal{O}_{\mathbb{C}_p}$ on the Tate module of the p-adic Kähler differentials $\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\mathbb{C}_p})$, and showed that the class $\beta_{\mathbb{C}_p}$ (with Bockstein image $\zeta_p := \epsilon(p^{-1})$ in $k^{\mathrm{alg}}_1(\mathbb{C}_p)$) maps to $(\zeta_p - 1)$ times a generator

$$\gamma_p := (\zeta_p - 1)^{-1} d\log\epsilon$$

of this Tate module. For reasons explained below, it will be convenient to define a variantly normalized generator

$$\gamma_{\mathbb{C}_p} := p_0^{-1} d\log\epsilon$$

such that $\beta_{\mathbb{C}_p} \mapsto p_0 \cdot \gamma_{\mathbb{C}_p}$.

Hesselholt further showed that (absolute) p-adic Galois group acts as (graded) ring automorphisms of $k^*_{\mathrm{alg}}(\mathbb{C}_p,\mathbb{Z}_p)$ through its abelian quotient

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p^{\hat{\infty}}/\mathbb{Q}_p) \cong \mathbb{Z}_p{}^\times \cong C_{p-1} \times \mathbb{Z}_p$$

via the cyclotomic character $\sigma(\beta_{\mathbb{C}_p}) = \chi(\sigma)\cdot\beta_{\mathbb{C}_p}$, compatibly with the action of σ on $p_0\gamma_{\mathbb{C}_p}$ by projection to the cyclic group $\mathrm{Gal}(\mathbb{Q}_p(p_0)/\mathbb{Q}_p)$ on the first term in the product, and by the identification of $\mathbb{Z}_p$ with $(1+p\mathbb{Z}_p)^\times$ on the second.

1.3.2.2 In 2012 Bhatt [4] showed that THH enjoys flat (*i.e.* fpqc) descent, and more recent work [6] of Bhatt, Morrow, and Scholze shows that for a perfectoid p-adic field P,

Theorem $\mathrm{THH}^*(\mathcal{O}_P, \mathbb{Z}_p)$ *is polynomial over* $\mathcal{O}_P$*, on a single generator of degree two: more precisely,*

$$\mathrm{THH}^*(\mathcal{O}_P, \mathbb{Z}_p) \cong \mathsf{S}^*_{\mathcal{O}_P}(\mathrm{THH}_2(\mathcal{O}_P, \mathbb{Z}_p))$$

is isomorphic (as $\mathrm{Gal}(P/\mathbb{Q}_p)$*-module) to the symmetric algebra on the Tate module*

$$\mathrm{THH}_2(\mathcal{O}_P, \mathbb{Z}_p) \cong T\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_P) := \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_P))$$

of the Kähler differentials of $\mathcal{O}_P$ *over* $\mathcal{O}_{\mathbb{Q}_p}$*.*

It is this result that makes this paper possible; it will be applied below to generalized cyclotomic fields, whose Galois theory is relatively well-understood. Perhaps we should explain here that, for a morphism $A \to B$ of commutative rings, we write

$$d : B \to \Omega^1_A(B)$$

for the universal A-module homomorphism satisfying $d(bc) = bdc + cdb$ (on the grounds that this notation may be easier than $\Omega^1_{B/A}$ to read when A and B are highly sub- or superscripted).

§II The p-adic Hopf line bundle

The complex Hopf line bundle $\eta_{\mathbb{C}} \in k(\mathbb{C}P^\infty)^0(B\mathbb{T})$ (classified by a map $\mathbb{C}P^\infty \to B\mathbb{T} \to B\,\mathrm{U}$) has a p-adic analog

$$\eta_{\mathbb{Q}_p^{\hat{\infty}}} \in k_{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}})^0(B\boldsymbol{\mu}, \mathbb{Z}_p)$$

defined by $B\boldsymbol{\mu} \to B\,\mathrm{Gl}_1(\mathbb{Q}_p^{\hat{\infty}}) \to B\,\mathrm{Gl}^+_\infty(\mathbb{Q}_p^{\hat{\infty}})$. It is $\mathrm{Gal}(\mathbb{Q}_p^\infty/\mathbb{Q}_p) \cong \mathcal{O}^\times_{\mathbb{Q}_p}$ invariant, and, with coproduct defined by multiplication

$$B\boldsymbol{\mu} \times B\boldsymbol{\mu} \to B\boldsymbol{\mu}\ ,$$

satisfies $\Delta(\eta_{\mathbb{Q}_p^{\hat{\infty}}}) = \eta_{\mathbb{Q}_p^{\hat{\infty}}} \otimes \eta_{\mathbb{Q}_p^{\hat{\infty}}}$; its image $\eta_{\mathbb{C}_p} \in k_{\mathrm{alg}}(\mathbb{C}_p)^0(B\boldsymbol{\mu}, \mathbb{Z}_p)$ has similar properties.

The sequence $\{p^{-n}, n \geq 1\}$, regarded via ϵ as a generator of $\pi_2^S(B\boldsymbol{\mu}, \mathbb{Z}_p)$, defines a homomorphism

$$k_{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}})^0(B\boldsymbol{\mu}, \mathbb{Z}_p) \to k_2^{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}}, \mathbb{Z}_p)$$

sending $\eta_{\mathbb{Q}_p^{\hat{\infty}}} - 1$ to a lift $\beta_{\mathbb{Q}_p^{\hat{\infty}}}$ of $\beta_{\mathbb{C}_p}$. Just as $\beta_{\mathbb{C}_p}$ generates $k_2^{\mathrm{alg}}(\mathbb{C}_p, \mathbb{Z}_p)$, $\beta_{\mathbb{Q}_p^{\hat{\infty}}}$ generates $k_2^{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}}, \mathbb{Z}_p)$, so both $k_{\mathrm{alg}}(\mathbb{Q}_p^\infty, \mathbb{Z}_p)$ and $k_{\mathrm{alg}}(\mathbb{C}_p, \mathbb{Z}_p)$ are complex-orientable ring spectra (which, in the case of $\mathbb{C}_p$, has been known since Suslin). We use this together with flat descent to construct coordinates $\boldsymbol{\kappa}$ (*i.e.* of cohomological degree two) for the related topological Hochschild groups, fitting in a diagram

$$\begin{array}{ccc} k_{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}})^*(B\boldsymbol{\mu},\mathbb{Z}_p) & \xrightarrow{\tau_{\mathbb{Q}_p^{\hat{\infty}}}} & \mathrm{THH}(\mathcal{O}_{\mathbb{Q}_p^{\hat{\infty}}})^*(B\boldsymbol{\mu},\mathbb{Z}_p) \cong \mathcal{O}_{\mathbb{Q}_p^{\hat{\infty}}}[\gamma_{\mathbb{Q}_p^{\hat{\infty}}}][[\boldsymbol{\kappa}_{\mathbb{Q}_p^{\hat{\infty}}}]] \\ \downarrow & & \downarrow \\ k_{\mathrm{alg}}(P)^*(B\boldsymbol{\mu},\mathbb{Z}_p) & \xrightarrow{\tau_P} & \mathrm{THH}(\mathcal{O}_P)^*(B\boldsymbol{\mu},\mathbb{Z}_p) \cong \mathcal{O}_P[\gamma_P][[\boldsymbol{\kappa}_P]] \\ \downarrow & & \downarrow \\ k_{\mathrm{alg}}(\mathbb{C}_p)^*(B\boldsymbol{\mu},\mathbb{Z}_p) & \xrightarrow{\tau_{\mathbb{C}_p}} & \mathrm{THH}(\mathcal{O}_{\mathbb{C}_p})^*(B\boldsymbol{\mu},\mathbb{Z}_p) \cong \mathcal{O}_{\mathbb{C}_p}[\gamma_{\mathbb{C}_p}][[\boldsymbol{\kappa}_{\mathbb{C}_p}]] \end{array}$$

of graded completed Hopf algebras[2] for any perfectoid field P containing $\mathbb{Q}_p^{\hat{\infty}}$ and contained in $\mathbb{C}_p$. The construction takes several steps.

2.1.1 Let $\overline{\boldsymbol{\eta}}_{\mathbb{Q}_p^{\hat{\infty}}}$ be a coordinate for $k_{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}})^2(B\boldsymbol{\mu},\mathbb{Z}_p)$ such that

$$\beta_{\mathbb{Q}_p^{\hat{\infty}}}\overline{\boldsymbol{\eta}}_{\mathbb{Q}_p^{\hat{\infty}}} = \eta_{\mathbb{Q}_p^{\hat{\infty}}} - 1 \text{ modulo } \overline{\boldsymbol{\eta}}^2_{\mathbb{Q}_p^{\hat{\infty}}} \ ,$$

with image $\overline{\boldsymbol{\eta}}_{\mathbb{C}_p} \in k_{\mathrm{alg}}(\mathbb{C}_p)^2(B\boldsymbol{\mu},\mathbb{Z}_p)$; then for some $a_i \in \mathbb{Z}_p$ we have

$$\eta_{\mathbb{C}_p} - 1 = \beta_{\mathbb{C}_p}\overline{\boldsymbol{\eta}}_{\mathbb{C}_p} + \sum_{i\geq 1} a_i(\beta_{\mathbb{C}_p}\overline{\boldsymbol{\eta}}_{\mathbb{C}_p})^{i+1} \ ;$$

so $\eta_{\mathbb{C}_p} = 1 + \beta_{\mathbb{C}_p}\boldsymbol{\eta}_{\mathbb{C}_p}$ with

$$\boldsymbol{\eta}_{\mathbb{C}_p} := \overline{\boldsymbol{\eta}}_{\mathbb{C}_p} + \sum_{i\geq 1} a_i\beta^i_{\mathbb{C}_p}\overline{\boldsymbol{\eta}}^{i+1}_{\mathbb{C}_p} \ .$$

We then have a graded formal group law

$$\begin{aligned} \Delta\boldsymbol{\eta}_{\mathbb{C}_p} &= \boldsymbol{\eta}_{\mathbb{C}_p} \otimes 1 + 1 \otimes \boldsymbol{\eta}_{\mathbb{C}_p} + \beta_{\mathbb{C}_p} \cdot \boldsymbol{\eta}_{\mathbb{C}_p} \otimes \boldsymbol{\eta}_{\mathbb{C}_p} \\ &:= \beta^{-1}_{\mathbb{C}_p}(\beta_{\mathbb{C}_p}\boldsymbol{\eta}_{\mathbb{C}_p} \otimes 1 +_{\hat{\mathbb{G}}_m} 1 \otimes \beta_{\mathbb{C}_p}\boldsymbol{\eta}_{\mathbb{C}_p}) \end{aligned}$$

on $k(\mathbb{C}_p)^*(B\boldsymbol{\mu},\mathbb{Z}_p)$ (where

$$X +_{\hat{\mathbb{G}}_m} Y = X + Y + XY$$

denotes the multiplicative formal group law, with formal power series

$$\log_{\hat{\mathbb{G}}_m}(X) = \log(1+X), \ \exp_{\hat{\mathbb{G}}_m}(X) = \exp(X) - 1$$

as logarithm and exponential). It follows that

$$\tilde{\eta}_{\mathbb{Q}_p^{\hat{\infty}}} = 1 + \beta_{\mathbb{Q}_p^{\hat{\infty}}}\tilde{\boldsymbol{\eta}}_{\mathbb{Q}_p^{\hat{\infty}}} \in k_{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}})^2(B\boldsymbol{\mu},\mathbb{Z}_p) \ ,$$

with $\tilde{\boldsymbol{\eta}}_{\mathbb{Q}_p^{\hat{\infty}}} := \overline{\boldsymbol{\eta}}_{\mathbb{Q}_p^{\hat{\infty}}} + \sum_{i\geq 1} a_i\beta^i_{\mathbb{Q}_p^{\hat{\infty}}}\overline{\boldsymbol{\eta}}^{i+1}_{\mathbb{Q}_p^{\hat{\infty}}}$, maps to $\eta_{\mathbb{C}_p}$ in $k_{\mathrm{alg}}(\mathbb{C}_p)^*(B\boldsymbol{\mu},\mathbb{Z}_p)$, and to $\beta_{\mathbb{Q}_p^{\hat{\infty}}}$ in $k_2^{\mathrm{alg}}(\mathbb{Q}_p^{\hat{\infty}},\mathbb{Z}_p)$.

2.1.2 Now by Bhatt descent

$$\begin{aligned} 0 \to \mathrm{THH}_2(\mathcal{O}_{\mathbb{Q}_p^{\hat{\infty}}},\mathbb{Z}_p)/p_0 \to (\mathrm{THH}_2(\mathcal{O}_{\mathbb{Q}_p^{\hat{\infty}}},&\mathbb{Z}_p)/p_0) \otimes_{\mathcal{O}_{\mathbb{Q}_p^{\hat{\infty}}}} \mathcal{O}_{\mathbb{C}_p} \\ &\cong \mathrm{THH}_2(\mathcal{O}_{\mathbb{C}_p},\mathbb{Z}_p)/p_0 \end{aligned}$$

[2]We thank the referee for noting that (following Bourbaki [Algebra, Ch II §11]) we are working with completions of graded rings, with respect to filtrations defined using this grading; or, alternatively, with affine formal group schemes.

is injective, so

$$p_0^{-1}\tau_{\mathbb{Q}_p^\infty}(\beta_{\mathbb{Q}_p^\infty}) =: \gamma_{\mathbb{Q}_p^\infty} \in \mathrm{THH}_2(\mathcal{O}_{\mathbb{Q}_p^\infty}, \mathbb{Z}_p)$$

is well-defined, and maps to $\gamma_{\mathbb{C}_p}$.

2.1.3 Lemma *The formal power series* $p_0^{-1}\log_{\hat{\mathbb{G}}_m}(p_0 X) =: \log_{\tilde{\mathbb{G}}_m}(X)$ *and* $p_0^{-1}\exp_{\hat{\mathbb{G}}_m}(p_0 X) =: \exp_{\tilde{\mathbb{G}}_m}(X)$ *have integral coefficients in* $\mathbb{Q}_p(p_0) = \widetilde{\mathbb{Q}_p}$*, so the associated groupscheme* $\tilde{\mathbb{G}}_m/\operatorname{Spec}\mathcal{O}_{\tilde{L}}$ *is of additive type.*

Proof These power series are mutually inverse, so it suffices to verify the assertion for one of them. We have

$$p_0^{-1}\exp_{\hat{\mathbb{G}}_m}(p_0 X) = \sum_{n\geq 1} p_0^{n-1}\frac{X^n}{n!} \in \mathbb{Q}_p(p_0)[[X]] \ ;$$

but

$$\mathrm{ord}_p\left(\frac{p_0^{n-1}}{n!}\right) = \frac{n-1}{p-1} - \frac{n-\alpha_p(n)}{p-1} = \frac{\alpha_p(n)-1}{p-1}$$

by Legendre (where $\alpha_p(n) = \sum a_k$ when $n = \sum a_k p^k$). This is non-negative when $n \geq 1$. □

Example $2_0 = -2$ so $\frac{1}{2}(e^{2x}-1) \in \mathbb{Z}_{(2)}[[x]]$, and is congruent mod two to $\sum_{n\geq 0} x^{2^n}$.

2.1.4 Definition Let

$$\mathbf{c}_{\mathbb{C}_p} := (p_0\gamma_{\mathbb{C}_p})^{-1}\log_{\hat{\mathbb{G}}_m}(p_0\gamma_{\mathbb{C}_p}\tau_{\mathbb{C}_p}(\boldsymbol{\eta}_{\mathbb{C}_p})) \in \mathrm{THH}(\mathcal{O}_{\mathbb{C}_p})^2(B\boldsymbol{\mu}, \mathbb{Z}_p) \ .$$

By the lemma, this analog of the Chern class is a power series with p-adically integral coefficients, so

$$\tau_{\mathbb{C}_p}(\eta_{\mathbb{C}_p}) = 1 + \exp_{\hat{\mathbb{G}}_m}(p_0\gamma_{\mathbb{C}_p}\mathbf{c}_{\mathbb{C}_p}) := 1 + p_0\gamma_{\mathbb{C}_p}\boldsymbol{\kappa}_{\mathbb{C}_p} \ ,$$

with $\boldsymbol{\kappa}_{\mathbb{C}_p} \in \mathrm{THH}(\mathcal{O}_{\mathbb{C}_p})^2(B\boldsymbol{\mu}, \mathbb{Z}_p)$ a coordinate for a formal group law with comultiplication

$$\Delta\boldsymbol{\kappa}_{\mathbb{C}_p} = \boldsymbol{\kappa}_{\mathbb{C}_p}\otimes 1 + 1\otimes\boldsymbol{\kappa}_{\mathbb{C}_p} + p_0\gamma_{\mathbb{C}_p}\boldsymbol{\kappa}_{\mathbb{C}_p}\otimes\boldsymbol{\kappa}_{\mathbb{C}_p} \ .$$

A completely analogous construction defines lifts $\mathbf{c}_{\mathbb{Q}_p^\infty}$ and $\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty}$ of $\mathbf{c}_{\mathbb{C}_p}$ and $\boldsymbol{\kappa}_{\mathbb{C}_p}$ such that

$$\tau_{\mathbb{Q}_p^\infty}(\tilde{\eta}_{\mathbb{Q}_p^\infty}) = 1 + p_0\gamma_{\mathbb{Q}_p^\infty}\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty} \in \mathrm{THH}(\mathcal{O}_{\mathbb{Q}_p^\infty})^2(B\boldsymbol{\mu}, \mathbb{Z}_p) \ ,$$

yielding a formal group law

$$\Delta\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty} = \sum a_{i,j}\boldsymbol{\kappa}^i_{\mathbb{Q}_p^\infty}\otimes\boldsymbol{\kappa}^j_{\mathbb{Q}_p^\infty}$$

on $\mathrm{THH}(\mathcal{O}_{\mathbb{Q}_p^\infty})^2(B\boldsymbol{\mu}, \mathbb{Z}_p)$. The inclusion $\mathbb{Q}_p^\infty \to \mathbb{C}_p$ defines an an injective map on THH, so

$$\Delta\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty} = \boldsymbol{\kappa}_{\mathbb{Q}_p^\infty}\otimes 1 + 1\otimes\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty} + p_0\gamma_{\mathbb{Q}_p^\infty}\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty}\otimes\boldsymbol{\kappa}_{\mathbb{Q}_p^\infty} \ ,$$

and thus similarly for any perfectoid field P sandwiched between them.

Corollary *For such fields, the parameter* κ_P *(such that* $\tau_P(\eta_P - 1) = p_0\kappa_P$*) defines a formal group law*

$$\mathrm{Spf}\ \mathrm{THH}(\mathcal{O}_P)^0(B\boldsymbol{\mu}, \mathbb{Z}_p) \cong \tilde{\mathbb{G}}_m \times_{\mathcal{O}_{\widetilde{\mathbb{Q}_p}}} \mathcal{O}_P$$

of additive type.

2.2 It follows from Hesselholt's results that $\sigma \in \mathrm{Gal}(P/\mathbb{Q}_p)$ acts, as in §1.3.2,1, on $\gamma_P \boldsymbol{\kappa}_P := \kappa_P$ as multiplication by the cyclotomic character $\chi(\sigma)$ via the map to $\mathrm{Gal}(\mathbb{Q}_p^\infty/\mathbb{Q}_p)$.

Inverting the Bott-Thomason class $\beta_P \in k_2^{\mathrm{alg}}(P, \mathbb{Z}_p)$ defines a 2-periodic p-adic algebraic K-theory functor

$$K_{\mathrm{alg}}(P)^*(-, \mathbb{Z}_p) = \beta_P^{-1}[X, k_{\mathrm{alg}}(P, \mathbb{Z}_p)]_{-*}$$

which evidently maps to $(p_0\gamma_P)^{-1}\,\mathrm{THH}(\mathcal{O}_P)^*(-, \mathbb{Z}_p)$; but since $\mathrm{ord}_p(p_0) > 0$, this localization defines a version

$$K_{\mathrm{alg}}(P)^*(-, \mathbb{Z}_p) \to H^*(-, \gamma_P^{-1}\,\mathrm{THH}^*(\mathcal{O}_P, \mathbb{Q}_p))$$

of the classical Chern-Dold character. There is a large literature (*e.g.* Segal [28], Snaith [31], Boyer, Lawson, Lima-Filho, Mann and Michelson [8], Totaro [34, 35] ...) on related integrality questions.

§III Generalized cyclotomic fields

Any $L \supset \mathbb{Q}_p$ as in §1.3.1 admits a maximal totally ramified Abelian extension $L^\infty \supset L$, with an Artin reciprocity homomorphism [29]

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/L) \longrightarrow \widehat{\mathbb{Z}} \times \mathrm{Gal}(L^\infty/L) \longrightarrow \mathrm{Gal}(L^\infty/L) \xrightarrow{\varkappa_L} {\mathcal{O}_L}^\times \ ,$$

where $\widehat{\mathbb{Z}} \cong \mathrm{Gal}(L^{nr}/L) \cong \mathrm{Gal}(\overline{k}_L/k_L)$, which classifies unramified extensions, has been supressed; they will play no part in this paper. The completion $L^{\hat{\infty}}$ of L^∞ in $\mathbb{C}_p$ is perfectoid; see [22, §1.4.17] or [37, Ex. 2.0.4, Ex. 2.1.1] for its tilt, along with much more information[3].

We argue below that the graded formal groups $\mathrm{THH}(\mathcal{O}_{L^\infty})^*(B\boldsymbol{\mu}, \mathbb{Z}_p)$ defined above have a natural interpretation in terms of the p-adic Fourier theory of Schneider and Teitelbaum, as a rigid analytic version [23, §5 intro.]

$$\begin{array}{ccc} \mathrm{LT}_L^{\mathrm{rigid}} & \xrightarrow{\varepsilon_L} & \mathrm{LT}_L \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ \mathbb{C}_p & \longrightarrow & \mathrm{Spec}\ \mathcal{O}_L \end{array}$$

of the Lubin-Tate formal groups LT_L used to construct these extensions, mapped by an analog of the exponential map of classical Lie theory (defined only in a neighborhood of the origin).

3.1.1 The construction of LT_L is elegant and in some sense quite elementary, but it depends (up to a canonical isomorphism) on some choices, *i.e.* of a uniformizing element π_L as in §1.3.1, as well as an element $[\pi_L](T) \in \mathcal{O}_L[[T]]$ equal to $\pi_L T$ modulo terms of higher order, and congruent modulo $\mathfrak{m}_L$ to T^q. In the following we will assume that LT_L is special in the sense of Lang, *i.e.* that

$$[\pi_L](T) = \pi_L T + T^q \ ;$$

[3] To a topologist it is tempting to call these 'chromatic' fields. There are interesting analogies with the Alexander cover of a link complement ...

this implies that $T \mapsto \omega T$, for $\omega \in W(k_L)^\times \subset \mathcal{O}_L{}^\times$, is an automorphism of LT_L, which simplifies issues of grading in homotopy theory. The resulting formal group

$$X, Y \mapsto F_L(X,Y) := X +_L Y \in \mathcal{O}_L[[X,Y]]$$

is in fact a formal $\mathcal{O}_L$-module, endowed with an isomorphism

$$a \mapsto [a]_L : \mathcal{O}_L \to \mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mathrm{LT}_L) \ .$$

The group

$$\mathrm{Hom}_{\mathcal{O}_L,c}(\mathcal{O}_L[[T]], \mathcal{O}_{\mathbb{C}_p}) =: \mathrm{LT}_L(\mathfrak{m}_{\mathbb{C}_p}) = (\mathfrak{m}_{\mathbb{C}_p}, +_L)$$

of points of LT_L (defined by continuous homomorphisms) is isomorphic (modulo $\mathbb{Q}$-vector spaces) to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$, and its Tate module

$$T_L := \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{LT}_L(\mathfrak{m}_{\mathbb{C}_p}))$$

is free of rank one over $\mathrm{End}(\mathrm{LT}_L) \cong \mathcal{O}_L$. Adjoining the torsion points of $\mathrm{LT}_L(\mathfrak{m}_{\mathbb{C}_p})$ to L defines the extension $L^\infty \supset L$; the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ acts on these torsion points, defining a reciprocity map

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/L) \to \mathrm{Gal}(L^\infty/L) \cong \mathrm{Aut}_{\mathcal{O}_L}(T_L) \cong \mathcal{O}_L{}^\times \ .$$

If $L_0 \subset L_1$ is Galois, then the diagram [33]

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}_p/L_1) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}_p/L_0) & \longrightarrow & \mathrm{Gal}(L_1/L_0) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \varkappa_{L_1}} & & \downarrow{\scriptstyle \varkappa_{L_0}} & & \downarrow & & \\
 & & (L_1^\times)\hat{} & \xrightarrow{N_0^1} & (L_0^\times)\hat{} & \longrightarrow & \mathrm{Gal}(L_1/L_0)^{\mathrm{ab}} & &
\end{array}$$

(with $N_0^1 = N_{L_0}^{L_1}$ the norm, and profinite completion denoted by a caret) commutes. In particular $\varkappa_{\mathbb{Q}_p}(\sigma) = N_{\mathbb{Q}_p}^L(\varkappa_L(\sigma))$ if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$.

Reducing modulo $\mathfrak{m}_L$ defines a formal group law on $k_L[[T]]$ of height n, and thus an embedding of $\mathcal{O}_L^\times$ in the automorphism group $\mathcal{O}_D^\times$ of the reduction as a kind of maximal torus.

3.1.2 A formal group law (*e.g.* LT_L) over a torsion-free ring (*e.g.* $\mathcal{O}_L$) has a unique logarithm and exponential, *e.g.*

$$\log_L(T),\ \exp_L(T) \in L[[T]]$$

such that

$$X +_L Y = \exp_L(\log_L(X) + \log_L(Y)) \in \mathcal{O}_L[[X,Y]] \ .$$

The formal multiplicative group of §2.1.2 is one classical example, and Honda's logarithm [13]

$$\log_\pi(T) := \sum_{n \geq 0} \pi^{-n} T^{q^n} \in L[[T]]$$

is another. Lemma 2.1.3 generalizes to special Lubin-Tate groups as follows:

Definition Let $\pi_0 \in \mathrm{LT}_L(\mathfrak{m}_{\mathbb{C}_p})$ be a primitive $[\pi]$-torsion point, *i.e.* a generator of the cyclic group of points satisfying

$$\pi_0^{q-1} + \pi = 0 \ ,$$

and let $\widetilde{L} := L(\pi_0)$; then $\mathrm{Gal}(\widetilde{L}/L) \cong k_L^\times \subset W(k_L)^\times$ (using Teichmüller representatives).

Proposition *The formal group law*

$$X +_{\tilde{L}} Y := \pi_0^{-1}(\pi_0 X +_L \pi_0 Y) \in \mathcal{O}_{\tilde{L}}[[X,Y]]$$

is of additive type, ie with $[p]_{\tilde{L}}(T) \equiv 0$ *modulo* π_0.

Proof Evidently $[\pi]_{\tilde{L}}(T) \cong \pi(T - T^q) \cong 0$ modulo π_0. But $[\pi]_L$ satisfies an Eisenstein equation

$$E_L([\pi]_L) = [\pi]_L^e \ +_L \ \cdots \ +_L \ [u]_L \circ [p]_L \ = \ 0$$

(with coefficients from $W(k_L)$, and u a unit) in $\mathrm{End}(\mathrm{LT}_L)$, so, similarly,

$$[\pi]_{\tilde{L}}^e \ +_{\tilde{L}} \ \cdots \ +_{\tilde{L}} \ [u]_{\tilde{L}} \circ [p]_{\tilde{L}} \ = \ 0$$

in the endomorphisms of $F_{\tilde{L}}$. But $[\pi]_{\tilde{L}} \equiv 0 \bmod \pi_0$, so $[p]_{\tilde{L}} \equiv 0 \bmod \pi_0$, as well. $\square$

Example Honda's logarithm is p-typical, so its renormalization

$$\pi_0^{-1} \log_\pi(\pi_0 T) = \sum_{n \geq 0} \pm \pi_0^{q^n - 1} \pi^{-n} T^{q^n}$$

has coefficients in $\mathcal{O}_L$, with

$$\mathrm{ord}_p(\pi_0^{q^n-1}\pi^{-n}) = e^{-1}(\frac{q^n - 1}{q - 1} - n) = e^{-1}(1 + \cdots + q^{n-1} - n) \geq 0 \ ,$$

which goes to ∞ as n does, making it a rigid analytic function.

Note that the completed Hopf $\mathcal{O}_{\tilde{L}}$-algebra defined by $F_{\tilde{L}}$ is the pushforward, under the homomorphism defined by

$$T \mapsto \pi_0 T : \mathcal{O}_L[[T]] \to \mathcal{O}_{\tilde{L}}[[T]] \ ,$$

of that defined by F_L; but (because this map does not preserve the coordinate) it is not a morphism of formal group laws.

3.2.1 A Lubin-Tate group has an associated p-divisible group with Cartier dual $\mathrm{Hom}(\mathrm{LT}_L, \hat{\mathbb{G}}_m)$; early work of Katz [14, §3] identifies its Tate module

$$T_L^\vee := \{\beta(T) \in (1 + T\mathcal{O}_{\mathbb{C}_p}[[T]])^\times \mid \beta(X +_L Y) = \beta(X) \cdot \beta(Y)\}$$

as free of rank one over $\mathcal{O}_L$. A homotopy theorist will recognize this as a specialization to Lubin-Tate groups of Ravenel and Wilson's almost simultaneous description [24] of the Hopf algebra $\mathrm{MU}_*(\mathbb{C}P^\infty)$ representing the Cartier dual of the universal p-divisible group: the canonical inclusions $\mathbb{C}P^i \subset \mathbb{C}P^\infty$ define bordism classes b_i with

$$b(T) = 1 + \sum_{i > 0} b_i T^i$$

satisfying the relation $b(X +_{\mathrm{MU}} Y) = b(X) \cdot b(Y)$ under the Pontryagin product.

Classical Fourier analysis identifies the dual $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of a finite-dimensional vector space V with its character group $V^{\mathcal{X}} := \mathrm{Hom}_c(V, \mathbb{T})$ by

$$V^* \ni \boldsymbol{\xi} \mapsto [\mathbf{x} \mapsto \exp(i\boldsymbol{\xi}(\mathbf{x}))] \in V^\chi \ .$$

In 2001 Schneider and Teitelbaum [26, §2] defined a p-adic analog of the Pontryagin dual of a free $\mathcal{O}_L$-module M as the rigid analytic group $M^{\mathcal{X}}$ of locally analytic characters

$$\theta : M \to (1 + \mathfrak{m}_{\mathbb{C}_p})^\times \ ,$$

and showed that the map

$$\mathrm{LT}_L(\mathfrak{m}_{\mathbb{C}_p}) \ni \alpha \mapsto [\beta \mapsto [a \mapsto \beta([a]_L(\alpha))]] \in \mathrm{Hom}_{\mathcal{O}_L}(T_L^\vee, \mathcal{O}_L^{\mathcal{X}}) \cong (T_L^\vee)^{\mathcal{X}}$$

is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$-modules. The inverse of this equivalence defines the morphism

$$\varepsilon_L : \mathrm{LT}_L^{\mathrm{rigid}} := (T_L^\vee)^{\mathcal{X}} \to \mathrm{LT}_L$$

of group-valued functors cited above, represented by a homomorphism of (completed) Hopf $\mathcal{O}_L$-algebras from $\mathcal{O}_L[[T]]$ to locally analytic homomorphisms [30, §6.1] from $T_L^\vee$ to $(1 + \mathfrak{m}_{\mathbb{C}_p})^\times$, sending T to

$$\varepsilon_L(T) = \exp_{\hat{\mathbb{G}}_m}(\Omega(L) \log_L(T)) \in L_{\mathrm{an}}(\mathcal{O}_L, \mathbb{C}_p) \ :$$

where $\Omega(L) \in \hat{\mathfrak{o}}_{L^\infty}$, with $\mathrm{ord}_p(\Omega(L)) = (p-1)^{-1} - e^{-1}(q-1)^{-1}$, is a 'period' of the formal group LT_L, with the remarkable property that

$$\Omega(L)^{\sigma - 1} = \{\frac{\varkappa_{\mathbb{Q}_p}}{\varkappa_L}\}(\sigma) \in \mathcal{O}_L^\times$$

for $\sigma \in \mathrm{Gal}(L^\infty/L)$ [3,26, lemma 3.4, 27, §6.2.3 Prop 6.4]. This can be reformulated as the assertion that

$$H^1_c(\mathrm{Gal}(L^\infty/L), {\mathcal{O}_L}^\times) \ni \{\frac{\varkappa_{\mathbb{Q}_p}}{\varkappa_L}\} \mapsto 0 \in H^1_c(\mathrm{Gal}(L^\infty/L), (\hat{L^\infty})^\times) \ ,$$

or as the

3.2.2 Proposition *The diagram (in which the vertical arrows multiply T by the indicated element)*

$$\begin{array}{ccc} L[[T]] & \xrightarrow{\varepsilon_L} & \hat{L^\infty}[[T]] \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle p_0} \\ \widetilde{L}[[T]] & \xrightarrow{\varepsilon_L^0} & \hat{L^\infty}[[T]] \end{array}$$

commutes, where

$$\varepsilon_L^0(T) := \exp_{\tilde{\mathbb{G}}_m}(\Omega_0(L) \log_{\widetilde{L}}(T)) = \Omega_0(L)T + \cdots \in \mathcal{O}_{\hat{L^\infty}}[[T]] \ ,$$

and $\Omega_0(L) := p_0 \pi_0^{-1} \Omega(L)$ is a unit. Moreover, the map $T \mapsto \varepsilon_L^0(T)$ is equivariant, in the sense that

$$[\varkappa_{\mathbb{Q}_p}(\sigma)]^{-1}_{\tilde{\mathbb{G}}_m} \circ \varepsilon_L^0 \circ [\varkappa_L(\sigma)]_{\widetilde{L}} = \sigma(\varepsilon_L^0) \ ,$$

with respect to the action of $\sigma \in \mathrm{Gal}(L^\infty/L)$ on the coefficients of $\varepsilon_L^0(T)$.

Proof We have $\log_{\widetilde{L}}([\varkappa]_L(\sigma)]_{\widetilde{L}}(T)) = \varkappa_L(\sigma) \log_{\widetilde{L}}(T)$ and $\Omega(L)\varkappa_L(\sigma) = \varkappa_{\mathbb{Q}_p}(\sigma)\sigma(\Omega(L))$; while $\exp_{\tilde{\mathbb{G}}_m}(\varkappa_{\mathbb{Q}_p}(\sigma) \cdots) = [\varkappa_{\mathbb{Q}_p}(\sigma)]_{\tilde{\mathbb{G}}_m}(\cdots)$. □

3.3.1 In 1982 Fontaine [9, Thm. 1] defined, for a Lubin-Tate group of L, a homomorphism

$$\xi_L : \overline{\mathbb{Q}}_p \otimes_{\mathcal{O}_L} T_L \to \Omega^1_{\mathcal{O}_L}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$$

of modules over the twisted group ring[4] $\mathcal{O}_{\overline{\mathbb{Q}}_p}\langle \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)\rangle$: if $h = \{h_n\} \in T_L$ with $[\pi^n]_L(h_n) = 0$, and $\overline{\mathbb{Q}}_p \ni \alpha = \alpha_n \pi^n$ with $\alpha_n \in \mathcal{O}_{\overline{\mathbb{Q}}_p}$ integral, then

$$\xi_L(\alpha \otimes h) := \alpha_n \cdot h_n^*(d \log_L(T)) \in \Omega^1_{\mathcal{O}_L}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$$

is well-defined; where $h_n \in \mathrm{LT}_L(\mathfrak{m}_{\overline{\mathbb{Q}}_p})$, and $d \log_L(T) = \log'_L(T) \cdot dT$. He then proves the exactness of the sequence

$$0 \longrightarrow \pi_0^{-1}\mathcal{O}_{\overline{\mathbb{Q}}_p} \otimes_{\mathcal{O}_L} T_L \longrightarrow \overline{\mathbb{Q}}_p \otimes_{\mathcal{O}_L} T_L \xrightarrow{\xi_L} \Omega^1_{\mathcal{O}_L}(\mathcal{O}_{\overline{\mathbb{Q}}_p}) \longrightarrow 0 \ ,$$

and taking Tate modules defines an isomorphism

$$T\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}) \cong p_0^{-1}\mathcal{O}_{\mathbb{C}_p} \otimes_{\mathcal{O}_{\mathbb{Q}_p}} T_{\mathbb{Q}_p}$$

of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-modules. This argument generalizes, without significant change, to imply the existence of an exact sequence

$$0 \longrightarrow \pi_0^{-1}\mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_L} T_L \longrightarrow L^\infty \otimes_{\mathcal{O}_L} T_L \xrightarrow{\boldsymbol{\xi}_L} \Omega^1_{\mathcal{O}_L}(\mathcal{O}_{L^\infty}) \longrightarrow 0$$

of $\mathrm{Gal}(L^\infty/L)$-modules, and hence an isomorphism

$$\Omega^1_{\mathcal{O}_L}(\mathcal{O}_{L^\infty}) \cong (L^\infty/\pi_0^{-1}\mathcal{O}_{L^\infty}) \otimes_{\mathcal{O}_L} T_L \ .$$

3.3.2 Now a sequence $\mathbb{Q}_p \subset L' \subset L \subset L'' \subset \mathbb{C}_p$ of extensions implies [9, §2.4 lemma 2] a monomorphism

$$0 \to \Omega^1_{\mathcal{O}_{L'}}(\mathcal{O}_L) \to \Omega^1_{\mathcal{O}_{L'}}(\mathcal{O}_{L''}) \ ,$$

and since $\mathcal{O}_{L^\infty}$ is flat over $\mathcal{O}_L$, the Jacobi-Zariski exact sequence

$$0 \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{L^\infty} \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{L^\infty}) \to \Omega^1_{\mathcal{O}_L}(\mathcal{O}_{L^\infty}) \to 0$$

implies that

$$T\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{L^\infty}) \cong (\pi_0\mathcal{D}_L)^{-1}\mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_L} T_L$$

as $\mathrm{Gal}(L^\infty/\mathbb{Q}_p)$-modules; where $\Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_L) \cong \mathcal{O}_L/\mathcal{D}_L\mathcal{O}_L \cong \mathcal{D}_L^{-1}/\mathcal{O}_L$ (the Dedekind different being the inverse of the fractional ideal $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_L, \mathbb{Z}_p)$) defined by the trace from $\mathcal{O}_L$ to $\mathcal{O}_{\mathbb{Q}_p}$.

Similarly, since $\overline{\mathbb{Q}}_p$ is flat over $\mathbb{Q}_p^\infty$, the monomorphism

$$0 \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\mathbb{Q}_p^\infty}) \otimes_{\mathcal{O}_{\mathbb{Q}_p^\infty}} \mathcal{O}_{\overline{\mathbb{Q}}_p} \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$$

implies a commutative diagram

$$\begin{array}{ccccccc} \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\mathbb{Q}_p^\infty}) \otimes_{\mathcal{O}_{\mathbb{Q}_p^\infty}} \mathcal{O}_{\overline{\mathbb{Q}}_p} & \longrightarrow & \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}) & \longrightarrow & \Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & & & \\ \overline{\mathbb{Q}}_p/p_0^{-1}\mathcal{O}_{\overline{\mathbb{Q}}_p} & \xrightarrow{\cong} & \overline{\mathbb{Q}}_p/p_0^{-1}\mathcal{O}_{\overline{\mathbb{Q}}_p} \ ; & & & & \end{array}$$

[4]with finite sums $\sum a_\sigma \cdot \sigma$, $a_\sigma \in \mathcal{O}_{L^\infty}$, $\sigma \in \mathrm{Gal}(L^\infty/L)$ as elements, and multiplication $(a \cdot \sigma)(b \cdot \tau) = a\sigma(b) \cdot \sigma\tau$

but then $\Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}) = 0$, and hence (since

$$0 \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{L^\infty}) \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{\overline{\mathbb{Q}}_p}) = 0$$

is injective), that $\Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{L^\infty}) = 0$.

Corollary *The exact sequence*

$$0 \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{\mathbb{Q}_p^\infty}) \otimes_{\mathcal{O}_{\mathbb{Q}_p^\infty}} \mathcal{O}_{L^\infty} \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p}}(\mathcal{O}_{L^\infty}) \to \Omega^1_{\mathcal{O}_{\mathbb{Q}_p^\infty}}(\mathcal{O}_{L^\infty}) = 0$$

implies an isomorphism

$$\phi : p_0^{-1}\mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_{\mathbb{Q}_p}} T_{\mathbb{Q}_p} \cong (\pi_0 \mathcal{D}_L)^{-1}\mathcal{O}_{L^\infty} \otimes_{\mathcal{O}_L} T_L$$

of Tate modules over $\mathcal{O}_{L^\infty}\langle\,\mathrm{Gal}(L^\infty/\mathbb{Q}_p)\rangle$.

Proposition *Let* $t_{\mathbb{Q}_p}, t_L$ *generate the Tate modules* $T_{\mathbb{Q}_p}, T_L$, *and let* $\sigma \in \mathrm{Gal}(L^\infty/L)$; *then*

$$t_L^{-1}\phi(t_{\mathbb{Q}_p}) := \Omega_\partial(L) \in \mathcal{O}_{L^\infty}$$

also satisfies

$$\Omega_\partial(L)^{\sigma-1} = \{\frac{\varkappa_{\mathbb{Q}_p}}{\varkappa_L}\}(\sigma) \ ,$$

and hence $\mathcal{D}_L \Omega_\partial(L) = u \cdot \Omega(L)$ *for some unit* $u = u(L : \mathbb{Q}_p) \in {\mathcal{O}_L}^\times$.

With $t_{\mathbb{Q}_p}$ as in §1.3, we can choose t_L so $\Omega_L^\partial \equiv 1 \bmod \mathfrak{m}_{L^\infty}$ (and in particular so $(\Omega_L^\partial)^k \to 1$ as $k \to \infty$). The Galois action on THH of the Teichmüller units in $W(k_L)$ is then consistent with the topological grading.

§IV Applications and speculations

Writing ε_L^∂ for the variant of ε_L^0 defined by replacing $\Omega_0(L)$ by $\Omega_0^\partial(L)$ defines a composition

$$\begin{array}{ccccccc}
\mathrm{Spf}\ \mathrm{THH}(\mathcal{O}_{L^\infty})^0(B\boldsymbol{\mu}, \mathbb{Z}_p) & \xrightarrow{\mathcal{D}_L} & \tilde{\mathbb{G}}_m \times_{\mathcal{O}_{\tilde{L}}} \mathcal{O}_{L^\infty} & \xrightarrow{\varepsilon_L^\partial} & F_{\tilde{L}} & \xrightarrow{\pi_0} & \mathrm{LT}_L \\
\downarrow & & & & & & \downarrow \\
\mathrm{Spec}\ \mathcal{O}_{L^\infty} & & \longrightarrow & & & & \mathrm{Spec}\ \mathcal{O}_L
\end{array}$$

of affine group schemes, with the top right morphism as in §3.1.2 and the top left morphism defined (following §2.1.4) by $T \mapsto \mathcal{D}_L T$. By §3.2.2, the morphism across the top takes the natural action of $\mathrm{Gal}(L^\infty/L)$ on the left to the action of $\mathcal{O}_L^\times$ by formal group automorphisms on the right, compatible with Artin reciprocity, and it seems natural to conjecture that when $L/\mathbb{Q}_p$ is Galois, this extends to equivariance with respect to an action of $\mathrm{Gal}(L^\infty/\mathbb{Q}_p)$; see Appendix III of [36].

The morphism χ_L from the Lazard ring to $\mathcal{O}_L$ which classifies LT_L lifts to a graded Hirzebruch genus

$$\mathrm{MU}^* \ni [M] \mapsto \chi_L[M] \cdot u^{\dim_{\mathbb{C}} M} \in \mathcal{O}_L[u] \ ,$$

defining a commutative diagram

$$\begin{array}{ccc} \mathrm{MU}^* & \xrightarrow{\chi_L} & \mathcal{O}_L[u] := k^*(L) \\ & \searrow^{\kappa_{L^\infty}} & \downarrow{\scriptstyle u\mapsto \pi_0\mathcal{D}_L\gamma_{L^\infty}} \\ & & \mathrm{THH}^*(\mathcal{O}_{L^\infty},\mathbb{Z}_p) \cong \mathcal{O}_{L^\infty}[\gamma_{L^\infty}] \end{array}$$

of ring homomorphisms. The appendix below summarizes a construction for weakly commutative complex-oriented cohomology theories with

$$\mathrm{Spf}\ \ k(L)^0(\mathbb{C}P^\infty) \cong \mathrm{LT}_L\ .$$

Hirzebruch's work from the 60s, interpreting multiplicative natural transformations

$$[\mathrm{MU}^* \to E^*] \in E^0(\mathrm{MU}) \cong E^0(B\,\mathrm{U})$$

of cohomology theories in terms of the Thom isomorphism and symmetric functions, sending an orientation $\boldsymbol{\kappa} \in E^0(B\mathbb{T})$ to the (grouplike, *i.e.* $\Delta\kappa = \kappa \otimes \kappa$) limit κ (as $k \to \infty$ of $\varphi_E^{-1}(\otimes^k \boldsymbol{\kappa}) \in E^0(M\,\mathrm{U}(k))$), defines a lift of this diagram to a diagram of multiplicative natural transformations between $\mathbb{Z}_2$-graded cohomology theories.

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Appendix: Integral lifts of $K(n)$ parametrized by local fields

For any $n \in \mathbb{N}$ and prime p (or, alternately, for every field with $q = p^n$ elements), there is a functor

$$K(n) : (\mathrm{Spaces}) \to (\mathbb{F}_q - \mathrm{modules})$$

(roughly, the residue field at a prime of the sphere spectrum) with

$$K(n)^*(B\mathbb{T}) = \mathbb{F}_q[u^{\pm 1}][[T]]$$

a graded complete Hopf algebra or formal group with addition $+_{K(n)}$, with $|u| = +2,\ |T| = -2$, such that the p-fold multiplication map is represented by

$$[p]_{K(n)}(T) = u^{q-1}T^q \ .$$

Araki's generators for $\mathrm{BP}_*(\mathrm{pt}) = \mathbb{Z}_{(p)}[\ldots, v_i, \ldots]$ satisfy

$$[p]_{\mathrm{BP}}(T) \ = \ \sum_{\mathrm{BP}} v_i T^{p^i} = pT +_{\mathrm{BP}} v_1 T^p +_{\mathrm{BP}} \ldots$$

($i \geq 0,\ v_0 = p$), and there is a Baas-Sullivan quotient ([18, §3.4]: the totalization of a suitable Koszul complex, or an iterated homotopy cofiber) with

$$\mathrm{MU}_*(\mathrm{pt}) \to \mathrm{BP}_*(\mathrm{pt}) \to \mathbb{Z}_p[u] \ ,$$

such that $v_i \mapsto 0,\ i \neq 0, n$ and $v_n \mapsto u^{q-1}$, classifying a graded group law $+_{k(n)}$ with

$$[p]_{k(n)}(T) = pT \ +_{k(n)} \ u^{q-1}T^{p^n} \ ,$$

i.e. $+_{k(n)} \equiv +_{K(n)} \bmod p$. Hazewinkel's functional equation [22] implies that

$$T_0 +_{k(n)} T_1 = u^{-1} \exp_{k(n)}(\log_{k(n)}(uT_0) + \log_{k(n)}(uT_1)) \in \mathbb{Z}_p[u][[T_0, T_1]]$$

with

$$\log_{k(n)}(uT) := uT + \sum_{k>0} \prod_{1 \leq i \leq k} (1 - p^{q^i - 1})^{-1} \cdot p^{-k} (uT)^{q^k} \in \mathbb{Q}_p[[uT]] \ .$$

Tensoring this Baas-Sullivan theory with $W(\mathbb{F}_q)$ defines a cohomology theory $k(L_0)$ (where $L_0 := W(\mathbb{F}_q) \otimes \mathbb{Q}$ is the unique **un**ramified extension of degree n of $\mathbb{Q}_p$), such that

$$k(L_0)^*(\mathrm{pt}) = W(\mathbb{F}_q)[u] \to \mathbb{F}_q[u] \to \mathbb{F}_q[u^{\pm 1}] = K(n)^*(\mathrm{pt})$$

defines a nice integral lift to a connective version of $K(n)$. More generally, for any $L \subset \overline{\mathbb{Q}}_p$ with $n = [L : \mathbb{Q}_p]$, there is a connective spectrum $k(L)$ with $k(L)^*(\mathrm{pt}) = \mathcal{O}_L[u]$, and such that $k(L)^*(B\mathbb{T})$ is a Lubin-Tate formal group for L; *e.g.* if $n = 1$, $k(\mathbb{Q}_p)$ is the p-adic completion of connective classical topological K-theory (associated to the multiplicative formal group).

Lubin-Tate groups of local number fields parametrize good integral lifts of $K(n)$. In particular, the group $\mathcal{O}_L^\times$ of units acts as stable multiplicative automorphisms of $k(L)^*(\mathbb{C}P^\infty)$, *e.g.* with $\alpha \in \mathcal{O}_L^\times$ sending u to $\alpha \cdot u$ (generalizing the action of ${\mathbb{Z}_p}^\times$ by p-adic Adams operations on classical (p-completed) topological K-theory). It seems natural to think of these lifts as indexed by maximal toruses in the unit group $D_n^\times$ of a division algebra with center $\mathbb{Q}_p$ and Brauer-Hasse invariant $1/n$. Recent work [18] of Hopkins and Lurie, using the modern theory of Thom spectra, has changed the geography of this subject: in particular, it raises the question of possible Azumaya multiplications on such lifts, which might support lifts of the $\mathrm{Gal}(L^\infty/L)$-action discussed above, to an action compatible with some such multiplicative structure.

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