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Mahowald square and Adams differentials

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Abstract. This article is a survey on the RP^∞-method that was used to compute differentials in the Adams spectral sequence of the sphere spectrum. This method was introduced in [Ann. of Math. (2) **186** (2017), no. 2, 501–580] by Guozhen Wang and the author and was also used in [Algebr. Geom. Topol. **18** (2018), no. 7, 3887–3906] to solve extension problems in the Adams spectral sequence. The method is based on the algebraic Kahn-Priddy theorem and the Mahowald square. In this article, we discuss the idea of the RP^∞-method and apply this method to prove certain Adams differentials in low stems. We also discuss a way of constructing Toda brackets for elements in the stable homotopy groups of sphere using the Kahn-Priddy transfer map.

Contents

1. Introduction

At the prime 2, the Adams spectral sequence is still the most effective tool for computing stable stems. The E_2-page of the Adams spectral sequence is comparably tractable—one may use the May spectral sequence, the Curtis algorithm for Lambda algebras, or computer programs such as the ones made by Bob Bruner, Christian Nassau and Amelia Perry to compute it in a large range. The more difficult part is to compute Adams differentials and to solve extension problems.

Many methods have been developed to compute Adams differentials. Historically, up to the 19-stem, the differentials are obtained by comparing with Toda's unstable computations—one deduces the differentials by knowing what to expect in the E_∞-page. One could also deduce these differentials using the solution of the Hopf invariant one problem [**1**] and computations of the image-of-J spectrum [**7**, **16**]. Up to the 28-stem, the differentials are obtained by May [**21**] using the multiplicative structure based on earlier differentials. The range from 30 to 45 is due to Barratt-Mahowald-Tangora [**3**]. They have two major techniques. The first one is Moss's theorem [**22**], which connects Massey products on any Adams E_r-page to Toda brackets in homotopy. The second one is a trick due to Mahowald, which translates differentials and extensions through certain finite CW complexes. One can generalize Mahowald's trick to the Mahowald square in the setting of the 4 spectral sequences situation—a combination of an Atiyah-Hirzebruch spectral sequence (AHSS), an algebraic Atiyah-Hirzebruch spectral sequence (AAHSS) and a (cellwise) Adams spectral sequence (ASS). This general method has been used by many experts for various spectra. See Mahowald's work [**17**] on the metastable computation, for example. Another method [**4**] involved in this range is Bruner's use of Steenrod operations in the ASS.

The Adams differentials in the range up to 59 have been proved rigorously by Isaksen [**10–12**] using motivic analogues of the classical methods. Recently, Gheorghe, Isaksen, Wang and the author [**9, 13**] have developed another method through the motivic world that computes Adams differentials using algebraic Novikov differentials. The computation has been pushed up to the 90 stem, but is orthogonal to the theme of this article.

Recently, based on the algebraic Kahn-Priddy theorem [**15**] and Mahowald's trick, Guozhen Wang and the author [**26**] introduced the RP^∞-method. The method has been applied to prove a notoriously hard Adams differential $d_3(D_3) = B_3$ in stem 61 (see [**26**]), which is one major reason why Isaksen's computations [**10**] stopped at stem 59. The method has also been applied to solve certain extension problems (see [**28**]), which include the last 2-extension problem that was left by Isaksen up to stem 59.

The starting point of the RP^∞-method is the following algebraic version of the Kahn-Priddy theorem which is due to W. H. Lin [**15**].

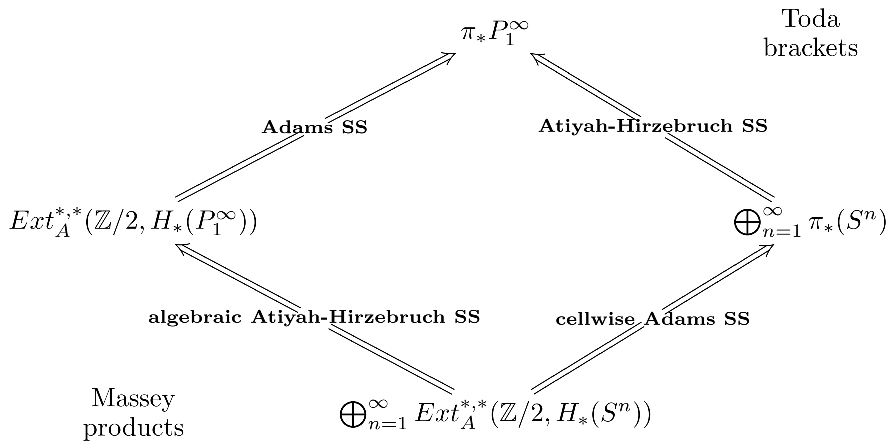
THEOREM 1.1. *Let $t : P_1^\infty \to S^0$ be the transfer map, where P_1^∞ is the suspension spectrum of RP^∞ and S^0 is the sphere spectrum. Then the induced map on the Adams E_2-page*

$$t_\sharp : Ext_A^{s,t}(\mathbb{Z}/2, H_*(P_1^\infty)) \longrightarrow Ext_A^{s+1,t+1}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is an epimorphism for bidegrees such that $t - s > 0$. Here A is the dual Steenrod algebra.

The observation we made is that according to the (algebraic) Kahn-Priddy theorem [**14, 15**], if there is a certain nontrivial d_r differential in the Adams spectral sequence for the sphere spectrum, then the pre-image of the source must support a nontrivial $d_{r'}$ differential for $r' \leq r$ and the pre-image of the target cannot be killed by a nontrivial $d_{r''}$ differential for $r'' < r$. Therefore, to compute nontrivial Adams differentials for the sphere spectrum, we can work with certain Adams differentials for P_1^∞ and use the Mahowald square to compute the differentials.

Here both the AHSS and the AAHSS come from the skeletal filtration on P_1^∞, and they both start from the E_1-page. The d_1-differentials in both spectral sequences correspond to a multiplication map by 2 or 0, depending on the dimension of the cells. In theory, the longer differentials in the AAHSS can be computed by (matrix) Massey products and the longer differentials in the AHSS can be computed by (matrix) Toda brackets. In fact, in [**27**], Wang and the author computed the Adams E_2-page of P_1^∞ in the range of $t < 72$ using the Lambda algebra. This Lambda algebra computation gives us a lot of information on the algebraic Atiyah-Hirzebruch spectral sequence. In particular, there is a one-to-one correspondence between the differentials in the Lambda algebra computation and the differentials in the algebraic Atiyah-Hirzebruch spectral sequence. From the Lambda algebra computation, one can also read off information about $Ext_A^{*,*}(\mathbb{Z}/2, H_*(P_n^{n+k}))$, where P_n^{n+k} is the suspension spectrum of RP^{n+k}/RP^{n-1}. We also computed the map $t_\sharp$ in the same range. The AHSS differentials on the other hand are much harder to compute.



Mahowald square

Now assuming we have enough information in the AAHSS, AHSS and the cellwise ASS, we can deduce differentials in the ASS by comparison. In practice, since the AAHSS computation tells us which cell in P_1^∞ a class in $Ext_A^{*,*}(\mathbb{Z}/2, H_*(P_1^\infty))$ comes from, we could work with an $H\mathbb{F}_2$-subquotient of P_1^∞. Here an $H\mathbb{F}_2$-subcomplex of P_1^∞ is a CW spectrum together with a map to P_1^∞ that induces a monomorphism on mod 2 homology and an $H\mathbb{F}_2$-quotient complex of P_1^∞ is a CW spectrum together with a map from P_1^∞ that induces an epimorphism on mod 2 homology. An $H\mathbb{F}_2$-subquotient means a "zigzag" of maps between spectra that induce a "zigzag" of monomorphisms and epimorphisms on mod 2 homology. (See Definition 4.1 of [**26**] for a precise definition.) Comparing with P_1^∞, its $H\mathbb{F}_2$-subquotient has the advantage of having fewer cells, chosen depending on the source and target of a proposed Adams differential. Therefore, it is easier to compute its Atiyah-Hirzebruch differentials. Ideally, we can keep track of Adams differentials in P_1^∞ through certain $H\mathbb{F}_2$-subquotients with as few cells as possible, establish the Adams differential in the final $H\mathbb{F}_2$-subquotient, and use naturality of Adams spectral sequences to deduce the differential in the ASS for the sphere.

The crucial part of the RP^∞-method is to know which Adams differentials and which spectra are to be considered. This is often a case-by-case situation and is usually suggested by a "zigzag" of differentials in the AAHSS, AHSS and the cellwise ASS. The idea of the "zigzag" process can be found, for example, in [**8**]. Note that the "zigzag" process only provides intuition—the rigorous proof has to be made using naturality of spectral sequences with careful checking of filtration jumps. We will give an example in Section 2.

Although the RP^∞-method is complicated and ad hoc, it is actually inductive in the following sense. Suppose that the source of an Adams differential in the sphere spectrum is in stem n. The pre-image of the source in stem n is a class in stem n for P_1^∞, which must come from a class on a cell in positive dimension in the E_1-page of the AAHSS, since P_1^∞ has cells in only positive dimensions. The internal stem of this class is in a stem strictly smaller than n. Therefore, if the ASS of all $H\mathbb{F}_2$-subquotients of P_1^∞ are well understood up to stem n, then one should be able to deduce Adams differentials for the sphere spectrum in stem $n+1$.

Similar ideas apply to extension problems as well.

The Kahn-Priddy map is also helpful in constructing Toda brackets. It is a theorem due to Cohen [**6**] that every element in the stable homotopy groups of spheres can be decomposed as Toda brackets using only 2, η, ν and σ, at the prime 2. Note that in Cohen's proof, the Toda brackets really mean mapping through certain finite cell complexes, in which all attaching maps can be constructed by iteration of Toda brackets of the Hopf maps. It does not necessarily give us a matric Toda bracket where all intermediate complexes are wedges of spheres. Nevertheless, in practice, there is no universal way of constructing such a Toda bracket for a given class in stable stems. The first element that the author does not know how to express in this form is the class $\theta_{4.5}$ defined by Isaksen [**10**] in stem 45, which is detected by h_4^3 in the ASS. Knowing a (matric) Toda bracket is in general very useful in computations of stable stems. Since many classes in the range after 45 are divisible by $\theta_{4.5}$, knowing a Toda bracket of it would be very helpful in proving certain relations and differentials in the ASS.

Suppose we have a class α in stem n. Then the Kahn-Priddy theorem tells us it maps through P_1^∞. Moreover, it must map through P_1^n. All primary attaching maps in P_1^∞ are well understood—they are in fact detected by the J-spectrum. The Kahn-Priddy map is well understood as well—when restricting to P_1^n, the cofiber is a desuspension of an $H\mathbb{F}_2$-subquotient of P_1^∞ due to James periodicity. Therefore, the resulting composite

$$S^n \longrightarrow P_1^n \longrightarrow S^0$$

realizing α could be viewed as a matric Toda bracket in a generalized sense. This again does not give us a matric Toda bracket where all intermediate complexes are wedges of spheres, but it is practical to get such a decomposition. We will give a few examples in Section 4 including a tentative one for the class $\theta_{4.5}$ in π_{45}.

The rest of the article is organized as follows. In Section 2, we will review Mahowald's trick on translation between differentials and extensions through an example—the mod 2 Moore spectrum at stem 14. In Section 3, we prove certain Adams differentials in low stems using the RP^∞-method. We will put emphasis on the "zigzag" process that predicts Adams differentials and $H\mathbb{F}_2$-subquotients. In Section 4, we give a few constructions of Toda brackets using the Kahn-Priddy map.

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2. The mod 2 Moore spectrum M at stem 14—a warmup

The first few nontrivial Adams differentials for the sphere spectrum are in stems 14 and 15. Namely,

$$d_2(h_4) = h_0h_3^2,\ d_3(h_0h_4) = h_0d_0,\ d_3(h_0^2h_4) = h_0^2d_0.$$

Knowing these differentials, there are two elements left in the E_∞-page of the ASS: h_3^2 and d_0. So there are two possibilities for π_{14}: $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4$, depending on a possible 2-extension from h_3^2 to d_0. For stem 15, there is a possible d_3 differential from h_1h_4 to h_1d_0. The purpose of this section is to show the following two statements are equivalent using the mod 2 Moore spectrum M.

(1) There is a nontrivial 2-extension in the 14-stem from h_3^2 to d_0.
(2) There is a nontrivial differential $d_3(h_1h_4) = h_1d_0$.

We start with the Adams E_2-page of M. The cofiber sequence

$$S^0 \longrightarrow M \longrightarrow S^1$$

gives us a short exact sequence on homology with $\mathbb{Z}/2$-coefficients

$$0 \longrightarrow H_*(S^0) \longrightarrow H_*(M) \longrightarrow H_*(S^1) \longrightarrow 0$$

and therefore a long exact sequence on Ext groups

$$\cdots \longrightarrow Ext_A^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow Ext_A^{s,t}(\mathbb{Z}/2, H_*M) \longrightarrow Ext_A^{s,t-1}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow \cdots$$

where the boundary homomorphism is multiplication by h_0:

$$h_0 : Ext_A^{s,t-1}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow Ext_A^{s+1,t}(\mathbb{Z}/2, \mathbb{Z}/2).$$

We also have the long exact sequence on homotopy groups with boundary homomorphism multiplication by 2, which is compatible with the long exact sequence of the Ext groups—this is the naturality of the ASS.

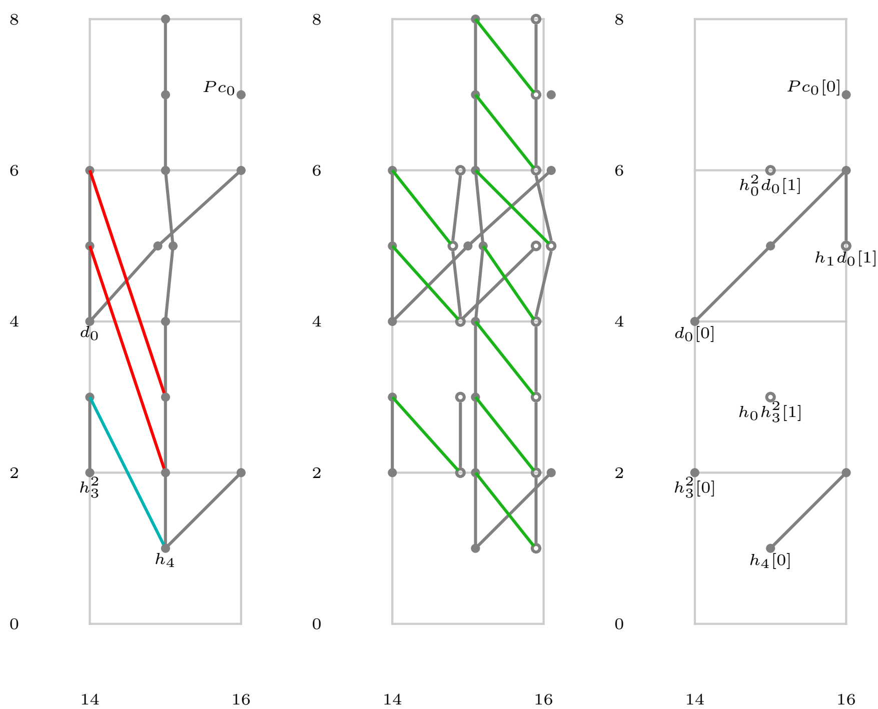
$$\begin{array}{ccccccccc} \cdots \longrightarrow & Ext_A^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) & \longrightarrow & Ext_A^{s,t}(\mathbb{Z}/2, H_*M) & \longrightarrow & Ext_A^{s,t-1}(\mathbb{Z}/2, \mathbb{Z}/2) & \longrightarrow \cdots \\ & \Downarrow & & \Downarrow & & \Downarrow & \\ \cdots \longrightarrow & \pi_{t-s}(S^0) & \longrightarrow & \pi_{t-s}(M) & \longrightarrow & \pi_{t-s}(S^1) & \longrightarrow \cdots \end{array}$$

The long exact sequence on homotopy groups can be viewed as the AHSS for M, while the long exact sequence of the Ext groups can be viewed as the AAHSS for M.

To fix notation, for any classes in $\pi_*(M)$ and $Ext_A^{*,*}(\mathbb{Z}/2, H_*M)$, we denote them by $\alpha[n]$ and $a[n]$, as the elements in the E_1-page of the (algebraic) AHSS detecting them, where $\alpha \in \pi_*(S^0)$, $a \in Ext_A^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ and n is the grading from the cellular filtration—so either 0 or 1.

In the next page, we have a chart of three spectral sequences for S^0 and M at stems 14-16. The one on the left is the ASS of S^0. There are three nontrivial differentials in this range. The one in the middle is the AAHSS for M. All AAHSS differentials in this range are d_1-differentials. The one on the right consists of the remaining dots in the middle AAHSS, which is the E_2-page of the ASS for M. It turns out there are no differentials in this range, so it is also the E_∞-page of the ASS for M.

The Adams spectral sequences of S^0 and M at stems 14-16



Now suppose that there is a nontrivial 2-extension from h_3^2 to d_0. This is the same thing as saying there is a nontrivial Atiyah-Hirzebruch d_1-differential that is not seen in the AAHSS. Therefore $\pi_{14}(M)$ must be $\mathbb{Z}/2$ generated by $\sigma^2[0]$ by the long exact sequence on homotopy groups. For the ASS of M, the only possibility is a d_3-differential:

$$d_3(h_4[0]) = d_0[0].$$

By the Leibniz rule of h_1-multiplication, we must have

$$d_3(h_1h_4[0]) = h_1d_0[0].$$

This further implies that

$$d_3(h_1h_4) = h_1d_0$$

in the ASS for the sphere spectrum by naturality.

Since each step of the above argument can be reversed, it follows that the two statements are equivalent.

Admittedly, there are other ways to see the two statements are equivalent and to prove the contrary of each statement is true. For example, if d_0 detects a class which is divisible by 2, then h_1d_0 must be killed since $2\eta = 0$ and h_1h_4 is the only possibility. This shows that (1) implies (2). For the other implication, using the differential

$$d_2(h_4) = h_0h_3^2,$$

we have a Massey product in the Adams E_3-page:

$$h_1h_4 = \langle h_3^2, h_0, h_1\rangle.$$

Then Moss's theorem tells us if $2 \cdot \sigma^2 = 0$ then $h_1 h_4$ is a permanent cycle. This shows that (2) implies (1).

For statement (1), since σ is in an odd stem, and the stable homotopy ring of spheres is commutative in the graded sense, then we must have $2 \cdot \sigma^2 = 0$. For statement (2), Mahowald [**18**] in fact proved that $h_1 h_j$ survives for all $j \geq 3$. One could also use the Hurewicz image of tmf to show that κ (the homotopy class detected by d_0) is not divisible by 2 and that $\eta\kappa$ is nonzero.

It is interesting to compare classes in the E_∞-page of the ASS of M and classes in the E_∞-page of the AHSS of M. Let ρ_{15} be a homotopy class in π_{15} that is detected by $h_0^3 h_4$. Since there are only d_1-differentials in the AHSS for M, it is clear that the 15-stem of the E_∞-page of the AHSS of M is generated by

$$\eta\kappa[0],\ \rho_{15}[0],\ \sigma^2[1],\ \kappa[1].$$

The 15-stem of the E_∞-page of the ASS of M is generated by

$$h_4[0],\ h_0 h_3^2[1],\ h_1 d_0[0],\ h_0^2 d_0[1].$$

To see the correspondence, we compare the Adams filtrations. The class $\sigma^2[1]$ has Adams filtration 2 in S^1, so its Adams filtration in M is at most 2. The only possibility is that $h_4[0]$ detects $\sigma^2[1]$. Similarly, the class $\kappa[1]$ is detected by $h_0 h_3^2[1]$, the class $\eta\kappa[0]$ is detected by $h_1 d_0[0]$, and the class $\rho_{15}[0]$ is detected by $h_0^2 d_0[1]$.

Knowing the correspondence from the naturality of the ASS, we explain how to use the "zigzag" process to predict the same answer. Start with the class $\sigma^2[1]$: it is supposed to be detected by $h_3^2[1]$. However, $h_3^2[1]$ supports an algebraic Atiyah-Hirzebruch differential that kills $h_0 h_3^2[0]$ so it is not present on the Adams E_2-page. Furthermore, the element $h_0 h_3^2[0]$ is killed by $h_4[0]$ in the cellwise ASS. In summary, we have two differentials

$$d_1(h_3^2[1]) = h_0 h_3^2[0] \text{ in the AAHSS,}$$

$$d_2(h_4[0]) = h_0 h_3^2[0] \text{ in the cellwise ASS.}$$

Combining them together, it leads us to deduce that $h_4[0]$ detects $\sigma^2[1]$.

As another example, the fact that the class $\kappa[1]$ is detected by $h_0 h_3^2[1]$ follows from the following "zigzag" of 4 differentials:

$$d_1(d_0[1]) = h_0 d_0[0] \text{ in the AAHSS,}$$

$$d_3(h_0 h_4[0]) = h_0 d_0[0] \text{ in the cellwise ASS,}$$

$$d_1(h_4[1]) = h_0 h_4[0] \text{ in the AAHSS,}$$

$$d_2(h_4[1]) = h_0 h_3^2[1] \text{ in the cellwise ASS.}$$

As a final comment on this computation, we actually have

$$\pi_{15}(M) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

The class $h_1^2 d_0[0]$ is killed by a d_2-differential and there is a nontrivial 2-extension from $h_0 h_3^2[1]$ to $h_1 d_0[0]$. It is easier to see this 2-extension from the AHSS. We have

$$2 \cdot \kappa[1] = \langle 2, \kappa, 2\rangle[0] = \eta\kappa[0].$$

In general, suppose that $\alpha \cdot \beta = 0, \beta \cdot \gamma = 0$. Then in the homotopy of the cofiber of α, we always have a γ-extension in terms of the Atiyah-Hirzebruch notations:

$$\gamma \cdot \beta[|\alpha| + 1] = \langle \alpha, \beta, \gamma\rangle[0].$$

Here $|\alpha|$ is the degree of α. (See, for example, Lemma 5.2 in [**28**] for a proof.)

3. RP^∞-method for some Adams differentials in low stems

We start with commenting on the Hopf invariant one differential $d_2(h_4) = h_0h_3^2$.

Since it is a d_2-differential, there is only one possibility—the pullback differential in P_1^∞ has to be a d_2-differential as well. Following computations in [**28**], we see that the differential in P_1^∞ is

$$d_2(1[15]) = h_1h_3[6].$$

This leads us to consider a certain $H\mathbb{F}_2$-subquotient of P_1^∞ that contains both the 15-cell and the 6-cell. In fact, by the solution of the Hopf invariant one problem, we know that the primary attaching map from the 15-cell goes down to the 6-cell by $\eta\sigma \in \pi_8$. In other words, we have a splitting of finite spectra

$$P_7^{15} \simeq P_7^{14} \vee S^{15},$$

and $\Sigma^6C(\eta\sigma)$ is an $H\mathbb{F}_2$-subcomplex of P_6^{15}:

$$\Sigma^6C(\eta\sigma) \hookrightarrow P_6^{15},$$

where $C(\eta\sigma)$ is the cofiber of $\eta\sigma$. Therefore, the Adams d_2-differential is compatible with the Atiyah-Hirzebruch differential

$$d_9(1[15]) = \eta\sigma[6].$$

One can first use the Atiyah-Hirzebruch differential to prove an Adams d_2-differential in $\Sigma^6C(\eta\sigma)$ and then use naturality to prove the same differential in P_1^∞ and therefore the one in the sphere. Of course this "proof" is circular due to the Hopf invariant one problem, but it is nice to see the connection through the Mahowald square.

In the rest of this section, we give a proof of the Adams differential in the 30-stem

$$d_3(r) = h_1d_0^2$$

by the RP^∞-method. We choose this differential for the following reason: up to the 45-stem, this is essentially the only Adams differential that cannot be computed by the recent motivic method. It can, however, be computed by Bruner's power operation arguments or by an ad hoc argument due to Mahowald-Tangora [**19**], for example.

Since d_0 is a permanent cycle, it is equivalent to give a proof of the Adams differential in the 44-stem:

$$d_3(d_0r) = h_1d_0^3.$$

Checking computations in [**28**], we see the pre-images in the ASS for P_1^∞ of d_0r and $h_1d_0^3$ are $v[2]$ and $d_0^3[1]$. This is a great sign that the cells involved in P_1^∞ are in very low dimensions. Now let's do the "zigzag":

$$d_2(v[2]) = h_1^2u[2] \text{ in the cellwise ASS,}$$

$$d_2(h_1u[4]) = h_1^2u[2] \text{ in the AAHSS.}$$

The class h_1u is a surviving cycle in π_{40} and detects $2\overline{\kappa}^2$. The minimal $H\mathbb{F}_2$-subcomplex of P_1^∞ that contains the 4-cell is P_1^4, and it has an $H\mathbb{F}_2$-quotient complex P_1^4/S^3 since S^3 is an $H\mathbb{F}_2$-subcomplex of P_1^4. The attaching maps in P_1^4/S^3 are η and 2. Therefore, in the AHSS for both P_1^4/S^3 and P_1^4, the class $2\overline{\kappa}^2[4]$ supports a differential:

$$d_3(2\overline{\kappa}^2[4]) = \langle 2\overline{\kappa}, \eta, 2\rangle[1] = \eta^2\overline{\kappa}^2[1].$$

This is great since $\eta^2\overline{\kappa}^2$ is exactly detected by d_0^3! This completes the "zigzag" process.

Now we move on to a rigorous proof using naturality. What we need is to prove a nontrivial Adams differential in P_1^∞:

$$d_3(v[2]) = d_0^3[1].$$

Since the "zigzag" of differentials are among elements from cells in dimensions between 1 and 4, checking computations in [**28**], we have the Adams E_2-page of P_1^4 in stems 43 and 44. One could also easily do the AAHSS computation by hand.

TABLE 1. The Adams E_2-page of P_1^4 and P_3^4 in the 43 and 44-stems for $s \leq 12$

	$Ext_A^{s,t}(\mathbb{Z}/2, H_*(P_1^4))$		$Ext_A^{s,t}(\mathbb{Z}/2, H_*(P_3^4))$
$s\backslash t-s$	43	44	44
12	$d_0^3[1]$		
11	$\bullet$	$\bullet$	
10	$\bullet$	$z[3]$	$z[3]$ $h_1u[4]$
9		$v[2]$	
8	$\bullet$		$g^2[4]$
7	$\bullet$	$h_1Ph_1h_5[3]$	$h_1Ph_1h_5[3]$
6	$\bullet$	$h_0^2f_1[4]$	$h_0^2f_1[4]$ $h_1Ph_1h_5[4]$
5	$h_1f_1[2]$		$h_1h_5c_0[4]$
4	$\bullet$ $\bullet$	$h_2^3h_5[4]$	$h_2^3h_5[4]$
3		$c_2[3]$	$c_2[3]$

Now since in the ASS of P_1^4 the element $d_0^3[1]$ detects $\eta^2\overline{\kappa}^2[1]$, which is killed by an Atiyah-Hirzebruch differential, the element $d_0^3[1]$ must be killed by an Adams differential!

To prove that it is $v[2]$ that kills it, we simply rule out other possibilities. Other elements that could support a differential and kill $d_0^3[1]$ are

$$z[3],\ h_1Ph_1h_5[3],\ h_0^2f_1[4],\ h_2^3h_5[4],\ c_2[3].\ (*)$$

They all come from the 3 and 4-cells. We therefore consider the quotient map $P_1^4 \to P_3^4$ and use the Mahowald square to prove that, except for one of them which supports an Adams differential—so is irrelevant for the differential we want to prove—the classes in $(*)$ are all permanent cycles. This then gives us a proof that

$$d_3(v[2]) = d_0^3[1]$$

in the ASS of P_1^4 and therefore in the ASS of P_1^∞, which completes the proof of the differential

$$d_3(d_0r) = h_1d_0^3$$

in the ASS of the sphere spectrum.

For the five classes, it is straightforward to check that they all survive in the ASS of P_3^4 by using the method of Mahowald square as explained in Section 2. In fact, they detect homotopy classes

$$\eta\overline{\kappa}^2[3],\ \eta\{Ph_1h_5\}[3],\ \eta\{f_1\}[3],\ \nu\{h_2^2h_5\}[4],\ \{f_1\}[4].$$

For the class $\{f_1\}[4]$, we have an Atiyah-Hirzebruch differential in P_1^4:

$$d_2(\{f_1\}[4]) = \eta\{f_1\}[2].$$

This shows that the element $h_1f_1[2]$ must be killed in ASS of P_1^4 and the following Adams differential is the only possibility:

$$d_2(c_2[3]) = h_1f_1[2].$$

For the other homotopy classes, it is straightforward to check that they all survive in the AHSS of P_1^4. Therefore, by the naturality of the AHSS, the corresponding homotopy classes must be detected by elements in the ASS of P_1^4, with Adams filtration at most the same as the ones in the ASS of P_3^4. This proves that the other 4 classes $z[3],\ h_1Ph_1h_5[3],\ h_0^2f_1[4],\ h_2^3h_5[4]$ are all permanent cycles.

Following the "zigzag," we could summarize the idea of the proof by the "road map":

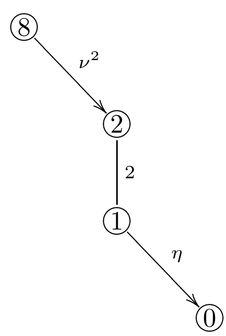
$$P_3^4 \longleftarrow P_1^4 \longrightarrow P_1^\infty \longrightarrow S^0$$

As a general comment on the RP^∞-method, it is clear that the method works better when the "spheres of origin" of the source and target are in nearby dimensions. When the difference of the dimensions is large, one usually needs to run a longer "zigzag." For example, in the proof of the Adams differential $d_3(D_3) = B_3$, the "spheres of origin" of the source and target are in dimensions 22 and 6. See Appendix II of [**26**] for the "zigzag" process of the proof of the Adams differential $d_3(D_3) = B_3$.

4. The Kahn-Priddy map and Toda brackets

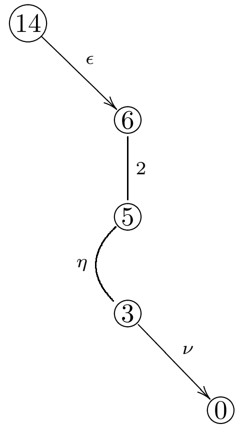
The Kahn-Priddy theorem implies that any homotopy class of the sphere spectrum in stem n must factor through P_1^n. We illustrate by examples that this observation and the computations in [**28**] lead us to Toda brackets in some generalized sense. We use the theory of cell diagrams to denote these Toda brackets. For background and uses of cell diagrams, see [**2**, **26**, **29**] for example.

Consider the homotopy class $\epsilon \in \pi_8$. It is detected by c_0. The pre-image under the algebraic Kahn-Priddy map is $h_2^2[2]$. Since the 1-cell in P_1^∞ maps to S^0 by η, we have the following cell diagram for ϵ.



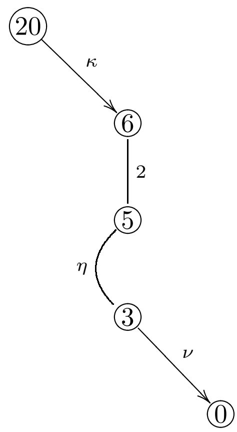
In other words, we have a Toda bracket $\epsilon \in \langle \nu^2, 2, \eta\rangle$ in π_8.

As another example, take $\kappa \in \pi_{14}$. It is detected by d_0. The pre-image under the algebraic Kahn-Priddy map is $c_0[6]$. Since the minimal $H\mathbb{F}_2$-subcomplex of P_1^∞ that contains the 6-cell is the 3 cell complex with cells in dimension 6, 5 and 3, and the 3-cell maps to S^0 by ν, we have the following cell diagram for κ.



In other words, we have a Toda bracket $\kappa \in \langle \epsilon, 2, \eta, \nu \rangle$ in π_{14}.

Let's consider $\overline{\kappa} \in \pi_{20}$. It is detected by g. The pre-image under the algebraic Kahn-Priddy map is $h_0h_3^2[6]$. Now note that $h_0h_3^2$ does not survive in the ASS of the sphere spectrum. However, as discussed in Section 2, $h_0h_3^2[1]$ survives in the mod 2 Moore spectrum and detects $\kappa[1]$. We have P_5^6 as a suspension of the mod 2 Moore spectrum. Therefore, we have the following cell diagram for $\overline{\kappa}$.



In other words, we have a Toda bracket $\overline{\kappa} \in \langle \kappa, 2, \eta, \nu \rangle$ in π_{20}.

It is clear that these observations work better when the "sphere of origin" is in low stems.

Now we consider the class $\theta_{4.5} \in \pi_{45}$ defined by Isaksen in [**10**]. It is detected by h_4^3. The pre-image under the algebraic Kahn-Priddy map is $h_4^2[15]$. The 15-cell is attached to the 6-cell by $\eta\sigma$. Since $\eta\sigma\theta_4 \neq 0$, we need to add in more cells to kill this obstruction. A natural candidate is the 8-cell, since it is attached to the 6-cell by η. This raises another problem: $\eta\sigma$ on the 6-cell would be killed by σ on the 8-cell. To avoid this problem, we add in the 7-cell as well, so σ on the 8-cell would kill 2σ on the 7-cell instead. Therefore, $\theta_{4.5}$ maps through the cofiber of $\eta\sigma[6]$ for P_1^8. The cell diagram is given below.

We can also form $\theta_{4.5}$ as a Toda bracket of the following maps:

$$S^{44} \xrightarrow{\left[\begin{array}{cc} \theta_4 & \sigma\theta_4 \end{array}\right]} S^{14} \vee S^7 \xrightarrow{\left[\begin{array}{cc} \eta\sigma[6] & 0 \\ \eta[6] & 2 \end{array}\right]} P_1^6 \vee S^7 \xrightarrow{\left[\begin{array}{c} KP_6 \\ \sigma \end{array}\right]} S^0$$

where KP_6 is the Kahn-Priddy map restricted to P_1^6.

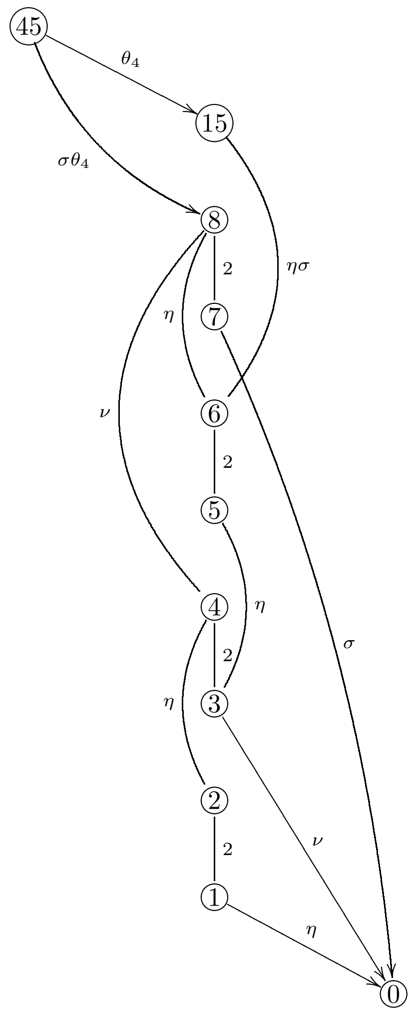
As a sanity check, we have

$$2\theta_{4.5} = \left\langle 2, \left[\begin{array}{cc} \theta_4 & \sigma\theta_4 \end{array}\right], \left[\begin{array}{c} \eta\sigma[6] \\ \eta[6] \end{array}\right] \right\rangle KP_6 + \left\langle 2, \left[\begin{array}{cc} \theta_4 & \sigma\theta_4 \end{array}\right], \left[\begin{array}{c} 0 \\ 2 \end{array}\right] \right\rangle \sigma.$$

The second term is $\langle 2, \sigma\theta_4, 2\rangle\sigma$, which contains $\eta\sigma^2\theta_4 = 0$. Note that since we know the exponent of P_1^6:

$$8 \cdot \pi_*(P_1^6) = 0,$$

we have that $16 \cdot \theta_{4.5} = 0$, which is compatible with our knowledge.



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