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The universal abelian regular ring

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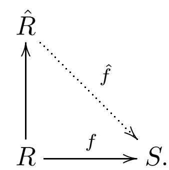
Dedicated to Mike Prest on the occasion of his 65th birthday

Abstract. Olivier's construction of a universal commutative regular ring is generalized to obtain for a noncommutative ring R the universal abelian regular R-ring $R \to \hat{R}$. This R-ring induces a homeomorphism from the spectrum of $\hat{R}$ to the constructible Cohn spectrum of R. An étale bundle is constructed over the Ziegler spectrum $\mathrm{Zg}^*(R)$, equipped with the Zariski topology, whose sheaf of sections is the sheafification of Prest's presheaf of definable scalars. When the bundle is restricted to $\mathrm{Zg}^*(\hat{R})$, which is homeomorphic to the spectrum of $\hat{R}$, one obtains an étale bundle whose R-ring of global sections is isomorphic to $\hat{R}$.

Let R be a commutative ring. The *universal commutative (von Neumann) regular* R-algebra is a ring morphism $R \to \hat{R}$ into a commutative regular ring that is universal with respect to this property: every ring morphism $f : R \to S$ to a commutative regular ring factors uniquely through $\hat{R}$ as in



Olivier [**14**] proved the existence of the universal commutative regular R-algebra by the formal adjunction, for every element $r \in R$, of a *reflexive inverse:*

$$\hat{R} = R[y_r \mid r \in R]/(ry_rr - r, y_rry_r - y_r \mid r \in R)$$

with $R \to \hat{R}$ given by the obvious morphism. The purpose of this article is to generalize Olivier's construction (Theorem 2.2) to the noncommutative case and study the universal abelian regular R-ring $R \to R^{\mathrm{ab}}$. One mimics the construction of Olivier, by formal adjunction of *group inverses*, defined as commuting reflexive inverses, iterated countably many times.

Wiegand [**22**, Thm 1] described $\hat{R}$ as the global sections of a sheaf on the Zariski spectrum $\mathrm{Spec}(R)$, equipped with the patch topology [**9**]. The universal abelian

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regular R-ring $R \to \hat{R}$ induces a homeomorphism $\mathrm{Spec}(\hat{R}) \to \widehat{\mathrm{Spec}}(R)$ between the spectrum of $\hat{R}$ and the Cohn spectrum $\widehat{\mathrm{Spec}}(R)$, equipped with the patch topology. There is a bijective correspondence $e \mapsto \mathcal{O}(e)$ between the idempotent elements of $\hat{R}$ and the clopen subsets of $\mathrm{Spec}(\hat{R})$. The Pierce sheaf [**15**, Thm 4.4] is a sheaf Γ of rings on $\mathrm{Spec}(\hat{R})$ with the property that $\Gamma[\mathcal{O}(e)] = e\hat{R}e$. In the last few sections of the paper, we invoke Prest's theory of the sheaf of definable scalars to show (Theorem 8.5) how the Pierce sheaf Γ is obtained by pullback of the sheaf of locally definable scalars on R along the homeomorphic embedding $\mathrm{Zg}^*(\hat{R}) \to \mathrm{Zg}^*(R)$ of Ziegler spectra, endowed with the Zariski topology.

Corollary 5.3 provides an axiomatization for the full subcategory $\hat{R}$-Mod $\subseteq$ R-Mod of modules over the universal abelian regular R-ring.

Throughout the paper, R will denote an associative ring with identity $1 \in R$. An R-ring is a morphism $f : R \to S$ of rings with domain R. An R-ring S may be thought of as a left R-module ${}_RS$ via the action $rs = f(r)s$. We may refer to the element $f(r)$ as $r \in S$. An R-field $R \to \Delta$ is an R-ring where Δ is a (not necessarily commutative) field. An R-algebra is an R-ring $f : R \to S$ with R commutative and the image of f contained in the center of S.

1. The group inverse

An element $r \in R$ is *regular* if there exists a $y \in R$ such that $ryr = r$; then y is called an *inner inverse* of r. Note that $y' = yry$ is also an inner inverse with the further property that $y'ry' = y'$; such an inner inverse is called a *reflexive inverse* of r and, if it commutes with r, a *group inverse* of r (see [**17**, §8.6]). This notion is important in the theory of *abelian regular rings*, which are rings in which every element has a group inverse. Equivalently, a regular ring R is abelian [**3**, §3] if and only if every idempotent $e \in R$ is central. If $r \in R$ is a unit, then the inverse of r in R is clearly a group inverse; and if a regular element $r \in R$ has a unique reflexive inverse then it must be a group inverse [**20**, Lemma 3.3]. A regular element $r \in R$ has at most one group inverse y, for if y' were another, then

$$y = y^2r = y^2r^2y' = y(yr^2)y' = yry'$$

and similarly $y' = yry'$. Here are some conditions equivalent to the existence of a group inverse.

Proposition 1.1. *Let R be an associative ring. The following are equivalent for an element $r \in R$:*

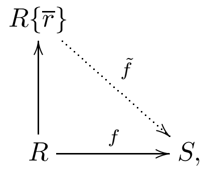
1. *r has a group inverse in R;*
2. *there is an idempotent element $e \in R$ such that $rR = eR$ and $Rr = Re$;*
3. *there exists a direct sum decomposition $R_R = rR \oplus r.ann(r)$; and*
4. *every left R-module ${}_RM$ admits a direct sum endodecomposition[1] $M = rM \oplus \mathrm{ann}_M(r)$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $ryr = r$ and $yry = y$, with $e = ry = yr$. Then $rR = eR$ and $Rr = Re$.
(2) $\Rightarrow$ (3). Since $Rr = Re$, we have that $\mathrm{r.ann}(r) = \mathrm{r.ann}(e) = (1-e)R$.
(3) $\Rightarrow$ (4). Let $e \in R$ be the projection onto rR parallel to $\mathrm{r.ann}(r)$. Thus $rR = eR$ and $\mathrm{r.ann}(r) = (1-e)R$, which imply that $M = eM \oplus (1-e)M = rM \oplus \mathrm{ann}_M(r)$.

[1] A decomposition as a module over its endomorphism ring.

(4) ⇒ (1). Take $M = {}_RR$ and let $e \in R$ be as in the previous implication. The equalities $rR = eR$ and $\mathrm{r.ann}(r) = (1-e)R$ imply that $r = er$, $r = re$ and that there exists a $y \in R$ such that $e = ry$; we may assume that $y = eye$. Then $ryr = er = r$ and $yry = ye = y$, so that y is a reflexive inverse of r. The idempotent $f = yr$ satisfies $r(1-f) = r - ryr = 0$, so that $1 - f \in (1-e)R$, whence $ry = e = ef = f = yr$. □

We will denote the group inverse of r, when it exists, by $\overline{r}$. Given an element $r \in R$, there exists an R-ring $R \to R\{\overline{r}\} := R\{y\}/I$ universal with respect to the property that r has a group inverse. It is obtained by freely adjoining a (usually noncentral) variable y and factoring out by the ideal $I = (ryr - r, yry - y, ry - yr)$ of relations satisfied by a group inverse. If $f : R \to S$ is any R-ring in which r has a group inverse, then there is a unique extension



given by $\tilde{f} : \overline{r} = y + I \mapsto \overline{r}$. In particular, any ring morphism $g : R\{\overline{r}\} \to S'$ is determined by the image of $R \to R\{\overline{r}\}$, so that this universal R-ring is epic.

Prest [**16**, Cor 5.5.4] has shown that whenever $f : R \to S$ is an epic R-ring, then the full subcategory S-Mod $\subseteq R$-Mod, obtained by restriction of scalars along f, is axiomatizable in the language of left R-modules (see §3). For the epic R-ring $R \to R\{\overline{r}\}$, Condition (4) of Proposition 1.1 provides such an axiom. The formula $r|u := \exists v\ (u \doteq rv)$ defines in ${}_RM$ the endosubmodule rM, while the formula $ru \doteq 0$ defines $\mathrm{ann}_M(r)$. Thus we can express Condition (4) as follows.

Proposition 1.2. *A left R-module M belongs to the subcategory $R\{\overline{r}\}$-*Mod $\subseteq$ *R-*Mod *if and only if*

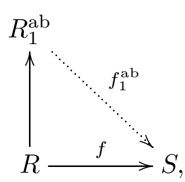
$$M \models \{\forall u\ \exists v, w\ [(u \doteq v + w) \wedge r|v \wedge rw \doteq 0]\} \wedge \forall u\ [(ru \doteq 0 \wedge r|u) \to u \doteq 0].$$

2. Olivier's construction

Let us carry out the construction of the previous section for all $r \in R$ at once, by adjoining a group inverse for every element of R. An R-ring is obtained by adjoining noncentral variables y_r, one for every $r \in R$,

$$R \to R_1^{\mathrm{ab}} := R\{y_r \mid r \in R\}/I,$$

modulo the ideal $I = (ry_rr - r, y_rry_r - y_r, ry_r - y_rr \mid r \in R)$ generated by the relations that ensure each $y_r + I = \overline{r}$ is a group inverse of r. As above, the R-ring $R \to R_1^{\mathrm{ab}}$ is universal with respect to the property that every $r \in R$ obtains a group inverse. In particular, every abelian regular R-ring $f : R \to S$ factors uniquely, as above, through R_1^{ab},



The universal property of the R-ring $R \to R_1^{\mathrm{ab}}$ ensures that it is also epic. As such, the full subcategory R_1^{ab}-Mod $\subseteq R$-Mod is axiomatizable and we obtain the following version of Proposition 1.2.

PROPOSITION 2.1. *A left R-module M belongs to the subcategory R_1^{ab}-*Mod $\subseteq$ R-Mod *if and only if for every $r \in R$,*

$$M \models \{\forall u\ \exists v, w\ [(u \doteq v + w) \wedge r|v \wedge rw \doteq 0]\} \wedge \forall u\ [(ru \doteq 0 \wedge r|u) \to u \doteq 0].$$

If $r \in R$ is a nilpotent element, then the existence of a group inverse in R_1^{ab} implies that $r = 0$ in R_1^{ab}. The kernel of $R \to R_1^{\mathrm{ab}}$ therefore contains the upper nilradical of R. Also note that if $e \in R$ is idempotent, then e will centralize R in R_1^{ab}, because the element $(1-e)re$ is nilpotent, for every $r \in R$. Thus $re = ere$ in R_1^{ab} and similarly $er = ere$.

The construction $R \to R_1^{\mathrm{ab}}$ is the first step in the noncommutative version of Olivier's construction. The process may be iterated to obtain a denumerable sequence $R = R_0^{\mathrm{ab}} \longrightarrow R_1^{\mathrm{ab}} \longrightarrow R_2^{\mathrm{ab}} \longrightarrow \cdots$, of ring morphisms defined recursively by $R_{n+1}^{\mathrm{ab}} := (R_n)_1^{\mathrm{ab}}$. Each of the compositions $R \to R_n^{\mathrm{ab}}$ is an epic R-ring and, therefore, so is the limit $R \to R^{\mathrm{ab}} := \varinjlim R_n^{\mathrm{ab}}$, which is abelian regular, because every element of R^{ab} has a group inverse. Indeed, if $r \in R^{\mathrm{ab}}$ is represented by some approximation $r_n \in R_n^{\mathrm{ab}}$, then the construction ensures that r_n obtains a group inverse in R_{n+1}^{ab}. Moreover, the abelian regular R-ring $R \to R^{\mathrm{ab}}$ is universal with respect to this property, because every abelian regular R-ring $f : R \to S$ factors uniquely through each of the $R \to R_n^{\mathrm{ab}}$ and therefore uniquely through $R \to R^{\mathrm{ab}}$.

THEOREM 2.2. *Every ring R admits a universal abelian regular R-ring $R \to R^{\mathrm{ab}}$.*

If a ring R is abelian regular to begin with, then $R \to R_1^{ab}$ is just the identity morphism, so that this noncommutative iterated version of Olivier's contruction halts at stage 0. More generally, if R_n^{ab} is abelian regular at some finite stage of the construction, then the process halts and $R^{\mathrm{ab}} = R_n^{\mathrm{ab}}$. It seems like an interesting general ring theoretic problem to determine the least n, if it exists, for which R_n^{ab} is abelian regular.

If R is a von Neumann regular ring (not necessarily abelian), let $N \subseteq R$ be the ideal generated by elements of the form $er(1-e)$, with e idempotent. These elements are nilpotent so that N is contained in the kernel of $R \to R_1^{\mathrm{ab}}$ and we obtain a morphism $R/N \to R_1^{\mathrm{ab}}$ of epic R-rings. On the other hand, R/N is abelian regular, by [**3**, Prop 3.3], so that we also have a morphisms $R_1^{\mathrm{ab}} \to R/N$. These morphisms of epic R-rings are therefore isomorphisms and we see that $R \to R_1^{\mathrm{ab}} = R/N$ is abelian regular.

COROLLARY 2.3. *There is a bijection $\mathcal{P} \mapsto R^{\mathrm{ab}}/\mathcal{P}$ between the prime ideals of R^{ab} and the epic R-fields $R \to R^{\mathrm{ab}} \to R^{\mathrm{ab}}/\mathcal{P}$. In particular, $R^{\mathrm{ab}} \neq 0$ if and only there exists a nonzero epic R-field.*

PROOF. If $\mathcal{P} \subseteq R^{\mathrm{ab}}$ is a prime ideal, then it is maximal and the quotient $R^{\mathrm{ab}}/\mathcal{P}$ is a field [**3**, Thm 3.2]. It follows that $R \to R^{\mathrm{ab}} \to \mathrm{R}^{\mathrm{ab}}/\mathcal{P}$ is an epic R-field. Conversely, every R-field $f : R \to \Delta$ is abelian regular, so that Theorem 2.2 implies that it factors through the universal abelian regular R-ring, with $f^{\mathrm{ab}} : \overline{r} \mapsto r^{-1} \in \Delta$ if $r \neq 0$ in Δ. Define $E_n(f) \subseteq \Delta$ by recursion on $n \geq 1$, so that $E_0(f) := \mathrm{Im}\, f$, the image of f, and E_{n+1} the subring of Δ generated by $E_n(f)$ and the elements r^{-1}, for nonzero $r \in E_n(f)$. A commutative diagram arises in the category of R-rings.

$$\begin{array}{ccccccccc} R = R_0^{\mathrm{ab}} & \longrightarrow & R_1^{\mathrm{ab}} & \longrightarrow & R_2^{\mathrm{ab}} & \longrightarrow & \cdots & \longrightarrow & R^{\mathrm{ab}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f^{\mathrm{ab}}} \\ E_0(f) & \longrightarrow & E_1(f) & \longrightarrow & E_2(f) & \longrightarrow & \cdots & \longrightarrow & E(f), \end{array}$$

where $E(f) = \cup_n \ E_n(f)$ is the subfield of Δ generated by the image of f. The universal property of the ring morphisms $f_n : R_n^{\mathrm{ab}} \to E_n(f)$ ensures that every f_n is onto. It follows that all the R-rings in the diagram are epic and the limit $f^{\mathrm{ab}} : R^{\mathrm{ab}} \to \Delta$ is onto. Therefore, there exists a prime ideal $\mathcal{P} \subseteq R^{\mathrm{ab}}$ for which $\Delta = R^{\mathrm{ab}}/\mathcal{P}$. □

Corollary 2.3 implies that there is a canonical morphism of R-rings, $R^{\mathrm{ab}} \to \prod_{\mathcal{P}} R^{\mathrm{ab}}/\mathcal{P}$, to the product of all epic R-fields. This morphism is an embedding, because the kernel is the Jacobson radical $J(R^{\mathrm{ab}})$, which is 0 in a von Neumann regular ring.

The *inversion height* of an epic R-field $f : R \to \Delta$ is the least n for which $E_n(f) = E(f) = \Delta$. The construction of the universal abelian regular R-ring was inspired by the thesis of Javier Sanchez [**5**, **19**], who—building on the work of Fisher [**4**] and Reutenauer [**18**]—gives examples of epic R-fields of various inversion height.

3. Definable scalars of R-rings.

The language of left R-modules $\mathcal{L}(R) = (+, -, 0, r)_{r \in R}$ is the language $\mathcal{L}(\mathbb{Z}) = (+, -, 0)$ of abelian groups expanded to include unary function symbols, one for every $r \in R$, also denoted as r. If ${}_RM$ is a left R-module, the symbols of $\mathcal{L}(\mathbb{Z})$ are intended to interpret the underlying abelian group, while the unary function symbols $r \in R$ interpret the scalar action of $r \in R$ on that group. The axioms for a left R-module are expressible in $\mathcal{L}(R)$ by finitely many axiom schemata.

The nonlogical symbols of $\mathcal{L}(R)$ are precisely those needed to express a linear equation

$$r_1u_1 + r_2u_2 + \cdots + r_nu_n \doteq 0.$$

A system of m linear equations in n variables is a conjunction of such formulae

$$\bigwedge_{i=1}^{m} \sum_{j=1}^{n} r_{ij}u_j \doteq 0$$

and is denoted as usual by $A\mathbf{u}^t \doteq 0$, where $A = (r_{ij})$ is the associated $m \times n$ matrix of coefficients and $\mathbf{u}^t$ is the column vector of free variables given by the transpose of the n-tuple $\mathbf{u} = (u_1, u_2, \ldots, u_n)$.

A formula $\varphi(\mathbf{u})$ in $\mathcal{L}(R)$ is *positive primitive* (pp) if it is obtained from a system $(A, B)\begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{pmatrix} \doteq 0$ of linear equations by existential quantification over certain of the free variables, say $\mathbf{v}$. Thus A is an $m \times n$ matrix and B an $m \times k$ matrix, where $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_k)$, and the pp formula is displayed as

$$\varphi(\mathbf{u}) = \exists \mathbf{v} \ (A, B)\begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{pmatrix} \doteq 0$$

with the free variables $\mathbf{u}$ explicitly shown. We will mostly be interested in pp formulae in one and two (free) variables. A pp formula in one variable has the form

$$\psi(u) = \exists \mathbf{v}\ (A_0, A) \begin{pmatrix} u \\ \mathbf{v}^t \end{pmatrix} \doteq 0,$$

where A_0 is a column matrix with as many rows as there are in A. For a pp formula in two variables, we conform to the convention of Cohn [**1**, Ch 7] and write the formula as

$$\rho(u, v) = \exists \mathbf{w}\ (A_0, A_*, A_\infty) \begin{pmatrix} u \\ \mathbf{w}^t \\ v \end{pmatrix} \doteq 0,$$

where A_0 and A_∞ are now column vectors with the same number of rows as A_*.

Given a pp formula $\varphi(\mathbf{u})$ as above and left R-module ${}_RM$, an n-tuple $\mathbf{x} \in M^n$ satisfies $\varphi(\mathbf{u})$ in ${}_RM$, written ${}_RM \models \varphi(\mathbf{x})$, if there exists a k-tuple $\mathbf{y} \in M^k$ such that $(A, B) \begin{pmatrix} \mathbf{x}^t \\ \mathbf{y}^t \end{pmatrix} \doteq 0$. For example, consider an R-ring $f : R \to S$ as a left R-module and take a pp formula $\psi(u)$ in one variable as above. Then ${}_RS \models \psi(1)$ if and only if the nonhomogenous system $A\mathbf{v}^t = -A_0$ has a solution in S if and only if $A\mathbf{v}^t = A_0$ does.

If ${}_RM$ is a left R-module and $\varphi(\mathbf{u})$ a pp formula in n variables, the collection of n-tuples $\mathbf{x} \in M^n$ that satisfy $\varphi(\mathbf{u})$ in ${}_RM$ form the *pp definable subgroup*

$$\varphi(M) := \{\mathbf{x} \in M^n \mid M \models \varphi(\mathbf{x})\}$$

of M^n, closed under the action of the endomorphism ring $\mathrm{End}_R\, M$. For example, if $R \to \Delta$ is an R-field, then ${}_R\Delta$ is endosimple, so it must be that for every $\psi(u)$ in one variable, $\psi(\Delta) = \Delta$ or $\psi(\Delta) = 0$, depending on whether the homogeneous system $A\mathbf{v} = A_0$ is solvable in Δ or not.

For a pp formula $\rho(u, v)$ in two free variables and a left R-module ${}_RM$ there is the prospect that ρ defines the graph of a $\mathbb{Z}$-linear map $\rho : M \to M$,

$$(\dagger) \qquad M \models \forall u \exists! v\ \rho(u, v).$$

This happens precisely when the two pp formulae $\exists v \rho(u, v)$ and $\rho(0, v)$ in one variable define in M the pp definable subgroups M and 0, respectively. A function $\rho : M \to M$ that is pp definable in this way is called a *definable scalar* on M. Let us explain how the definable scalars on M form an R-ring $R \to R_M$. If $\rho, \sigma : M \to M$ are scalars on M defined by

$$\rho(u, v) = \exists \mathbf{w}\ (A_0, A_*, A_\infty) \begin{pmatrix} u \\ \mathbf{w}^t \\ v \end{pmatrix} \doteq 0 \quad \text{and} \quad \sigma(u, v) = \exists \mathbf{x}\ (B_0, B_*, B_\infty) \begin{pmatrix} u \\ \mathbf{x}^t \\ v \end{pmatrix} \doteq 0,$$

respectively, then $\rho - \sigma$ and $\rho \cdot \sigma$ are defined by the pp formulae

$$(\rho - \sigma)(u, v) = \exists \mathbf{y} \begin{pmatrix} B_0 & B_* & B_\infty & 0 & 0 \\ A_0 & 0 & A_\infty & A_* & A_\infty \end{pmatrix} \begin{pmatrix} u \\ \mathbf{y}^t \\ v \end{pmatrix} \doteq 0, \quad \text{and}$$

$$(\rho\sigma)(u, v) = \exists \mathbf{y} \begin{pmatrix} B_0 & B_* & B_\infty & 0 & 0 \\ 0 & 0 & A_0 & A_* & A_\infty \end{pmatrix} \begin{pmatrix} u \\ \mathbf{y}^t \\ v \end{pmatrix} \doteq 0.$$

These matrices can also be found on [**1**, p. 385]. The ring R_M of definable scalars is an R-ring, because the action of every $r \in R$ on M is itself pp definable, by the formula $(r,-1)\begin{pmatrix} u \\ v \end{pmatrix} \doteq 0$.

Consider an R-ring $f : R \to S$ as a left R-module and suppose that the pp formula $\rho(u,v)$ defines a scalar on ${}_RS$. Then the formula $\exists v\rho(u,v)$ defines in ${}_RS$ the entire endo-submodule S. But this occurs if and only if ${}_RS \models \exists v\rho(1,v)$, which is equivalent to the condition that the system $(A_*, A_\infty)\begin{pmatrix} \mathbf{w}^t \\ v \end{pmatrix} \doteq A_0$ has a solution in S. The other condition is that every solution to the corresponding homogeneous system $(A_*, A_\infty)\begin{pmatrix} \mathbf{w}^t \\ v \end{pmatrix} \doteq 0$ satisfies $v = 0$. Both of these conditions will certainly hold if the matrix (A_*, A_∞) is a square matrix, invertible in S. Cohn [**1**, p. 384] calls these scalars *admissible.*

Because the graph of ρ is an $\mathrm{End}_S(S_S)$-submodule of S^2, it follows that the scalar defined on ${}_RS$ is nothing more than multiplication by $s = \rho(1)$ on the left. Prest (cf. proof of [**16**, Thm 6.1.8]) has characterized the elements $s \in S$ whose left action is a definable scalar on ${}_RS$ as precisely those elements for which $s \otimes 1 = 1 \otimes s$ in $S \otimes_R S$. Schofield [**21**, Lemma 7.15] has shown that this is the R-ring given by the dominion of f, $R \to \mathrm{dom}(f) \subseteq S$. The *dominion* of an R-ring $f : R \to S$ is defined to be the maximal subring of S with the property that two ring morphisms from S that agree on the image of f agree on $\mathrm{dom}(f)$. Inspection of the matrices above shows that the admissible scalars form an R-subring $R^f(S)$ of the dominion; this is the *rational closure* [**1**, p. 382] of f.

THEOREM 3.1. *Let R be a ring and $f : R \to S = \prod_{\mathcal{P}} R^{\mathrm{ab}}/\mathcal{P}$ the product of all epic R-fields. The dominion of f is given by the universal abelian regular ring $R \to R^{\mathrm{ab}} \subseteq \prod_{\mathcal{P}} R^{\mathrm{ab}}/\mathcal{P}$.*

PROOF. The R-ring $R \to R^{\mathrm{ab}}$ is epic, so necessarily lies in the dominion of $R \to S = \prod_{\mathcal{P}} R^{\mathrm{ab}}/\mathcal{P}$. On the other hand, suppose that $s \in \mathrm{dom}(f)$. There is a pp formula $\rho(u,v)$ in the language $\mathcal{L}(R)$ that defines in ${}_RS$ the action given by left multiplication by s. This implies that ${}_RS \models \rho(1,s)$ and s is the unique element of S for which this satisfaction holds. In particular, ${}_RS \models \exists v\ \rho(1,v)$. The result follows if we can establish that R^{ab} is a pure left R-submodule of S. For then ${}_RR^{\mathrm{ab}} \models \exists v\ \rho(1,v)$, and the witness must be s. This puts s in R^{ab}. But the inclusion $R^{\mathrm{ab}} \subseteq {}_RS$ may be seen as an inclusion in R^{ab}-Mod, where it is necessarily a pure monomorphism, because R^{ab} is von Neumann regular ring. Since $R \to R^{\mathrm{ab}}$ is epic, it is also a pure monomorphism in R-Mod. □

Theorem 3.1 gives a characterization of the elements of $R^{\mathrm{ab}} \subseteq \prod\{\Delta \mid R \to \Delta \text{ epic}\}$. A pp formula $\rho(u,v)$ defines a scalar on the product $\prod \Delta$ if and only if it defines a scalar $\rho(\Delta) : \Delta \to \Delta$ on every epic R-field. It follows that the elements of R^{ab} are those of the form $(\rho(\Delta)(1_\Delta))_\Delta$, where ρ is such a pp formula.

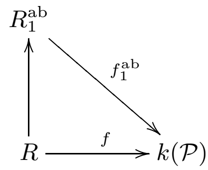
4. The commutative case

Let us show that if R is a commutative ring, then $R_1^{\mathrm{ab}} = \hat{R}$ is the *commutative* universal regular R-algebra.

LEMMA 4.1. *If C is a commutative ring and R is a C-algebra, then so is every epic R-ring $f : R \to S$.*

PROOF. Because $f : R \to S$ is epic, the dominion of f is S, and the left action of every $s \in S$ is a definable scalar on ${}_RS$. The graph of this action is closed under the endomorphisms of ${}_RS$. It follows that the left action of s commutes with that of C. □

The lemma implies that if R is commutative, then every epic R-field $f : R \to k$ is commutative. This is easy to see, because the center of k is a field containing the image of f. In fact, the kernel of f is a prime ideal $\mathcal{P} \subseteq R$ and $k = k(\mathcal{P})$ is the field of fractions of $R/\mathcal{P}$. This implies that $E_1(f) = k(\mathcal{P})$ and that the extension



is onto.

Theorem 4.2. *If R is a commutative ring, then $R_1^{\mathrm{ab}} = \hat{R}$ is the universal commutative regular R-algebra.*

Proof. To show that R_1^{ab} is commutative, first observe that if R is commutative, then so is every epic R-ring of the form $R \to R\{\overline{r}\}$ constructed in §1. This is clear, because the lemma implies that $R\{\overline{r}\}$ is an R-algebra generated by the element $\overline{r}$. So if we adjoin one group inverse at a time, $R \to R\{\overline{r}\} \to R\{\overline{r}\}\{\overline{s}\}$, a commutative R-algebra is obtained. The universal property of $R\{\overline{r}\}\{\overline{s}\}$ ensures that there is morphism $R\{\overline{r}\}\{\overline{s}\} \to R_1^{\mathrm{ab}}$ that respects the group inverses of r and s. It follows that any two group inverses of elements in R commute in R_1^{ab}. As these group inverses generate the R-algebra R_1^{ab}, it must be commutative. The universal properties of $R \to R_1^{\mathrm{ab}}$ and $R \to \hat{R}$ now ensure that these two R-algebras are isomorphic. To see that $R_1^{\mathrm{ab}} = \hat{R}$ is regular, look at the proof of [**10**, Prop 4.2] which implies that every element of $\hat{R}$ has a reflexive inverse. It works even if R is not reduced, as the hypothesis there assumes. $\square$

Theorem 4.2 implies that the iterated noncommutative variant of Olivier's construction is a generalization of his commutative construction. **It justifies the notation $R \to \hat{R} := R^{\mathrm{ab}}$ that we will use from now on.**

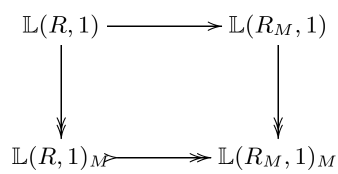
5. The lattice of pp definable subgroups

In general, we consider two pp formulae $\varphi(u)$ and $\psi(u)$ equivalent if for every left R-module ${}_RM$, $\varphi(M) = \psi(M)$, or, what amounts to the same thing, that the sentence $\varphi(u) \leftrightarrow \psi(u)$ is provable from the axioms for a left R-module. The lattice of pp formulae $\psi(u)$ in one variable is denoted by $\mathbb{L}(R, 1)$. This is a modular lattice where the infimum is given by $(\psi_1 \wedge \psi_2)(u) = \psi_1(u) \wedge \psi_2(u)$ and the supremum by

$$(\psi_1 + \psi_2)(u) = \exists v\ [\psi_1(u - v) \wedge \psi_2(v)].$$

There is a maximum element $u \doteq u$ in $\mathbb{L}(R, 1)$ and a minimum element $u \doteq 0$. A morphism $f : R \to S$ of rings induces a morphism of languages $\mathcal{L}(f) : \mathcal{L}(R) \to \mathcal{L}(S)$, which induces the obvious morphism $\mathbb{L}(f, 1) : \mathbb{L}(R, 1) \to \mathbb{L}(S, 1)$ of pp lattices. Similar considerations hold for the lattice $\mathbb{L}(R, n)$ of pp formulae in n variables.

If ${}_RM$ is a left R-module, the pp definable subgroups $\psi(M) \subseteq M$ represent the elements of the quotient lattice $\mathbb{L}(R, 1) \twoheadrightarrow \mathbb{L}(R, 1)_M,\ \psi(u) \mapsto \psi(M)$, modulo the congruence given by equivalence relative to M, $\varphi(M) = \psi(M)$. If $R \to R_M$ is the ring of definable scalars on M, then M can be considered as a left module over R_M and every pp definable subgroup $\psi(M)$ in $\mathbb{L}(R_M, 1)_M$ is pp definable over R. This is expressed by the commutative diagram



of bounded modular lattices, where the bottom horizontal arrow is an isomorphism. These considerations hold even if the number of variables is $n \geq 1$ and the case $n = 2$ includes an isomorphism $R_M \to (R_M)_M$ of the respective R-rings of definable scalars.

If R is a regular ring, then every pp formula $\psi(u)$ in one variable is equivalent to one of the form $e|u$ for some idempotent $e \in R$. Furthermore, the localization $\mathbb{L}(R,1) \to \mathbb{L}(R,1)_R$, $e|u \mapsto eR$ is an isomorphism. The lattice $\mathbb{L}(R,1)$ is therefore isomorphic to the lattice of principal right ideals of R. A bounded complemented modular lattice $\mathbb{L}$ is said to be *coordinatized* by a given von Neumann regular ring R, if $\mathbb{L}$ is isomorphic to the lattice $\mathbb{L}(R,1)_R$ of principal right ideals of the ring R. More generally, if M is a left R-module over a regular ring, then the vertical arrow in the diagram

$$\begin{array}{ccc} \mathbb{L}(R,1) & \longrightarrow & \mathbb{L}(R_M,1) \\ & \searrow & \downarrow \\ & & \mathbb{L}(R,1)_M \end{array}$$

is an isomorphism of lattices, so that $R_M = R/\operatorname{ann}(M)$ coordinatizes $\mathbb{L}(R,1)_M$. The following proposition provides a kind of converse.

Proposition 5.1. *Let R be an associative ring and M a left R-module for which $\mathbb{L}(R,1)_M$ is complemented. Then the vertical arrow in the diagram*

$$\begin{array}{ccc} \mathbb{L}(R,1) & \longrightarrow & \mathbb{L}(R_M,1) \\ & \searrow & \downarrow \\ & & \mathbb{L}(R,1)_M \end{array}$$

is an isomorphism and $R \to R_M$ is a regular epic R-ring that coordinatizes $\mathbb{L}(R,1)_M$.

Proof. This is proved in [**7**, Prop 9] or in [**16**, p. 28], the salient point being that if $\psi(u)$ is the complement of $\varphi(u)$ in $\mathbb{L}(R,1)_M$, i.e., that $\varphi(M) \oplus \psi(M) = M$, then there is a pp definable projection $E(\varphi) \in R_M$ onto $\varphi(M)$ parallel to $\psi(M)$; it is defined by the pp formula $E(\varphi)(u,v) = \varphi(v) \wedge \psi(u-v)$. ☐

Abelian regular rings can also be characterized by the property that every element in the lattice $\mathbb{L}(R,1)_R$ of principal right ideals has a unique complement.

Theorem 5.2. *The following are equivalent for a left R-module ${}_RM$:*

(1) $M \in \hat{R}$-Mod;
(2) *the R-ring $R \to R_M$ of definable scalars is abelian regular; and*
(3) *every pp definable subgroup $\psi(M) \in \mathbb{L}(R,1)_M$ has a unique complement.*

Proof. (1) $\Leftrightarrow$ (2). If M belongs to $\hat{R}$-Mod, then $R_M = \hat{R}_M = \hat{R}/\operatorname{ann}_{\hat{R}}(M)$ is abelian regular. Conversely, if $R \to R_M$ is abelian regular, then it factors through the universal abelian regular R-ring $R \to \hat{R}$, and M acquires the structure of a left $\hat{R}$-module. (2) $\Leftrightarrow$ (3). If the R-ring $R \to R_M$ of definable scalars is abelian regular, the considerations above imply that $\mathbb{L}(R,1)_M = \mathbb{L}(R_M,1)_M$, which is coordinatized by the abelian regular ring R_M. Conversely, if the lattice $\mathbb{L}(R,1)_M$ is complemented, then R_M is a regular ring that coordinatizes it. If these complements are unique, then R_M must be abelian. ☐

Important examples of complemented lattices have been explored in the context of the universal enveloping algebra U associated to the Lie algebra $\mathrm{sl}(2,k)$ (of all 2×2 traceless matrices with entries in a field k). Prest and the second author [**12**] investigated the infinite-dimensional Verma modules over U and proved that the ring of definable scalars is (non-abelian) regular. The second author and Macintyre [**11**] examined the decidability issue for the theory of finite dimensional U-modules and brought out a connection to fundamental problems in the diophantine geometry of curves. Finally, Herzog's analysis [**7**] of the theory of finite dimensional representations of U was generalized by the authors [**8**] for quantized universal enveloping algebras U_q, (often called quantum groups) when the parameter q is not a root of unity. Coming back to our framework, note that condition (2) of Theorem 5.2 can be used to axiomatize the elementary class $\hat{R}$-Mod $\subseteq R$-Mod in a way that is analogous to Propositions 1.2 and 2.1.

Corollary 5.3. *A module M belongs to $\hat{R}$-Mod if and only if for every definable scalar $\rho(u,v) \in R_M$,*

$$M \models \{\forall u\ \exists v, w\ [(u \doteq v + w) \wedge \exists u'\ \rho(u', v) \wedge \rho(w, 0)]\} \wedge \forall u\ [(\rho(u, 0) \wedge \exists v \rho(v, u)) \to u \doteq 0].$$

Proof. The sentence expresses the condition that $M = \rho M \oplus \mathrm{ann}_M(\rho)$. Let $E \in R_M$ be the pp definable projection onto $\rho(M)$ parallel to $\mathrm{ann}_M(\rho)$. It follows that $EM = \rho M$, and therefore $E\rho = \rho$, and that $\rho(1 - E) = 0$, or $\rho = \rho E$. Let us note that the restriction $\rho|_{EM} : EM \to EM$ is invertible: for every $y \in \rho M = EM$, there is a unique $x \in \rho M = EM$ such that $\rho x = y$. For if $x' \in EM$ were another, then $\rho(x - x') = 0$ and $x - x' = E(x - x')$. Because $\mathrm{ann}_M(\rho)\ \cap\ EM = 0$, we have that $x - x' = 0$. This observation shows that the group inverse of $\rho \in R_M$ is defined by the pp formula

$$\overline{\rho}(u, v) := \exists u', v'\ [\rho(v, u') \wedge \rho(u - u', 0) \wedge \rho(v', v)],$$

because its restriction to EM is the inverse of $\rho|_{EM} : EM \to EM$, while the restriction to $(1 - E)M = \mathrm{ann}_M(\rho)$ is 0. □

The sentences given in Corollary 5.3 form a system of axioms for $\hat{R}$-Mod, because the condition that $\rho(u, v)$ define a scalar on ${}_RM$ is expressible by (†) in the language $\mathcal{L}(R)$. A nicer system of axioms could be given if we can find, for every φ an explicit form for a pp formula $\varphi^\perp$ that defines in M the unique complement of $\varphi(M)$ in $\mathbb{L}(R, 1)_M$. The axiom schema would then be of the form $\varphi(M) \oplus \varphi^\perp(M) = M$.

It would suffice to find such a $\varphi^\perp$ for the R-ring $R \to T = \prod_{\mathcal{P}} \hat{R}/\mathcal{P}$, because $\mathbb{L}(R, 1)_T$ is coordinatized by $\hat{R}$. Given a pp formula $\varphi(u)$, the task therefore is to find a complementary pp formula $\varphi^\perp(u)$ such that for every epic R-field Δ, $\varphi^\perp(\Delta) = \Delta$ if and only if $\varphi(\Delta) = 0$. More concretely, Condition (3) of Theorem 5.2 implies that for every nonhomogeneous system $A\mathbf{v}^t \doteq A_0$, there exists a complementary system $B\mathbf{w}^t \doteq B_0$ such that for every epic R-field Δ, the first has a solution in Δ if and only if the second does not. It would be interesting to find an explicit form of the second, obtained constructively from the first, because it could be used as the starting point of the present theory.

This is possible when the ring R is commutative. We make implicit use of the Prest dual [**16**, §1.3.1] to see that a complementary system is given by the nonhomogeneous system $A^t\mathbf{w}^t \doteq 0 \wedge A_0^t\mathbf{w}^t \doteq 1$. For if k is an epic R-field, then the system $A\mathbf{v}^t \doteq A_0$ has a solution in k if and only if A_0 belongs to the column space of $A = (A_1, \dots, A_k)$, $A_0 = \sum_{i=1}^k c_i A_i$, which holds if and only if $\mathbf{w}A = 0$ implies $\mathbf{w}A_0 = 0$. The complementary system $\mathbf{w}A = 0 \wedge \mathbf{w}A_0 = 1$ is thus obtained, but the scalars of R act on the right.

6. The constructible Cohn spectrum

Denote by Spec(R) the Cohn spectrum [**1**, p. 410] of a ring R. The points of Spec(R) are the epic R-fields $R \to \Delta$, with a basis of quasi-compact open subsets given by

$$\mathcal{O}(A) := \{\Delta \mid A \text{ is invertible in } \Delta\},$$

as A ranges over the square matrices with entries in R. If R is commutative, then the points of Spec(R) are in bijective correspondence with the Zariski spectrum $\mathcal{P} \mapsto k(\mathcal{P})$. If $r \in R$, then $\mathcal{O}(r) = \{\mathcal{P} \mid r \notin \mathcal{P}\}$ corresponds to a basic Zariski open subset, so that the Cohn topology is at least as fine as the Zariski topology. On the other hand, if A is a square matrix over R, then $\mathcal{O}(A) = \mathcal{O}(\det(A))$ so that the Cohn topology is no finer than the Zariski topology. It follows that the Cohn spectrum $\mathrm{Spec}(R)$ is a generalization of the Zariski spectrum to the noncommutative case.

Like the Zariski spectrum, the Cohn spectrum satisfies the axioms [**1**, Exercise 13, p. 141] for a spectral topological space, in the sense of Hochster [**9**]. The rule $R \mapsto \mathrm{Spec}(R)$ is a contravariant functor from the category of rings to that of spectral topological spaces: if $f : R \to S$ is a morphism of rings, and $S \to \Delta$ is an epic S-field, then $R \to S \to \Delta$ is an R-field, whose image generates the epic R-field $R \to \mathrm{Spec}(f)(\Delta)$. The map $\mathrm{Spec}(f) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is continuous; if A is a square matrix over R, then $\mathrm{Spec}(f)^{-1}\mathcal{O}(A) = \mathcal{O}(f(A))$ in $\mathrm{Spec}(S)$.

There is another notion of spectrum $\mathrm{JSpec}(R)$ whose points are the prime ideals of the ring R, equipped with the Jacobson topology: the closed sets are given by $V(I) := \{\mathcal{P} \mid \mathcal{P} \supseteq I\}$, as I ranges over the ideals of R. If R is abelian regular, then the epic R-rings correspond to maximal ideals, $\mathcal{P} \mapsto R/\mathcal{P}$ and every prime ideal is maximal, so there is a bijective function $\mathrm{ann} : \mathrm{Spec}(R) \to \mathrm{JSpec}(R)$, $\Delta \mapsto \mathrm{ann}(\Delta)$. By [**3**, Prop 3.12], the topological space $\mathrm{JSpec}(R)$ is a totally disconnected compact space, with a clopen basis given by $\mathcal{O}(e) = \{\mathcal{P} \mid e \notin \mathcal{P}\}$, where e ranges over the (central) idempotent elements of R. This implies that ann is a continuous bijection from a quasi-compact space to a Hausdorff space. By an exercise in Topology, it must be a homeomorphism, and we have established the following.

Proposition 6.1. *If R is abelian regular, then the Cohn spectrum* $\mathrm{Spec}(R)$ *is a totally disconnected compact space with a clopen basis given by* $\mathcal{O}(e)$, *as e ranges over the idempotent elements in R.*

Because the Cohn spectrum is spectral, we may introduce the patch topology on it, as in [**9**]. This space is the *constructible* Cohn spectrum of R, denoted by $\widehat{\mathrm{Spec}}(R)$; an open basis is given by the boolean combinations of quasi-compact open subsets of $\mathrm{Spec}(R)$.

Theorem 6.2. (cf. [**22**, Thm 1]) *The universal abelian regular R-ring $R \to \hat{R}$ induces a homeomorphism* $\mathrm{Spec}(\hat{R}) \to \widehat{\mathrm{Spec}}(R)$, $\mathcal{P} \mapsto \hat{R}/\mathcal{P}$, *of constructible Cohn spectra.*

Proof. The induced function $\mathcal{P} \mapsto \hat{R}/\mathcal{P}$ is a bijection, by Corollary 2.3. The preimage in $\mathrm{Spec}(\hat{R})$ of a basic open subset $\mathcal{O}(A)$ in $\mathrm{Spec}(R)$ is quasi-compact, and therefore clopen in $\mathrm{Spec}(\hat{R})$, which implies that $\mathrm{Spec}(\hat{R}) \to \widehat{\mathrm{Spec}}(R)$, $\mathcal{P} \mapsto \hat{R}/\mathcal{P}$, is a bijective continuous function between compact Hausdorff spaces. By that same exercise from Topology, it is a homeomorphism. □

7. The Ziegler spectrum

The first author noticed [**6**, Thm 9.5] that the constructible Cohn spectrum $\widehat{\mathrm{Spec}}(R)$ of a ring R may be identified with the closed subspace $\mathrm{Zg}_1(R)$ of endosimple points of the Ziegler spectrum $\mathrm{Zg}(R)$. In this section, we observe that this embedding is natural, induced by the universal abelian regular R-ring.

The Ziegler spectrum $\mathrm{Zg}(R)$ of a ring R is the space [**16, 23**] whose points are given by indecomposable pure injective left R-modules, topologized so that a basis of open subsets is given by

$$\mathcal{O}(\varphi/\psi) := \{U \in \mathrm{Zg}(R) \mid \varphi(U)/\psi(U) \neq 0\},$$

as $\psi \leq \varphi$ range over $\mathbb{L}(R, 1)$. The quasi-compact open subsets of this topology have the form $\mathcal{O}(\varphi/\psi)$, as $\psi \leq \varphi$ range over the various $\mathbb{L}(R, n)$, $n \geq 1$.

A module ${}_RU$ is *endosimple* if it is simple as a module over its endomorphism ring $\mathrm{End}_R U$. An endosimple module U is pure injective [**16**, Cor 4.4.24] so that every such indecomposable module represents a point $U \in \mathrm{Zg}(R)$. For example, every epic R-field $R \to \Delta$ becomes, by restriction of scalars, an indecomposable endosimple left R-module and it is shown in [**6**, §9] how every indecomposable endosimple left R-module arises in this way. The subspace $\mathrm{Zg}_1(R)$ of endosimple points of $\mathrm{Zg}(R)$ forms a closed subset [**16**, Example 5.1.16]. If we endow $\mathrm{Zg}_1(R)$ with the relative subspace topology, then the quasi-compact open subsets of $\mathrm{Zg}_1(R)$ are also closed, for suppose that $\psi \leq \varphi$ in $\mathbb{L}(R,1)$ and $\Delta \in \mathcal{O}(\varphi/\psi)$. Then $\varphi(\Delta) = \Delta$ and $\psi(\Delta) = 0$, which implies that in $Zg_1(R)$,

$$\mathcal{O}(\varphi/\psi)^c = \mathcal{O}(u \doteq u/\varphi(u)) \ \cup \ \mathcal{O}(\psi(u)/u \doteq 0)$$

is also open. Now every quasi-compact open subset is a finite union of such open subsets and therefore has an open complement. It follows that the Ziegler topology restricted to $\mathrm{Zg}_1(R)$ is its own patch topology.

If R is abelian regular, then the points of the Ziegler spectrum are given by the endosimple modules $R/\mathcal{P}$, as $\mathcal{P}$ ranges over the prime, i.e., maximal, ideals. To see that every indecomposable pure injective left R-module is of this form, note that if M is indecomposable, then every idempotent $e \in R$ induces a decomposition $M = eM \oplus (1-e)M$ so that either e or $1-e$ belongs to $\mathrm{ann}(M)$. This implies that $\mathrm{ann}(M) = \mathcal{P}$ is a prime ideal and that $M = R/\mathcal{P}$. Thus $\mathrm{Zg}(R) = \mathrm{Zg}_1(R)$ is totally disconnected.

Proposition 7.1. *If R is abelian regular, then* $\mathrm{Zg}(R) = \mathrm{Zg}_1(R)$ *and the function* $\mathrm{ann} : \mathrm{Zg}(R) \to \mathrm{Spec}(R)$, $\Delta \mapsto \mathrm{ann}(\Delta)$, *is a homeomorphism.*

Proof. The first equality has been established. To prove that ann is a homeomorphism, it suffices to verify that it is continuous, because it is then a bijective continuous function of compact Hausdorff spaces. The preimage of a basic open subset $\mathcal{O}(e) \subseteq \mathrm{Spec}(R)$ consists of those Δ for which $e\Delta = \Delta$. This is the Ziegler open subset $\mathcal{O}(e|u/u \doteq 0)$. □

If $f : R \to S$ is an epic R-ring and $U \in \mathrm{Zg}(S)$ is an indecomposable pure injective left S-module, then the left R-module ${}_RU$ obtained by restriction of scalars is also indecomposable pure injective, ${}_RU \in \mathrm{Zg}(R)$. The function ${}_SU \mapsto {}_RU$ is then a homeomorphic embedding $\mathrm{Zg}(f) : \mathrm{Zg}(S) \to \mathrm{Zg}(R)$ onto a closed subset. These ideas are clearly explained in [**16**, §5.5.1]. For the universal abelian regular R-ring $R \to \hat{R}$, these consideration take on the following form.

Theorem 7.2. *The universal abelian regular R-ring $R \to \hat{R}$ induces a homeomorphism*

$$\widehat{\mathrm{Spec}}(R) = \mathrm{Zg}(\hat{R}) \to \mathrm{Zg}_1(R) \subseteq \mathrm{Zg}(R)$$

from the constructible Cohn spectrum of R to the closed subset of endosimple points in the Ziegler spectrum.

Proof. The only claim that requires a comment is that $\mathrm{Zg}(\hat{R})$ is homeomorphic to the constructible Cohn spectrum. By Proposition 7.1, it is homeomorphic to $\mathrm{Spec}(\hat{R})$, which is homeomorphic to $\widehat{\mathrm{Spec}}(R)$, by Theorem 6.2. □

8. The étale bundle of definable scalars

The theory of the Pierce spectrum [**15**, Thm 4.4] (but see [**2**]) shows how to present an abelian regular ring R as the ring of global sections of a sheaf over $\mathrm{Spec}(R)$. In this section, we indicate how this sheaf is an instance of Prest's notion of the sheaf of locally definable scalars over the Ziegler spectrum $\mathrm{Zg}^*(R)$ - of an arbitrary ring - endowed with the Zariski topology. The universal abelian R-ring $\iota : R \to \hat{R}$ is then presented as the the ring of global sections of the inverse image sheaf along the homeomorphic embedding $\mathrm{Zg}^*(\iota) : \mathrm{Zg}^*(\hat{R}) \to \mathrm{Zg}^*(R)$ of the constructible Cohn spectrum.

The Ziegler spectrum $\mathrm{Zg}(R)$ of a ring R satisfies enough of the axioms of a spectral topological space so that a dual topology may be given, whose basic open subsets are the complements $\mathcal{O}(\varphi/\psi)^c$, $\psi \leq \varphi \in \mathbb{L}(R,n)$, of the quasi-compact open subsets of $\mathrm{Zg}(R)$. This is explained nicely in [**16**, §5.6] and this topology on $\mathrm{Zg}(R)$ is called the Zariski topology and denoted by $\mathrm{Zg}^*(R)$. In general, it is not so clear what the quasi-compact open subsets of $\mathrm{Zg}^*(R)$ are, but it is obvious that every clopen Ziegler subset is also clopen in the the Zariski topology. If R is abelian regular, Proposition 7.1 implies that $\mathrm{Zg}^*(R) = \mathrm{Zg}(R)$.

If ${}_RU \in \mathrm{Zg}(R)$, then the endomorphism ring $\mathrm{End}_R\, U$ is local. If R is commutative, there is a continuous function $\mathcal{P} : \mathrm{Zg}^*(R) \to \mathrm{Spec}(R)$, given by

$$U \mapsto \mathcal{P}(U) := \{r \in R \mid r : U \to U \text{ is not invertible}\}.$$

To verify continuity, let $\mathcal{O}(r) = \{\mathcal{P} \mid r \notin \mathcal{P}\} \subseteq \mathrm{Spec}(R)$ be a basic open subset and note that

$$\mathcal{P}^{-1}(\mathcal{O}(r)) = \mathcal{O}(u \doteq u/\exists v\ (rv \doteq u))^c \ \cap\ \mathcal{O}(rv \doteq 0/v \doteq 0)^c$$

is open in $\mathrm{Zg}^*(R)$. This is a feature of the Zariski topology that the Ziegler topology does not enjoy.

More generally, let $\rho(u, v)$ be a pp formula in two variables. Then ρ defines a scalar on every point in the Zariski open subset

$$\mathcal{O}_{\mathrm{Zar}}(\rho(u, v)) := \mathcal{O}(u \doteq u/\exists v\ \rho(u, v))^c \ \cap\ \mathcal{O}(\rho(0, v)/v \doteq 0)^c$$

of $\mathrm{Zg}^*(R)$. Thus $U \in \mathcal{O}_{\mathrm{Zar}}(\rho(u, v))$ if and only if $\rho(U) \in R_U$. For example, if $r \in R$, then $\mathcal{O}_{\mathrm{Zar}}(rv \doteq u)$ is $\mathcal{P}^{-1}(\mathcal{O}(r))$, which consists of those indecomposable pure injective modules ${}_RU$ on which r^{-1} is a definable scalar. Let $\mathrm{Bun}(R) := \dot{\bigcup}\, \{R_U \mid U \in \mathrm{Zg}^*(R)\}$ be the disjoint union of the R_U and define $p : \mathrm{Bun}(R) \to \mathrm{Zg}^*(R)$ to be the function whose fiber over U is $p^{-1}\{U\} = R_U$. There is a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Bun}(R) \\ & \overset{\mathrm{Ev}(\rho)}{\nearrow} & \big\downarrow p \\ \mathcal{O}_{\mathrm{Zar}}(\rho(u, v)) & \xrightarrow{\ \subseteq\ } & \mathrm{Zg}^*(R), \end{array}$$

where $\mathrm{Ev}(\rho)(U) := \rho(U) \in R_U$. The function p acquires the structure of a *topological bundle* [**13**, §II.4] over $\mathrm{Zg}^*(R)$ if we endow $\mathrm{Bun}(R)$ with the coarsest topology for which p and all the maps $\mathrm{Ev}(\rho) : \mathcal{O}_{\mathrm{Zar}}(\rho(u, v)) \to \mathrm{Bun}(R)$, $\rho(u, v) \in \mathbb{L}(R, 2)$, are open continuous.

PROPOSITION 8.1. *The topological bundle $p : \mathrm{Bun}(R) \to \mathrm{Zg}^*(R)$ is an étale bundle, with a subbasis of open subsets for* $\mathrm{Bun}(R)$ *given by the images* $\mathrm{Im}\ \mathrm{Ev}(\rho)$, $\rho(u, v) \in \mathbb{L}(R, 2)$, *and preimages $p^{-1}(\mathcal{O})$, as $\mathcal{O}$ ranges over a basis for $\mathrm{Zg}^*(R)$. A basis of open subsets is given by finite intersections of these: $\bigcap_i$ $\mathrm{Im}\ \mathrm{Ev}(\rho_i) \cap p^{-1}(\mathcal{O})$.*

PROOF. Let us first check that the given subsets are open. If $\mathrm{Ev}(\rho) : \mathcal{O}_{\mathrm{Zar}}(\rho) \to \mathrm{Bun}(R)$ is open, then its image must be open and if $p : \mathrm{Bun}(R) \to \mathrm{Zg}^*(R)$ is continuous, then all of the preimages $p^{-1}(\mathcal{O})$, with $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ are open.

Conversely, let us show that for any topology containing this open basis, the maps $\mathrm{Ev}(\rho)$ and p are open continuous. All the subsets $p^{-1}(\mathcal{O})$ are open in $\mathrm{Bun}(R)$, so that p is continuous. To see that p is open, let us see that the image under p of a basic open subset $\bigcap_i \mathrm{Im}\ \mathrm{Ev}(\rho)_i \cap p^{-1}(\mathcal{O}) \subseteq \mathrm{Bun}(R)$ is open. To that end, let $\rho(U)$ belong to this basic open subset. It means that $U \in \mathcal{O}$ and that for all i, $\rho(U) = \rho_i(U)$. Consider the Zariski open subset $\mathcal{O}' \subseteq \mathcal{O}$ given by

$$\mathcal{O}' = \bigcap_i\ \mathcal{O}_{\mathrm{Zar}}(\rho_i)\ \cap\ \bigcap_{i<j} \mathcal{O}(u \doteq u/(\rho_i - \rho_j)(u, 0))^c\ \cap\ \mathcal{O}.$$

Then $U \in \mathcal{O}' \subseteq p[\bigcap_i \mathrm{Im}\ \mathrm{Ev}(\rho)_i \cap p^{-1}(\mathcal{O})]$ and $\mathcal{O}'$ is the open subset of $\mathcal{O}$ on which all of the ρ_i's define the same scalar.

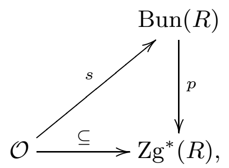
Now let us prove that $\mathrm{Ev}(\rho)$ is continuous. It is injective, so it suffices to prove that for every open subset $\mathrm{Im}\ \mathrm{Ev}(\sigma) \cap p^{-1}(\mathcal{O})$ in the subbasis, $\mathrm{Ev}(\rho)^{-1}[\mathrm{Im}\ \mathrm{Ev}(\sigma) \cap p^{-1}(\mathcal{O})]$ is open in $\mathcal{O}_{\mathrm{Zar}}(\rho)$. Well, the elements of $\mathrm{Im}\ \mathrm{Ev}(\rho) \cap \mathrm{Im}\ \mathrm{Ev}(\sigma) \cap p^{-1}(\mathcal{O})$ are those of the form $\rho(U)$, where $U \in \mathcal{O}$ and $\rho(U) = \sigma(U)$. As above, the preimage of that is the open subset

$$\mathcal{O}_{\mathrm{Zar}}(\rho) \cap \mathcal{O}_{\mathrm{Zar}}(\sigma) \cap \mathcal{O}(u \doteq u/(\rho - \sigma)(u, 0))^c \cap \mathcal{O}.$$

That $\mathrm{Ev}(\rho)$ is open follows from the fact that its image is open in $\mathrm{Bun}(R)$ and that $p|_{\mathrm{Im\ Ev}(\rho)}$ is a continuous inverse.

Finally, recall that a topological bundle $p : E \to X$ is *étale* if for every point $e \in E$, there is an open neighborhood $e \in \mathcal{O} \subseteq E$ such that the restriction $p|_{\mathcal{O}} : \mathcal{O} \to X$ is a homeomorphism onto an open subset $p(\mathcal{O}) \subseteq X$. Such a neighborhood$(\mathcal{O}, e)$ is called *étale.* Because the section $\mathrm{Ev}(\rho)$ on $\mathcal{O}_{\mathrm{Zar}}(\rho)$ is a homeomorphism onto an open image, it is clear that if $\rho(U) \in R_U$, then the image $(\mathrm{Im\ Ev}(\rho), \rho(U))$ is an étale neighborhood. □

There is a natural correspondence [**13**, Cor II.6.3] between étale bundles and sheaves over a topological space B. For the étale bundle $p : \mathrm{Bun}(R) \to \mathrm{Zg}^*(R)$, the corresponding sheaf is the *sheaf* Def_R *of sections.* It assigns to an open subset $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ the R-ring $R \to \mathrm{Def}_R(\mathcal{O})$ of continuous maps $s : \mathcal{O} \to \mathrm{Bun}(R)$ for which the diagram



commutes. It is clear that the ring $\mathrm{Def}_R(\mathrm{Zg}^*(R))$ of global sections carries an R-ring structure, because the action of every $r \in R$ gives a global scalar. The R-ring structure on $\mathrm{Def}_R(\mathcal{O})$ is then inherited via the restriction map $R \to \mathrm{Def}_R(\mathrm{Zg}^*(R)) \to \mathrm{Def}_R(\mathcal{O})$.

A section $s \in \mathrm{Def}_R(\mathcal{O})$ is *definable* if there is a pp formula $\rho(u, v)$ such that $\mathcal{O} \subseteq \mathcal{O}_{\mathrm{Zar}}(\rho)$ and $s = \mathrm{Ev}(\rho)|_{\mathcal{O}}$. Prest [**16**, Chs 6 & 14] defines the notion of a *presheaf-on-a-basis* of definable scalars, which assigns to a basic open subset $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ the R-ring of definable sections on $\mathcal{O}$. The sheaf Def_R on $\mathrm{Zg}^*(R)$ is the sheafification [**13**, Thm 11.2.3] of this presheaf; the following proposition shows that it is the *sheaf of locally definable scalars* on $\mathrm{Zg}^*(R)$.

Proposition 8.2. *If $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ is open, then $s \in \mathrm{Def}_R(\mathcal{O})$ if and only if for every point $U \in \mathcal{O}$, there is an open neighborhood $U \in \mathcal{O}' \subseteq \mathcal{O}$ such that $s|_{\mathcal{O}'}$ is definable.*

Proof. The collation property of the sheaf Def_R ensures that every section s of the kind described belongs to $\mathrm{Def}_R(\mathcal{O})$. On the other hand, let $s \in \mathrm{Def}_R(\mathcal{O})$ and pick $\rho(U) \in R_U$ in the image $\mathrm{Im}\ s$. The intersection of an étale neighborhood $(\mathrm{Im\ Ev}(\rho), \rho(U))$ with the image $\mathrm{Im}\ s$ of s is itself an étale neighborhood of $\rho(U)$. Then $\mathcal{O}' = p(\mathrm{Im\ Ev}(\rho) \cap \mathrm{Im}\ s) \subseteq \mathcal{O}$ is a neighborhood such that $s|_{\mathcal{O}'} = \mathrm{Ev}(\rho)|_{\mathcal{O}'}$, as required. □

If the ring R is abelian regular, then, according to the next proposition, the sheaf Def_R over $\mathrm{Zg}^*(R)$ coincides with the sheaf Γ over $\mathrm{Spec}(R)$ introduced in [**2**, **15**].

Proposition 8.3. *If R is abelian regular, then* $\mathrm{Bun}(R) = \dot{\bigcup} \{R/\mathcal{P} \mid \mathcal{P} \in \mathrm{Spec}(R)\}$, *and for a quasi-compact open subset $\mathcal{O}(e) \subseteq \mathrm{Zg}^*(R)$,*

$$\mathrm{Def}_R(\mathcal{O}(e)) = eRe.$$

In particular, the ring R is represented as the R-ring of global sections $R \to \mathrm{Def}_R(\mathrm{Zg}^*(R))$.

Proof. If R is abelian regular, then $\mathrm{Zg}(R) = \mathrm{Zg}^*(R)$ is the space whose points are the division rings $\Delta = R/\mathcal{P}$, $\mathcal{P} \in \mathrm{Spec}(R)$, and a basis of quasi-compact open subsets is given by $\mathcal{O}(e) = \{\Delta \mid e\Delta = \Delta\}$, with $e \in R$ idempotent. Each $R \to \Delta$ is an epic R-ring, so that its dominion is given by $R_\Delta = \Delta$; the topological space $\mathrm{Bun}(R)$ is therefore the disjoint union of these Δ.

Let us first show that the statement holds for every definable section $s \in \mathrm{Def}_R(\mathcal{O}(e))$. So suppose that $\rho(u,v)$ defines a scalar on every division ring $\Delta = R/\mathcal{P}$ such that $e\Delta = \Delta$. Then it defines a definable scalar on the R-ring $f : R \to \prod_{e\Delta=\Delta} \Delta$, and hence belongs to the dominion of f. The kernel of f is $(1-e)R$, so that the dominion of f is the dominion of $eRe = R/(1-e)R \to \prod_{e\Delta=\Delta} \Delta$. But $\mathrm{Spec}(eRe) = \mathcal{O}(e)$, so that the claim follows from Theorem 3.1.

The foregoing considerations imply that every definable section $s \in \mathrm{Def}_R(\mathcal{O}(e))$ extends to a global section of the form ere. This global section ere has the property that $ere|_{\mathcal{O}(e)} = s$, while $ere|_{\mathcal{O}(1-e)} = 0$. If $t \in \mathcal{O}(e)$, then Proposition 8.2 implies that there is a finite partition $\mathcal{O}(e) = \dot{\bigcup}_i \mathcal{O}(e_i)$ and that $t|_{\mathcal{O}(e_i)}$ is the restriction of a global scalar $e_i r_i e_i \in e_i R e_i$. Then t is the restriction of $\sum_i e_i r_i e_i \in eRe$ to $\mathcal{O}(e)$, as required. □

If $f : R \to S$ is an epic R-ring, then the induced homeomorphic embedding $\mathrm{Zg}(f) : \mathrm{Zg}(S) \to \mathrm{Zg}(R)$ is also continuous with respect to the Zariski topology $\mathrm{Zg}(f) = \mathrm{Zg}^*(f) : \mathrm{Zg}^*(S) \to \mathrm{Zg}^*(R)$. The action of every element $s \in S$ on a left S-module ${}_SM$ is a definable scalar over R. So if $U \in \mathrm{Zg}^*(S)$ is an indecomposable pure injective, then $S_U = R_U$ and we get the obvious map $\mathrm{Bun}(f) : \mathrm{Bun}(S) \to \mathrm{Bun}(R)$ of étale bundles.

PROPOSITION 8.4. *The morphism of étale bundles given by*

$$\begin{array}{ccc} \mathrm{Bun}(S) & \xrightarrow{\mathrm{Bun}(f)} & \mathrm{Bun}(R) \\ \downarrow{\scriptstyle p_S} & & \downarrow{\scriptstyle p_R} \\ \mathrm{Zg}^*(S) & \xrightarrow{\mathrm{Zg}^*(f)} & \mathrm{Zg}^*(R), \end{array}$$

where $p_S = p_R|_{\mathrm{Bun}(S)}$, is a pullback diagram.

PROOF. It suffices to verify that the topology on $\mathrm{Bun}(S) \subseteq \mathrm{Bun}(R)$ is the relative subspace topology, so we just need to check that an open subset from the subbasis of $\mathrm{Bun}(S)$ given in Proposition 8.1 is open in the relative subspace topology. The topology on $\mathrm{Zg}^*(S) \subseteq \mathrm{Zg}^*(R)$ is given by the relative subspace topology, so that the Zariski open subsets of $\mathrm{Zg}^*(S)$ have the form $\mathcal{O} \cap \mathrm{Zg}^*(S)$, where $\mathcal{O} \subseteq \mathrm{Zg}^*(R)$ is Zariski open. Together with the fact that $p_S = p_R|_{\mathrm{Bun}(S)}$, these considerations imply

$$p_S^{-1}(\mathcal{O} \cap \mathrm{Zg}^*(S)) = p_R^{-1}(\mathcal{O}) \cap \mathrm{Bun}(S).$$

The fact that the action of S is definable over R, implies that if $\rho(u,v)$ is a pp formula over S, then there is a pp formula $\rho'(u,v)$ over R such that $\mathcal{O}_{\mathrm{Zar}}(\rho) = \mathcal{O}_{\mathrm{Zar}}(\rho') \cap \mathrm{Zg}^*(S)$ and the section $\mathrm{Ev}(\rho) : \mathcal{O}_{\mathrm{Zar}}(\rho) \to \mathrm{Bun}(S)$ is the restriction of $\mathrm{Ev}(\rho') : \mathcal{O}_{\mathrm{Zar}}(\rho') \to \mathrm{Bun}(R)$ to $\mathrm{Zg}^*(S)$. It follows that $\mathrm{Im}\ \mathrm{Ev}(\rho) = \mathrm{Im}\ \mathrm{Ev}(\rho') \cap \mathrm{Bun}(S)$. □

If $f : R \to S$ is an epic R-ring, then the correspondence [**13**, Cor II.6.3] between sheaves and topological bundles, implies that the sheaf of locally definable scalars on $\mathrm{Zg}^*(S)$ is given by the pullback along $\mathrm{Zg}^*(f)$ of the sheaf of locally definable scalars on $\mathrm{Zg}^*(R)$. This is also referred to as the inverse image sheaf [**13**, §II.9],

$$\mathrm{Def}_S = [\mathrm{Zg}^*(f)]^*(\mathrm{Def}_R).$$

Let us combine the contents of the previous two propositions for the universal abelian regular R-ring.

THEOREM 8.5. *Let R be a ring with universal abelian regular R-ring $R \to \hat{R}$. The sheaf $\mathrm{Def}_{\hat{R}}$ of locally definable scalars over the constructible Cohn spectrum $\widehat{\mathrm{Spec}}(R) = \mathrm{Zg}^*(\hat{R})$ is obtained by pullback of the sheaf Def_R along the homeomorphic embedding $\mathrm{Zg}^*(\hat{R}) \to \mathrm{Zg}^*(R)$. It has the property that for every quasi-compact open subset $\mathcal{O}(e)$, $\mathrm{Def}_{\hat{R}}(\mathcal{O}(e)) = e\hat{R}e$.*

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