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Pure projective modules over non-singular serial rings

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Abstract. We characterize non-singular serial rings possessing a pure projective module which is not a direct sum of finitely presented modules. As a consequence we obtain that every pure projective module over a non-singular serial ring with Krull dimension is a direct sum of finitely presented modules.

1. Introduction

A module is said to be uniserial if the lattice of its submodules is a chain. A module is called serial if it is a direct sum of uniserial modules. A ring is serial if its regular (left and right) modules are serial. A theorem of Drozd and Warfield (see Theorem 3.1) shows that finitely presented modules over serial rings have a reasonable structure. Indeed, every finitely presented module over a serial ring is a direct sum of cyclically presented uniserial modules. A natural question, posed by Warfield, was whether the Krull-Remak-Schmidt theorem holds in the category of finitely presented modules over a serial ring, i.e., whether every finitely presented module over a serial ring has up to isomorphism only one decomposition as a direct sum of indecomposable modules. Facchini [**3**] showed that the answer is 'no' in general. However, Facchini's work gives tools to understand when the Krull-Remak-Schmidt holds in the category of finitely presented modules over a serial ring. Puninski [**9**, Question 2.24] asked for a characterization of such serial rings. To answer this question it is enough to translate Facchini's conditions on homomorphisms between uniserial modules to conditions on elements of the ring.

Recall that a module is pure projective if it is a direct summand of a (possibly infinite) direct sum of finitely presented modules. Another natural question is to characterize serial rings whose category of pure projective modules consists only of direct sums of finitely presented modules. In [**13**] an answer was given for chain domains, i.e., domains having the (left and right) regular modules uniserial: If R is a chain domain, then every pure projective R-module is a direct sum of finitely presented modules if and only if there is no idempotent ideal of the form RrR, where $0 \neq r \in J(R)$. The main result of this paper extends this characterization to non-singular serial rings. Note that a similar question in terms of RD-projectivity has been considered in [**5**].

The paper is organized as follows. Section 2 is a collection of basic facts about uniserial modules. These facts are used frequently throughout the paper, often

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without giving detailed references to them. Section 3 characterizes basic serial rings with essentially unique indecomposable direct sum decompositions of finitely presented modules. Section 4 is a brief overview of the dimension theory established in [**8**]. This theory is then used in the last section where a characterization of non-singular serial rings possessing only 'trivial' pure projective modules is given. To prove the result, we followed the approach of [**13**] closely, but we had to overcome several technical difficulties in the proof. Therefore we included almost complete proofs of Proposition 5.2 and Proposition 5.6 except for implications $(c) \Rightarrow (a)$ which have similar proofs as the corresponding implications in [**13**, Proposition 6.1, Proposition 7.3].

Throughout the paper a ring means an associative ring with unit, an R-module means a right module over a ring R, mod-R is the category of finitely presented R-modules, and $J(R)$ is the Jacobson radical of R.

2. Basics on uniserial modules

Let R be a ring. An R-module is *uniserial* if the lattice of its submodules is a chain. Below we list several important properties of uniserial modules. Usually we do not give a detailed reference when a result from this list is used.

(i) If U is a nonzero uniserial R-module, then the ring $S := \mathrm{End}_R(U)$ contains the following ideals $I := \{f \in S \mid f \text{ is not surjective }\}$ and $K := \{f \in S \mid f \text{ is not injective }\}$. If I and K are comparable in inclusion, then S is local and $J(S) = I \cup K$. If I and K are incomparable, then I, K are the only (left, right or two-sided) maximal ideals of S. In this case $J(S) = I \cap K = \{f \in S \mid f \text{ is neither injective nor surjective }\}$. If S is local, U is said to be *of type 1*, otherwise U is said to be *of type 2* (see [**4**, Theorem 9.1]).

(ii) If R is commutative or right noetherian, then every nonzero uniserial R-module is of type 1 (see [**4**, Proposition 9.24]).

(iii) Let U, V be uniserial R-modules such that there are $f, g \in \mathrm{Hom}_R(U, V)$, f injective and g surjective. Then $U \simeq V$, in fact, at least one of $f, g, f + g$ is an isomorphism (see [**4**, Lemma 9.2]).

(iv) Let U, V, W be nonzero uniserial modules, $f \in \mathrm{Hom}_R(U, V), g \in \mathrm{Hom}_R(V, W)$. Then gf is injective (surjective) if and only if f and g are both injective (surjective) (see [**4**, Lemma 6.26]).

Let U, V be uniserial modules. We say that U and V are of the same *monogeny class* ($[U]_m = [V]_m$) if there exist monomorphisms $f\colon U \to V$ and $g\colon V \to U$. We also use $[U]_m$ to denote the class of all modules having the same monogeny class as U. Similarly, U and V are of the same *epigeny class* ($[U]_e = [V]_e$) if there are epimorphisms $f\colon U \to V$ and $g\colon V \to U$. We also use $[U]_e$ to denote the class of modules having the same epigeny class as U. Observe that the property (iii) implies that $U \simeq V$ if and only if $[U]_m = [V]_m$ and $[U]_e = [V]_e$. The following theorem (known as the weak Krull-Schmidt theorem) clarifies the importance of monogeny and epigeny classes.

THEOREM 2.1. [**4**, Theorem 9.12] *Let $U_1, \dots, U_n, V_1, \dots, V_m$ be nonzero uniserial R-modules. Then $\oplus_{i=1}^n U_i \simeq \oplus_{j=1}^m V_j$ if and only if $n = m$ and there are permutations $\sigma, \tau \in S_n$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in \{1, \dots, n\}$.*

Recall that an R-module N is said to be *quasi-small* if for every family $\{M_i, i \in I\}$ of R-modules such that N is isomorphic to a direct summand of $\oplus_{i\in I} M_i$ there exists a finite set $F \subseteq I$ such that N is isomorphic to a direct summand of $\oplus_{i\in F} M_i$. The existence of uniserial modules which are *not* quasi-small (proved by Puninski [**10**]) is the main obstacle for a straightforward generalization of the weak Krull-Schmidt theorem to infinite direct sums of uniserial modules. In this paper we study how the existence of a uniserial module which is not quasi-small is related to the existence of a pure projective module which is not a direct sum of finitely presented modules.

Let U be a nonzero uniserial R-module, S, I, K as in (i). Set $U_e := \sum_{f\in S\setminus I} \operatorname{Ker} f$, $U_m := \cap_{f\in S\setminus K} \operatorname{Im} f$. With this notation we add the following properties to the list:

(v) U_m and U_e are fully invariant submodules of U (see [**11**, Lemma 2.2,2.3]).
(vi) U_e is not finitely generated unless $U_e = 0$ (see [**11**, Lemma 2.3]).
(vii) If $U_e \neq 0$ and $V \subseteq U$, then $[U/V]_e = [U]_e$ if and only if $V \subsetneq U_e$ (see [**11**, Lemma 2.3]).
(viii) The monogeny class of U contains a uniserial module V which is not quasi-small if and only if $U_m \subsetneq U_e$. This V is unique up to isomorphism ([**12**, Theorem 1.1] and [**11**, Lemma 2.8]) and V is isomorphic to a direct summand of $uR^{(\omega)}$ for every $u \in U \setminus U_m$.

3. Finitely presented modules over serial rings

The aim of this section is to review basic facts about the category of finitely presented modules over a serial ring. As an easy consequence of known facts we derive a criterion to determine when the Krull-Remak-Schmidt theorem holds in the category mod-R (the category of finitely presented right R-modules) if R is a basic serial ring.

Recall that an R-module is called *serial* if it is a direct sum of uniserial modules. A ring R is *serial* if R_R and ${}_RR$ are (right and left) serial modules. Throughout the paper we consider a serial ring R *always* with a complete set of its primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$. An easy application of the Krull-Remak-Schmidt theorem gives that modules $e_1R, \ldots, e_nR$ and $Re_1, \ldots, Re_n$ are (right and left) uniserial modules.

The following result proved by Warfield [**14**] and Drozd [**2**] describes the objects of this category.

THEOREM 3.1. [**9**, Theorem 2.3] *Every finitely presented module over a serial ring R is isomorphic to a direct sum of modules of the form $e_iR/e_irR, r \in R$.*

In particular, every finitely presented module over a serial ring is serial. Note that $e_irR = e_ire_jR$ whenever $e_ire_jR = \max_{\subseteq}\{e_ire_kR \mid k = 1, \ldots, n\}$. If e_iR/e_irR and e_jR/e_jsR are indecomposable finitely presented R-modules, the elements of $\operatorname{Hom}_R(e_iR/e_irR, e_jR/e_jsR)$ can be described by elements from $X = \{x \in e_jRe_i \mid xe_irR \subseteq e_jsR\}$. We say that $x \in X$ (or, more precisely, left multiplication by x) induces $f \in \operatorname{Hom}_R(e_iR/e_irR, e_jR/e_jsR)$ if $f(u + e_irR) = xu + e_jsR$ for every $u \in e_iR$.

The following lemma describes when a homomorphism induced by an element of the ring is injective or surjective.

LEMMA 3.2. *Let e_iR/e_irR and e_jR/e_jsR be indecomposable modules from* mod-*R, where R is a serial ring.*

(a) *If $e_js \neq 0$, then $x \in e_jRe_i$ induces an epimorphism if and only if $x \notin e_jJ(R)e_i$ and $xe_irR \subseteq e_jsR$.*
(b) *If $e_js \neq 0$, then $x \in e_jRe_i$ induces a monomorphism if and only if $xe_irR = e_jsR$*

PROOF. (a) is immediate since e_jR/e_jsR is a local module. To prove (b), we can follow the idea of [**13**, Corollary 2.2]: Suppose that $x \in e_jRe_i$ induces a homomorphism $f\colon e_iR/e_irR \to e_jR/e_jsR$, i.e., $xe_irR \subseteq e_jsR$. If $f \neq 0$, then $e_jsR \subsetneq xe_iR$. So if f is a monomorphism, then the inclusion $xe_irR \subseteq e_jsR$ cannot be strict. Conversely, assume that $xe_irR = e_jsR$. Then f is monic since if $e_it \in e_iR \setminus e_irR$ satisfies $xe_it \in e_jsR = xe_irR$, then $x(e_it - e_iru) = 0$ for some $u \in R$ and, consequently, the right annihilator of x contains e_irR. But then $xe_irR = e_jsR \neq 0$, a contradiction. □

Let R be a serial ring. If $e_ir, e_is \in e_iR$, then either $\mathrm{Hom}_R(e_iR/e_irR, e_iR/e_isR)$ or $\mathrm{Hom}_R(e_iR/e_isR, e_iR/e_irR)$ contains an epimorphism. This is immediate since e_iR is uniserial. Similarly, if e_ire_k and $e_jse_k \in Re_k$ then either $Re_ire_k \subseteq Re_jse_k$ or $Re_jse_k \subseteq Re_ire_k$. As a consequence of the previous lemma we get:

COROLLARY 3.3. *Let R be a serial ring, e_ire_k, e_jse_k nonzero elements of $J(R)$. Then either* $\mathrm{Hom}_R(e_iR/e_ire_kR, e_jR/e_jse_kR)$ *or* $\mathrm{Hom}_R(e_jR/e_jse_kR, e_iR/e_ire_kR)$ *contains a monomorphism.*

LEMMA 3.4. *Let R be a serial ring and let e_iR/e_irR, e_jR/e_jsR be indecomposable R-modules. If $e_js \neq 0$, then the image of every homomorphism $f\colon e_iR/e_irR \to e_jR/e_jsR$ is finitely presented.*

PROOF. Suppose that f is induced by $x \in e_jRe_i$ and $f \neq 0$. Then $xe_irR \subseteq e_jsR \subsetneq xe_iR$. Let $r' \in R$ be such that $xe_ir' = e_js$. Then it is easy to see that $\mathrm{Ker}\, f = e_ir'R/e_irR$, thus $\mathrm{Im}\, f \simeq e_iR/e_ir'R$. □

The serial ring R is called *basic* if $e_iR \simeq e_jR$ implies $i = j$ for every $i, j \in \{1, \dots, n\}$. The following lemma simplifies the description of basic serial rings with unique indecomposable decompositions of finitely presented modules.

LEMMA 3.5. *Assume that R is a serial ring and let $r, s \in R$.*

(a) *If there exists an epimorphism in* $\mathrm{Hom}_R(e_iR/e_irR, e_jR/e_jsR)$ *and $s \in J(R)$, then $e_iR \simeq e_jR$. If R is basic, then $i = j$.*
(b) *If there exists a monomorphism in* $\mathrm{Hom}_R(e_iR/e_ire_mR, e_jR/e_jse_nR)$ *and $r \in J(R)$, $0 \neq e_jse_n \in J(R)$, then $e_mR \simeq e_nR$. If R is basic, then $m = n$.*

PROOF. (a) Let $f \in \mathrm{Hom}_R(e_iR/e_irR, e_jR/e_jsR)$ be an epimorphism. If $s \in J(R)$, the modules $e_jR/e_jsR, e_iR/e_irR$ are nonzero. If $x \in e_jRe_i$ induces f, then $x \notin e_jJ(R)e_i$ and therefore $e_iR \simeq e_jR$. This implies $i = j$ if R is basic.

(b) Assume $f \in \mathrm{Hom}_R(e_iR/e_ire_mR, e_jR/e_jse_nR)$ is a monomorphism. Then $e_iR/e_ire_mR, e_jR/e_jse_nR$ are nonzero and f is induced by an element $x \in e_jRe_i$ such that $xe_ire_mR = e_jse_nR$. So there are $u, v \in R$ such that $xe_ire_mue_n = e_jse_n$ and $xe_ire_m = e_jse_nve_m$. Then $e_jse_n = e_jse_nve_mue_n$. Since $e_jse_n \neq 0$, $e_nve_mue_n \notin J(R)$. It follows that $e_mR \simeq e_nR$ and, if R is basic, $m = n$. □

Let R be a serial ring. Consider the following graph Γ_R. Its set of vertices is $\mathcal{V}_m \dot{\cup} \mathcal{V}_e$, where $\mathcal{V}_m$ is the set of all monogeny classes of indecomposable modules from mod-R and $\mathcal{V}_e$ is the set of all epigeny classes of indecomposable modules from mod-R. The graph Γ_R is bipartite, i.e., each edge connects a vertex from $\mathcal{V}_m$ and a vertex from $\mathcal{V}_e$: The vertices $v_m \in \mathcal{V}_m$ and $v_e \in \mathcal{V}_e$ are connected in Γ_R if and only if there exists an indecomposable module $U \in$ mod-R such that $[U]_m = v_m$ and $[U]_e = v_e$. Observe that there is a natural bijective correspondence between the isoclasses of indecomposable modules in mod-R and the edges of Γ_R, and we use this bijection to identify isoclasses of finitely presented indecomposables and edges of Γ_R. Repeating the argument of [**6**, Proposition 7.1], one can prove:

LEMMA 3.6. *Let R be a serial ring. Then every connected component of Γ_R is a complete bipartite graph.*

If U and V are indecomposable modules from mod-R, define $U \sim V$ if and only if the edges of Γ_R corresponding to $[U]$ and $[V]$ are in the same connected component of Γ_R. Since each connected component of Γ_R is a complete bipartite graph, $U \sim V$ if and only if there exists an indecomposable module $W \in$ mod-R such that $[W]_m = [U]_m$ and $[W]_e = [V]_e$, if and only if there exists an indecomposable module $W \in$ mod-R such that $[W]_e = [U]_e$ and $[W]_m = [V]_m$.

By [**7**, Lemma 5.2], if $U \sim V$ for $U, V \in$ mod-R indecomposable, then U is of type 1 if and only if V is of type 1. Therefore modules corresponding to edges of a connected component $\mathcal{C}$ are of the same type. The connected component $\mathcal{C}$ is said to be of *type 2* if all the modules corresponding to its edges are of type 2.

LEMMA 3.7. *Let $U \sim V$ be indecomposable finitely presented modules over a serial ring R.*

(a) *If* $\mathrm{Hom}_R(U, V)$ *contains a monomorphism, then* $[U]_m = [V]_m$.
(b) *If* $\mathrm{Hom}_R(U, V)$ *contains an epimorphism, then* $[U]_e = [V]_e$.

PROOF. (a) Consider a uniserial module W such that $[W]_m = [V]_m$ and $[W]_e = [U]_e$. The existence of a monomorphism in $\mathrm{Hom}_R(U, V)$ implies the existence of a monomorphism in $\mathrm{Hom}_R(U, W)$ and $[W]_e = [U]_e$ guarantees the existence of an epimorphism in $\mathrm{Hom}_R(U, W)$. Therefore $U \simeq W$ and $[U]_m = [V]_m$. (b) Consider a uniserial module W such that $[W]_e = [V]_e$ and $[W]_m = [U]_m$ and conclude making use of arguments dual to those in (a). □

We say that the Krull-Remak-Schmidt theorem holds in mod-R if every finitely presented R-module M can be written as $M = \oplus_{i=1}^n U_i$, where U_i are indecomposable and if $M = \oplus_{i=1}^m V_i$ is another such a decomposition, then $m = n$ and there exists a permutation $\pi \in S_n$ such that $U_i \simeq V_{\pi(i)}$ for every $i \in \{1, \dots, n\}$.

We can apply the weak Krull-Schmidt theorem to get a condition for unique decompositions in mod-R in terms of Γ_R.

COROLLARY 3.8. *Let R be a serial ring. The following statements are equivalent*

(a) *The Krull-Remak-Schmidt theorem holds in* mod-R.
(b) *The graph Γ_R does not contain a cycle.*
(c) *There are no indecomposable modules $U, V, W \in$ mod-R such that $[U]_m = [W]_m \neq [V]_m$ and $[V]_e = [W]_e \neq [U]_e$.*

PROOF. $(a) \Rightarrow (b)$. Assume (a) and consider a cycle in Γ_R. Since every connected component of Γ_R is a complete bipartite graph, there exists a cycle of length 4. Consider the edges $[U_1],[U_2],[U_3],[U_4]$ of such a cycle. Assume that $[U_1]_e = [U_2]_e, [U_2]_m = [U_3]_m, [U_3]_e = [U_4]_e$ and $[U_4]_m = [U_1]_m$. The isoclasses of U_1,U_2,U_3,U_4 are different edges of Γ_R, so these modules are pair-wise non-isomorphic. But the weak Krull-Schmidt theorem gives $U_1 \oplus U_3 \simeq U_2 \oplus U_4$ so the Krull-Remak-Schmidt theorem does not hold in mod-R.

$(b) \Rightarrow (a)$ Consider a connected component $\mathcal{C}$ of Γ_R. If there are no cycles in Γ_R, either $\mathcal{C} \cap \mathcal{V}_m$ or $\mathcal{C} \cap \mathcal{V}_e$ is a one-element set. If $|\mathcal{C} \cap \mathcal{V}_m| = 1$, then $[U]_e = [V]_e$ implies $U \simeq V$ whenever $[U],[V] \in \mathcal{C}$. In this case we say that $\mathcal{C}$ has one monogeny class. Similarly, if $|\mathcal{C} \cap \mathcal{V}_e| = 1$, we say that the component $\mathcal{C}$ has one epigeny class and, in this case $[U]_m = [V]_m \Rightarrow U \simeq V$ if $[U],[V] \in \mathcal{C}$.

Let $U_1,\dots,U_n,V_1,\dots,V_m$ be indecomposable modules in mod-R such that $\oplus_{i=1}^n U_i \simeq \oplus_{j=1}^m V_j$. By the weak Krull-Schmidt theorem, $n = m$ and there are $\sigma,\tau \in S_n$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in \{1,\dots,n\}$. We have to show that there is $\pi \in S_n$ such that $U_i \simeq V_{\pi(i)}, i \in \{1,\dots,n\}$. For $i \in \{1,\dots,n\}$ define $\pi(i) := \tau(i)$ if $[U_i]$ is in a connected component having one monogeny class and $\pi(i) := \sigma(i)$ if $[U_i]$ is not in a connected component of Γ_R having one monogeny class. Thus if $[U_i]$ is in a connected component having one monogeny class, then $[U_i]_e = [V_{\pi(i)}]_e$, hence $V_{\pi(i)} \sim U_i$ and $U_i \simeq V_{\pi(i)}$. If $[U_i]$ is not in a connected component having one monogeny class, it is in a connected component having one epigeny class. Therefore $[U_i]_m = [V_{\pi(i)}]_m$, $V_{\pi(i)} \sim U_i$ and $U_i \simeq V_{\pi(i)}$. It remains to show that π is a bijection. But this is obvious since if $\pi(i) = \pi(i')$ for $i \neq i'$, then exactly one element from $\{[U_i],\ [U_{i'}]\}$ corresponds to an edge of a connected component having one monogeny class. Consequently, $[V_{\pi(i)}]$ and $[V_{\pi(i')}]$ correspond to edges in different connected components of Γ_R, thus $\pi(i) \neq \pi(i')$.

$(b) \Leftrightarrow (c)$ is an easy consequence of Lemma 3.6. □

In the remaining part of this section we rewrite (b),(c) in terms of the ring R and obtain a characterization of when the Krull-Remak-Schmidt theorem holds in mod-R provided R is a basic serial ring.

If R is a chain ring and $U = R/rR$ for some $r \in R$, then, by [**11**, Lemma 4.3], $U_e = RrR/rR$. The following lemma extends this result to the case of serial rings. Recall that, with any serial ring, we consider a complete set of indecomposable orthogonal idempotents $\{e_1,\dots,e_n\}$.

Lemma 3.9. *Let R be a serial ring, $i \in \{1,\dots,n\}$ and $U = e_iR/e_irR$ for some $r \in J(R)$. Then $U_e = e_iRe_irR/e_irR$.*

Proof. If $e_ir = 0$, then U is an indecomposable projective R-module, therefore $U_e = 0$. Assume $e_ir \neq 0$. Let $E := e_iRe_i$ be the endomorphism ring of e_iR. Then E is local and $J(E) = e_iJ(R)e_i$. It follows that every epimorphism from $\mathrm{End}_R(U)$ is induced by some $x \in E \setminus J(E)$ satisfying $xe_irR \subseteq e_irR$. Then it is easy to check that the kernel of every epimorphism in $\mathrm{End}_R(U)$ is contained in e_iRe_irR/e_irR, hence $U_e \subseteq e_iRe_irR/e_irR$.

Conversely, suppose that $u = e_ise_irt \in e_iRe_irR \setminus e_irR$. We want to check the existence of an epimorphism $f \in \mathrm{End}_R(U)$ satisfying $f(u + e_irR) = 0$. If $e_ise_i \notin J(E)$, then left multiplication by $(e_ise_i)^{-1} \in E$ induces such an epimorphism, so assume $e_ise_i \in J(E)$. Then $e_i + e_ise_i \in E \setminus J(E)$. Observe that $u + e_irR = (e_ise_i + e_i)e_irt + e_irR$. Therefore it suffices to find $e \in E \setminus J(E)$ such that $e(e_ise_i + e_i)e_irt \in$

$e_i rR$. Obviously, the inverse of $e_i s e_i + e_i$ in E satisfies this, therefore we can choose f to be the endomorphism of U induced by left multiplication by $(e_i s e_i + e_i)^{-1}$. □

Corollary 3.10. *Let R be a serial ring, $i \in \{1, \ldots, n\}$, $U = e_iR/e_irR$ and $V = e_iR/e_isR$ for some $r, s \in J(R)$. Then $[U]_e = [V]_e$ if and only if $e_iRe_irR = e_iRe_isR$, if and only if $Re_irR = Re_isR$.*

Proof. We prove only the first equivalence, the proof of the second one is straightforward. Assume that $[U]_e = [V]_e$, i.e., there are epimorphisms in $\mathrm{Hom}_R(U, V)$ and $\mathrm{Hom}_R(V, U)$. If $e_ir = 0$, then U is projective and V contains a direct summand isomorphic to U. Since V is uniserial, $V \simeq U$, but then e_isR is a direct summand of e_iR, hence also $e_is = 0$. Similarly $e_is = 0$ implies $e_ir = 0$. Therefore we may assume $e_ir, e_is \neq 0$.

Without loss on generality suppose $0 \neq e_irR \subseteq e_isR \subsetneq e_iR$. We are left to check that $e_iRe_isR \subseteq e_iRe_irR$. Let $\pi\colon U \to V$ be the canonical projection and $g\colon V \to U$ be an epimorphism. Then $f = g\pi \in \mathrm{End}_R(U)$ is an epimorphism such that $f(e_is + e_irR) = 0$. By Lemma 3.9, $e_is \in e_iRe_irR$ and therefore $e_iRe_isR \subseteq e_iRe_irR$.

Now assume $e_iRe_irR = e_iRe_isR$. Obviously, $e_ir = 0$ if and only if $e_is = 0$. Therefore suppose $0 \neq e_irR \subseteq e_isR \subsetneq e_iR$. Since $e_irR \subseteq e_isR$, the canonical projection is an epimorphism $\pi\colon U \to V$. Since $e_is \in e_iRe_irR$ there exists an epimorphism $f \in \mathrm{End}_R(U)$ such that $f(e_is + e_irR) = 0$, i.e., $\mathrm{Ker}\,\pi \subseteq \mathrm{Ker}\, f$. Therefore there exists $g\colon V \to U$ such that $f = g\pi$. In particular, g is an epimorphism in $\mathrm{Hom}_R(V, U)$, thus $[U]_e = [V]_e$. □

We give a similar criterion for monogeny classes. Recall that if R is a serial ring, then every indecomposable finitely presented R-module is of the form e_iR/e_ire_jR, $i, j \in \{1, \ldots, n\}$. As observed in Lemma 3.5, if R is basic and $e_ire_m, e_jse_n \in J(R)$ are nonzero, then $\mathrm{Hom}_R(e_iR/e_ire_mR, e_jR/e_jse_nR)$ can contain a monomorphism only if $m = n$.

Lemma 3.11. *Suppose that R is a serial ring $0 \neq e_ire_k, e_jse_k \in J(R)$. Then $[e_iR/e_ire_kR]_m = [e_jR/e_jse_kR]_m$ if and only if $Re_ire_kRe_k = Re_jse_kRe_k$, if and only if $Re_ire_kR = Re_jse_kR$.*

Proof. Suppose that $[e_iR/e_ire_kR]_m = [e_jR/e_jse_kR]_m$. Then there are $a \in e_jRe_i$ and $b \in e_iRe_j$ satisfying $ae_ire_kR = e_jse_kR$ and $be_jse_kR = e_ire_kR$. In particular, $e_ire_k \in Re_jse_kRe_k$ and $e_jse_k \in Re_ire_kRe_k$. This implies that $Re_ire_kRe_k = Re_jse_kRe_k$.

Conversely, assume $Re_ire_kRe_k = Re_jse_kRe_k$. Write e_jse_k as a sum of elements of the form $v'e_ire_ku'e_k$, for $u', v' \in R$. Since the left module Re_k is uniserial, there are $u \in R$ and $v \in e_jRe_i$ such that $e_jse_k = ve_ire_kue_k$. Left multiplication of v induces a monomorphism in $\mathrm{Hom}_R(e_iR/e_ire_kue_kR, e_jR/e_jse_kR)$. Observe that $e_iRe_ire_kue_kR$ contains e_ire_k. By Lemma 3.9, we get $[e_iR/e_ire_kR]_e = [e_iR/e_ire_kue_kR]_e$. Furthermore, $e_ire_kue_k$ and e_ire_k are nonzero elements of e_iRe_k. Therefore, by Corollary 3.3, at least one of $\mathrm{Hom}_R(e_iR/e_ire_kR, e_iR/e_ire_kue_kR)$, $\mathrm{Hom}_R(e_iR/e_ire_kue_kR, e_iR/e_ire_kR)$ contains a monomorphism. In any case the existence of such a monomorphism together with $[e_iR/e_ire_kR]_e = [e_iR/e_ire_kue_kR]_e$ implies $e_iR/e_ire_kR \simeq e_iR/e_ire_kue_kR$. So there exists an injective homomorphism from e_iR/e_ire_kR to e_jR/e_jse_kR. Similarly we prove that there exists a monomorphism from e_jR/e_jse_kR to e_iR/e_ire_kR. □

Combining the previous results, we obtain the following generalization of [**9**, Prop 2.21]. Observe that, for basic serial rings, it gives a classification of indecomposable finitely presented R-modules up to isomorphism.

COROLLARY 3.12. *Suppose that R is a serial ring and $0 \neq e_i r e_j, e_i s e_j \in e_i J(R) e_j$. Then the following are equivalent*

(a) $[e_i R / e_i r e_j R]_m = [e_i R / e_i s e_j R]_m$
(b) $[e_i R / e_i r e_j R]_e = [e_i R / e_i s e_j R]_e$
(c) $e_i R / e_i r e_j R \simeq e_i R / e_i s e_j R$
(d) $R e_i r e_j R = R e_i s e_j R$.

We conclude the section giving an answer to [**9**, Question 2.24] for basic serial rings.

THEOREM 3.13. *Let $\{e_1, \dots, e_n\}$ be a complete set of orthogonal indecomposable idempotents of a basic serial ring R. The following statements are equivalent:*

(a) *The Krull-Remak-Schmidt theorem does not hold in* mod-R.
(b) *There are nonzero $e_i r e_j, e_i s e_k, e_l u e_k, e_l t e_j \in J(R)$ such that $j \neq k$, $i \neq l$ and $e_i R e_i r e_j R = e_i R e_i s e_k R$, $R e_i s e_k R e_k = R e_l u e_k R e_k$, $e_l R e_l u e_k R = e_l R e_l t e_j R$, $R e_l t e_j R e_j = R e_i r e_j R e_j$.*
(c) *There are nonzero $e_i r e_j, e_i s e_k, e_l u e_k \in J(R)$ such that $j \neq k, i \neq l$ and $e_i R e_i r e_j R = e_i R e_i s e_k R$, $R e_i s e_k R e_k = R e_l u e_k R e_k$.*

PROOF. $(a) \Rightarrow (b)$ By Corollary 3.8, (a) is equivalent to the existence of a cycle in Γ_R. Since every connected component of Γ_R is a complete bipartite graph, (a) is in fact equivalent to the existence of a cycle of length 4. Let $[U_1], [U_2], [U_3], [U_4]$ be edges of such a cycle, say $[U_1]_e = [U_2]_e$, $[U_2]_m = [U_3]_m$, $[U_3]_e = [U_4]_e$ and $[U_4]_m = [U_1]_m$. Furthermore, $[U_1]_m \neq [U_3]_m$ and $[U_2]_e \neq [U_4]_e$. Note that if a connected component of Γ_R contains an edge corresponding to a projective module, then the component has only one monogeny class. Therefore U_1, U_2, U_3, U_4 are not projective. Let $U_1 = e_i R / e_i r e_j R$, $U_2 = e_i R / e_i s e_k R$, (to see that U_1 and U_2 have to be factors of the same $e_i R$ use Lemma 3.5 (a)), $U_3 = e_l R / e_l u e_k R$ (the occurence of the same primitive idempotent e_k in the presentations of U_2 and U_3 is a consequence of Lemma 3.5(b)), $U_4 = e_l R / e_l t e_j R$. Since U_1, U_2, U_3, U_4 are not projective, $e_i r e_j, e_i s e_k, e_l u e_k, e_l t e_j$ are nonzero elements of $J(R)$. The relations between monogeny and epigeny classes of U_1, U_2, U_3, U_4 translate to relations in (b). Finally, if $j = k$ then, by Corollary 3.3, either $\mathrm{Hom}_R(U_1, U_3)$ or $\mathrm{Hom}_R(U_3, U_1)$ contains a monomorphism. But then $[U_1]_m = [U_3]_m$ follows from Lemma 3.7 since $U_1 \sim U_3$. This is not possible, so $j \neq k$. Similarly one can show that $[U_2]_e \neq [U_4]_e$ implies $i \neq l$.

$(b) \Rightarrow (a)$ Using the same arguments as in the proof of the previous implication one can see that $[e_i R / e_i r e_j R], [e_i R / e_i s e_k R], [e_l R / e_l u e_k R], [e_l R / e_l t e_j R]$ give a cycle in Γ_R.

The proof of $(a) \Leftrightarrow (c)$ is similar. □

Note that [**9**, Question 2.24] also aks whether the condition (a) of Theorem 3.13 is left-right symmetric. It is possible to use condition (b) to check that this is indeed the case. However, it is much easier to apply Corollary 3.8 and the Auslander-Bridger transpose to see that (cf. [**1**, Proposition 7.2]).

4. Dimension theory for pure projective modules over serial rings

In this section we briefly review the dimension theory developed in [**8**] to study direct summands of serial modules. We are interested only in a particular case of pure projective modules over serial rings.

Let R be a serial ring, $\mathcal{P} := \mathrm{Add}(\text{mod-}R)$ the full subcategory of Mod-R whose objects are pure projective R-modules. Let U be an indecomposable finitely presented R-module of type 1. Let $\mathcal{J}_U$ be the largest ideal of $\mathcal{P}$ such that $\mathcal{J}_U(U,U) = J(\mathrm{End}_R(U))$. If $X, Y \in \mathcal{P}$, then

$$\mathcal{J}_U(X,Y) := \{f \in \mathrm{Hom}_R(X,Y) \mid \forall \alpha \in \mathrm{Hom}_R(U,X), \\ \forall \beta \in \mathrm{Hom}_R(Y,U)\ \beta f \alpha \in J(\mathrm{End}_R(U))\}\,.$$

The factor category $\mathcal{P}/\mathcal{I}_U$ is equivalent to the category of right modules over the division ring $\mathrm{End}_R(U)/J(\mathrm{End}_R(U))$ and therefore there exists a natural way to assign a cardinal $\dim_U(M)$ to every pure projective module M (see [**8**, page 30] for details). Note that if $V \in \text{mod-}R$ is indecomposable and $U \not\simeq V$, then $1_V \in \mathcal{J}_U$, so V is a zero object in $\mathcal{P}/\mathcal{I}_U$. Thus $\dim_U(V) = 0$. On the other hand, if $V \simeq U$, then $\dim_U(V) = 1$.

Similarly, if U is an indecomposable finitely presented R-module of type 2 and $I \subseteq \mathrm{End}_R(U)$ ($K \subseteq \mathrm{End}_R(U)$) is the ideal of $\mathrm{End}_R(U)$ consisting of nonmonomorphisms (nonepimorphisms), let $\mathcal{I}_U$ ($\mathcal{K}_U$) be the ideals of $\mathcal{P}$ such that $\mathcal{I}_U$ ($\mathcal{K}_U$) is the largest ideal of $\mathcal{P}$ satisfying $\mathcal{I}_U(U,U) = I$ ($\mathcal{K}_U(U,U) = K$). The category $\mathcal{P}/\mathcal{I}_U$ (resp. $\mathcal{P}/\mathcal{K}_U$) is equivalent to the category of modules over the division ring $\mathrm{End}_R(U)/I$ ($\mathrm{End}_R(U)/K$). Therefore we may assign a cardinal $\mathrm{mdim}_U(M)$ ($\mathrm{edim}_U(M)$) to every pure projective R-module M. Let $V \in \text{mod-}R$ be indecomposable. Similarly as above, it is possible to check that $\mathrm{mdim}_U(V) = 0$ ($\mathrm{edim}_U(V) = 0$) if $[U]_m \neq [V]_m$ ($[U]_e \neq [V]_e$) and $\mathrm{mdim}_U(V) = 1$ ($\mathrm{edim}_U(V) = 1$) if $[U]_m = [V]_m$ ($[U]_e = [V]_e$).

The cardinal numbers defined by indecomposable objects of mod-R determine objects of $\mathcal{P}$ up to isomorphism, because from [**8**, Theorem 7.4] it is easy to derive:

Theorem 4.1. *Let R be a serial ring, M, N pure projective R-modules. Then $M \simeq N$ if and only if $\dim_U(M) = \dim_U(N)$ for every indecomposable $U \in$ mod-R of type 1, and $\mathrm{mdim}_V(M) = \mathrm{mdim}_V(N)$, $\mathrm{edim}_V(M) = \mathrm{edim}_V(N)$ for every indecomposable $V \in$ mod-R of type 2.*

We need this theorem only for countably generated pure projective modules M, N. In this case the theorem is a particular case of [**8**, Lemma 7.2].

If $U, V \in \text{mod-}R$ are of type 2 and $[U]_m = [V]_m$, then $\mathrm{mdim}_U(M) = \mathrm{mdim}_V(M)$ for every $M \in \mathcal{P}$. If v is a vertex of a connected component of type 2 in Γ_R contained in $\mathcal{V}_m$, we define a dimension function mdim_v on objects of $\mathcal{P}$ by $\mathrm{mdim}_v(M) := \mathrm{mdim}_U(M)$, where $U \in \text{mod-}R$ is indecomposable and v consists of modules having the same monogeny class as U. Similarly we define a dimension function edim_v for every vertex $v \in \Gamma_R$ of a connected component of type 2 which is contained in $\mathcal{V}_e$.

The cardinal invariants are compatible with direct sums, i.e., if $*\dim_U$ is a function described as above and $M_i, i \in I$, is a family of modules in $\mathcal{P}$, then $*\dim_U(\oplus_{i\in I} M_i) = \sum_{i\in I} *\dim_U(M_i)$ (see [**8**, Corollary 2.7]).

If $U_i, i \in I$, is a family of indecomposable modules from mod-R and N is a direct summand of $\oplus_{i\in I} U_i$, then N is an image of an idempotent endomorphism

$\pi \in \mathrm{End}_R(\oplus_{i\in I} U_i)$. We display π as a column-finite matrix $\pi = (p_{i,j})_{i,j\in I}$, where $p_{i,j} \in \mathrm{Hom}_R(U_j, U_i)$.

If $V \in$ mod-R is indecomposable and of type 2, it is easy to see that $\mathrm{mdim}_V(N) > 0$ if and only if there are $i, j \in I$, such that $[U_i]_m = [U_j]_m = [V]_m$ and $p_{i,j}$ is a monomorphism. Similarly, $\mathrm{edim}_V(N) > 0$ if and only if there are $i, j \in I$ such that $[U_i]_e = [U_j]_e = [V]_e$ and $p_{i,j}$ is an epimorphism.

5. The main result

Using the dimension theory it is easy to prove the following criterion.

LEMMA 5.1. *Let M be a pure projective module over a serial ring R. The following are equivalent:*

(a) *M is a direct sum of finitely presented modules.*
(b) *The equality $\sum_{v_m\in\mathcal{V}_m\cap\mathcal{C}} \mathrm{mdim}_{v_m}(M) = \sum_{v_e\in\mathcal{V}_e\cap\mathcal{C}} \mathrm{edim}_{v_e}(M)$ holds for every connected component $\mathcal{C} \subseteq \Gamma_R$ of type 2.*

PROOF. $(a) \Rightarrow (b)$ Suppose $M = \oplus_{i\in I} U_i$, where every $U_i \in$ mod-R is indecomposable. Then (b) follows from $\sum_{v_m\in\mathcal{V}_m\cap\mathcal{C}} \mathrm{mdim}_{v_m}(M) = |\{i \in I \mid [U_i] \in \mathcal{C}\}| = \sum_{v_e\in\mathcal{V}_e\cap\mathcal{C}} \mathrm{edim}_{v_e}(M)$.

$(b) \Rightarrow (a)$ For every vertex $v \in \mathcal{V}_m$ in a connected component of type 2 set $\kappa_v := \mathrm{mdim}_v(M)$ and for every vertex $v \in \mathcal{V}_e$ in a connected component of type 2 let $\lambda_v := \mathrm{edim}_v(M)$. If $\mathcal{C}$ is a connected component of type 2, set $\kappa_{\mathcal{C}} := \dot\bigcup_{v\in\mathcal{C}\cap\mathcal{V}_m} \kappa_v$ and $\lambda_{\mathcal{C}} := \dot\bigcup_{v\in\mathcal{C}\cap\mathcal{V}_e} \lambda_v$. (b) implies the existence of a bijection $b_{\mathcal{C}} \colon \kappa_{\mathcal{C}} \to \lambda_{\mathcal{C}}$ for every connected component $\mathcal{C}$ of type 2. Using $b_{\mathcal{C}}$ we construct a module $M_{\mathcal{C}}$ which is a direct sum of indecomposable modules from mod-R, $\mathrm{mdim}_v(M_{\mathcal{C}}) = \kappa_v$ for every $v \in \mathcal{C} \cap \mathcal{V}_m$ and $\mathrm{edim}_v(M_{\mathcal{C}}) = \lambda_v$ for every $v \in \mathcal{C} \cap \mathcal{V}_e$. Let $i \in \kappa_{\mathcal{C}} = \dot\bigcup_{v\in\mathcal{C}\cap\mathcal{V}_m} \kappa_v$. Then $i \in \kappa_v$ for unique $v \in \mathcal{C} \cap \mathcal{V}_m$ and $b_{\mathcal{C}}(i) \in \lambda_w$ for unique $w \in \mathcal{C} \cap \mathcal{V}_e$. Let U_i be a module in mod-R corresponding to the edge of Γ_R connecting v and w. Define $M_{\mathcal{C}} := \oplus_{i\in\kappa_{\mathcal{C}}} U_i$ and $M_2 := \oplus_{\mathcal{C}} M_{\mathcal{C}}$, where the direct sum is over all connected components of type 2.

Finally, let $\mathcal{I}$ be the set of isoclases of indecomposable finitely presented modules of type 1, and let $M_1 := \oplus_{[U]\in\mathcal{I}} U^{(\mathrm{dim}_U(M))}$. Consider $M' := M_1 \oplus M_2$. It is easy to check that for every indecomposable $W \in$ mod-R the dimensions related to W coincides on M and M'. Therefore $M \simeq M'$ and M is isomorphic to a direct sum of finitely presented modules. □

PROPOSITION 5.2. *Let R be a serial ring. The following are equivalent*

(a) *There exists a countably generated pure projective module M and a connected component $\mathcal{C}$ of Γ_R of type 2 such that $\sum_{v_m\in\mathcal{V}_m\cap\mathcal{C}} \mathrm{mdim}_{v_m}(M) < \sum_{v_e\in\mathcal{V}_e\cap\mathcal{C}} \mathrm{edim}_{v_e}(M)$.*
(b) *There exists a countably generated pure projective module M and a connected component $\mathcal{C}$ of Γ_R of type 2 such that $0 = \sum_{v_m\in\mathcal{V}_m\cap\mathcal{C}} \mathrm{mdim}_{v_m}(M)$ and $0 < \sum_{v_e\in\mathcal{V}_e\cap\mathcal{C}} \mathrm{edim}_{v_e}(M)$.*
(c) *There exist $U, W \in$ mod-R indecomposable, U of type 2 and $W \not\sim U$, and homomorphisms $\alpha\colon U \to W, \beta\colon W \to U$ such that $U \not\sim W$ and $\mathrm{Ker}\ \beta\alpha \subsetneq U_e$.*

PROOF. Suppose (a) is true and $0 < \sum_{v_m\in\mathcal{V}_m\cap\mathcal{C}} \mathrm{mdim}_{v_m}(M)$. Since M is countably generated, it is a direct summand of a countable direct sum of finitely presented modules. Therefore, $\sum_{v_e\in\mathcal{V}_e\cap\mathcal{C}} \mathrm{edim}_{v_e}(M)$ is at most countable and

$\sum_{v_m \in \mathcal{V}_m \cap \mathcal{C}} \operatorname{mdim}_{v_m}(M)$ is finite. Since components of Γ_R are complete bipartite, there exists $U \in \text{mod-}R$ indecomposable such that $[U] \in \mathcal{C}$, $\operatorname{mdim}_U(M) > 0$ and $\operatorname{edim}_U(M) > 0$. Then, using the same arguments as in [**13**, Lemma 4.3], $M \simeq U \oplus M'$ and $0 \leq \sum_{v_m \in \mathcal{V}_m \cap \mathcal{C}} \operatorname{mdim}_{v_m}(M') = \sum_{v_m \in \mathcal{V}_m \cap \mathcal{C}} \operatorname{mdim}_{v_m}(M) - 1$ and $1 + \sum_{v_e \in \mathcal{V}_e \cap \mathcal{C}} \operatorname{edim}_{v_e}(M') = \sum_{v_e \in \mathcal{V}_e \cap \mathcal{C}} \operatorname{edim}_{v_e}(M)$. Repeating this procedure, after finitely many steps we get a pure projective module satisfying (b).

Suppose that (b) is true. Let M be a direct summand of $N = \oplus_{i \in \mathbb{N}} U_i$, where the modules $U_i \in \text{mod-}R$ are indecomposable. Let $\pi = (p_{i,j})_{i,j \in \mathbb{N}} \in \operatorname{End}_R(N)$ be a projection of N onto M, so $\pi^2 = \pi$ and $\pi(N) = M$. Since $0 < \sum_{v_e \in \mathcal{V}_e \cap \mathcal{C}} \operatorname{edim}_{v_e}(M)$, there are $i, j \in \mathbb{N}$ such that $[U_i]_e = [U_j]_e$, $[U_i], [U_j] \in \mathcal{C}$ and $p_{i,j} \colon U_j \to U_i$ is surjective. Fix such i, j and consider the set $I := \{t \in \mathbb{N} \mid [U_t] \in \mathcal{C}\}$. Observe that for every $t \in I$ the homomorphism $p_{t,j} \colon U_j \to U_t$ cannot be a monomorphism, since otherwise, according to Lemma 3.7(a), $[U_j]_m = [U_t]_m$ would hold and $\operatorname{mdim}_{U_j}(M) > 0$ would contradict the assumption (b). Let $i_0 \in I$ be such that $\operatorname{Ker}\, p_{i_0,j} \subseteq \operatorname{Ker}\, p_{t,j}$ for every $t \in I$. Observe that $i \in I$ and $p_{i,j}$ onto imply $\operatorname{Ker}\, p_{i_0,j} \subsetneq (U_j)_e$.

Now use the identity $\pi^2 = \pi$, in particular $p_{i_0,j} = \sum_{k \in \mathbb{N}} p_{i_0,k} p_{k,j}$. Of course, there are only finitely many nonzero summands on the right hand side. Then there exists k such that $\operatorname{Ker}\, p_{i_0,k} p_{k,j} \subseteq \operatorname{Ker}\, p_{i_0,j}$. If $U_k \in \mathcal{C}$, i.e., $k \in I$, then $\operatorname{Ker}\, p_{k,j} = \operatorname{Ker}\, p_{i_0,j}$ and $p_{i_0,k}$ is a monomorphism. Since $U_{i_0} \sim U_k$, $[U_{i_0}]_m = [U_k]_m$ and, again, we get a contradiction since $\operatorname{mdim}_{U_{i_0}}(M) > 0$. Therefore $W := U_k$ is not in $\mathcal{C}$.

Consider the homomorphism $\gamma = p_{i_0,k} p_{k,j}$. Since $\operatorname{Ker}\, \gamma \subsetneq (U_j)_e$, $[\operatorname{Im}\, \gamma]_e = [U_j]_e$. In particular, $U := \operatorname{Im}\, \gamma \sim U_{i_0}$ and since $U \subseteq U_{i_0}$, an application of Lemma 3.7(a) gives $[U_{i_0}]_m = [U]_m$. Let $\mu \colon U_{i_0} \to U$ be a monomorphism and $\delta \colon U \to U_j$ be an epimorphism. Then it is enough to put $\alpha := p_{k,j}\delta$ and $\beta := \mu p_{i_0,k}$. This proves (c) since, by [**11**, Lemma 2.3(4)], $\delta^{-1}((U_j)_e) = U_e$ and all modules corresponding to edges of the connected component $\mathcal{C}$ are of type 2.

Finally suppose (c) is true. Consider an epimorphism $f \in \operatorname{End}_R(U)$ such that $\operatorname{Ker}\, \beta\alpha \subsetneq \operatorname{Ker} f$. Then $\operatorname{Ker}\, (f^2 + \beta\alpha) \subsetneq \operatorname{Ker}\, f$, so there exists an epimorphism $\varphi \in \operatorname{End}_R(U)$ such that $f = \varphi f^2 + (\varphi\beta)\alpha$. Now since f is onto and not mono, $W \not\sim U$ one can construct as in [**13**, Lemma 5.2] a countably generated pure projective module M such that $\operatorname{mdim}_U(M) = 0$, $\operatorname{edim}_U(M) = 1$ and dimensions related to W are infinite (in the construction [**13**, page 201] we set $M := U$, $N := W$, $f_i := f$ for every $i \in \mathbb{N}$ and the arguments of [**13**, Lemma 5.2] apply also in this more general case). In fact, if $\mathcal{C}$ is the connected component of Γ_R containing the edge $[U]$, then $\sum_{v_m \in \mathcal{V}_m \cap \mathcal{C}} \operatorname{mdim}_{v_m}(M) = 0$ and $\sum_{v_e \in \mathcal{V}_e \cap \mathcal{C}} \operatorname{edim}_{v_e}(M) = 1$ since M is a direct summand of $U^{(\omega)} \oplus W^{(\omega)}$. Therefore (a) and (b) hold.

□

LEMMA 5.3. *Let R be a serial ring and let U, V be indecomposable modules of* mod-R *such that $U \sim V$. If $[U]_m$ contains a module which is not quasi-small, then the same holds for $[V]_m$.*

PROOF. In other words, we have to prove that $U_m \subsetneq U_e$ implies $V_m \subsetneq V_e$. The claim is obvious if $[U]_m = [V]_m$, so it is enough to prove it under the assumption $[U]_e = [V]_e$. Let $f \in \operatorname{End}_R(U)$ and $g \in \operatorname{End}_R(U)$ be a monomorphism and an epimorphism such that $gf = 0$. Furthermore, let $\alpha \colon V \to U$, $\beta \colon U \to V$ be epimorphisms. Consider $V' = \alpha^{-1}(\operatorname{Im}\, f)$. It is a submodule of V, $\alpha(V') \simeq U$

implies that there exists an epimorphism in $\mathrm{Hom}_R(V', V)$. Hence $V' \simeq V$. If $\gamma \in \mathrm{Hom}_R(V, V')$ is a monomorphism, then $(\beta g\alpha)\gamma = 0$, where $\beta g\alpha \in \mathrm{End}_R(V)$ is an epimorphism. Therefore $V_m \subsetneq V_e$. ☐

PROPOSITION 5.4. *Let R be a serial ring. Then there exists a pure projective R-module which is uniserial and not quasi-small if and only if there exists $0 \neq e_ire_j \in J(R)$ such that the ideal Re_ire_jR is idempotent.*

PROOF. Suppose that $I = Re_ire_jR$ is a nonzero idempotent ideal contained in $J(R)$. Observe that, by Nakayama's lemma, e_iI is not a finitely generated right ideal. Consider $U := e_iR/e_ire_jR$. It is a uniserial module with $U_e = e_iI/e_ire_jR \neq 0$. Since $I = I^2$, there are $a \in e_iI$, $b \in Ie_j$ such that $e_ire_j = ab$. Write $a = se_ire_jv$, $b = ue_ire_jw$, where $s \in e_iR, v, u \in R, w \in Re_j$. Then $e_ire_j = (se_ire_jvue_i)(e_ire_jw)$. Set $V := e_iR/e_ire_jwR$ and note $V_e = e_iI/e_ire_jwR$, hence $[V]_e = [U]_e$. Consider the homomorphism $f\colon V \to U$ induced by $se_ire_jvue_i$. Since $e_ire_jR = (se_ire_jvue_i)(e_ire_jw)R$, f is injective and $U \simeq V$ follows. On the other hand $se_ire_jvue_i \in e_iI$, so there exists an epimorphism $g \in \mathrm{End}_R(U)$ such that $gf = 0$. As $U \simeq V$, we conclude $U_m \subsetneq U_e$ and therefore $\mathrm{Add}(U)$ contains a uniserial module which is not quasi-small.

Conversely, assume that there exists a pure projective R-module X which is uniserial and not quasi-small. Then X contains a finitely generated submodule W' such that $W'_m \subsetneq W'_e$. Consider W' of the form e_iR/e_iI, where I is a right ideal of R. Since $W'_e \neq 0$ there exists $W = e_iR/e_ire_jR$ such that $[W]_e = [W']_e$, for example choose e_ire_j such that $e_ire_j + e_iI$ is a nonzero element of W'_e. The argument from the proof of Lemma 5.3 shows that $W_m \subsetneq W_e$, so there are a monomorphism $f \in \mathrm{End}_R(W)$, an epimorphism $g \in \mathrm{End}_R(W)$ such that $gf = 0$. Suppose that g is induced by $u \in e_iRe_i$ and f is induced by $v \in e_iRe_i$. Then $ve_ire_jR = e_ire_jR$ and $uv \in e_ire_jR$. But u is invertible in e_iRe_i, so $v \in Re_ire_jR$ and consequently the ideal Re_ire_jR is idempotent. Finally note that e_ire_j is has to be an element of $\mathrm{rad}(e_iR) = e_iJ(R)$, so $Re_ire_jR \subseteq J(R)$. ☐

LEMMA 5.5. *Let W be a non-projective indecomposable finitely presented module over a serial ring R, and $f, g \in \mathrm{End}_R(W)$ monomorphisms. Then there exists a monomorphism $h \in \mathrm{End}_R(W)$ such that $f = hg$ or $g = hf$.*

PROOF. Let $W = e_kR/e_kwR$ for some primitive idempotent $e_k \in R$, and let $u, v \in e_kRe_k$ be elements inducing f, g. Since $0 \neq e_kw \in J(R)$, we have $ue_kwR = e_kwR$ and $ve_kwR = e_kwR$. Since e_kRe_k is a chain ring, there exists $t \in e_kRe_k$ such that either $tu = v$ or $u = tv$. In the former case we have $te_kwR = tue_kwR = ve_kwR = e_kwR$. Therefore t induces $h \in \mathrm{End}_R(W)$ such that $hf = g$. Similarly, in the latter case t induces $h \in \mathrm{End}_R(W)$ such that $f = hg$. ☐

PROPOSITION 5.6. *Let R be a serial ring such that there is no pure projective R-module which is uniserial and not quasi-small. The following are equivalent*

(a) *There exists a countably generated pure projective module M and a connected component $\mathcal{C}$ of Γ_R of type 2 such that $\sum_{v_m \in \mathcal{V}_m \cap \mathcal{C}} \mathrm{mdim}_{v_m}(M) > \sum_{v_e \in \mathcal{V}_e \cap \mathcal{C}} \mathrm{edim}_{v_e}(M)$.*
(b) *There exists a countably generated pure projective module M and a connected component $\mathcal{C}$ of Γ_R of type 2 such that $\sum_{v_m \in \mathcal{V}_m \cap \mathcal{C}} \mathrm{mdim}_{v_m}(M) > 0$ and $\sum_{v_e \in \mathcal{V}_e \cap \mathcal{C}} \mathrm{edim}_{v_e}(M) = 0$.*

(c) *There exist* $U, W \in \text{mod-}R$ *indecomposable,* U *of type* 2 *and* $W \not\sim U$*, and homomorphisms* $\alpha\colon U \to W, \beta\colon W \to U$*,* $f, g\colon U \to U$ *such that* f *is mono but not epi and* $gf^2 - f = \beta\alpha$*.*

Proof. $(a) \Rightarrow (b)$: It can be proved as $(a) \Rightarrow (b)$ in Proposition 5.2.

$(b) \Rightarrow (c)$: Suppose there exists M satisfying (b). Then M is a direct summand of $N = \oplus_{i\in\mathbb{N}} U_i$, where the modules $U_i := e_{l_i}R/e_{l_i}s_ie_{r_i}R$ are indecomposable and finitely presented. Let $\pi = (p_{i,j}) \in \text{End}_R(N)$ be an idempotent endomorphism with image M. Then there are $i, j \in \mathbb{N}$ such that $[U_i]_m = [U_j]_m$, $p_{i,j}$ is a monomorphism and $[U_i], [U_j] \in \mathcal{C}$. Moreover, if $[U_k]_e = [U_l]_e$ and $[U_k], [U_l] \in \mathcal{C}$, then $p_{k,l}$ is not onto.

Fix i, j such that $[U_i], [U_j] \in \mathcal{C}$ and $p_{i,j}$ is a monomorphism. By Lemma 3.7, $[U_i]_m = [U_j]_m$. Let $x_{i,j} \in e_{l_i}Re_{l_j}$ be an element inducing $p_{i,j}$. We claim that the left ideal $Rx_{i,j}$ is independent of the choice of $x_{i,j}$, i.e., if $y_{i,j} \in e_{l_i}Re_{l_j}$ is another element inducing $p_{i,j}$, then $Rx_{i,j} = Ry_{i,j}$. Indeed, in this case $x_{i,j} - y_{i,j} \in e_{l_i}s_ie_{r_i}Re_{l_j}$. If $x_{i,j} \in Re_{l_i}s_ie_{r_i}R$, the image of $p_{i,j}$ would be contained in $(U_i)_e$ and it is not possible since in this case $\text{Add}(U_i)$ would contain a uniserial module which is not quasi-small. Therefore the submodule $Rx_{i,j} \subseteq Re_{l_j}$ is strictly bigger than $R(x_{i,j} - y_{i,j})$ and, since Re_{l_j} is uniserial, $Rx_{i,j} = Ry_{i,j}$ follows.

Consider the set $I = \{k \in \mathbb{N} \mid [U_k] \in \mathcal{C}\}$ and its subset $I' = \{k \in I \mid p_{k,j}$ can be induced by $x_{k,j} \in e_{l_k}Re_{l_j}$ such that $Rx_{i,j} \subseteq Rx_{k,j}\}$. Observe that $i \in I'$, so I' is nonempty. We claim that if $k \in I'$, then $Rx_{k,j}$ does not depend on the choice of the element inducing $p_{k,j}$. Assume that $x_{k,j}$ induces $p_{k,j}$ and $Rx_{i,j} \subseteq Rx_{k,j}$. Let $y_{k,j} \in e_{l_k}Re_{l_j}$ be another element inducing $p_{k,j}$, again $y_{k,j} - x_{k,j} \in e_{l_k}s_ke_{r_k}R$ and therefore it suffices to prove $x_{k,j} \notin Re_{l_k}s_ke_{r_k}R$. Since $U_k \sim U_j$ there exists a finitely presented module $W := e_{l_k}R/e_{l_k}wR$ such that $[W]_e = [U_k]_e$ and $[W]_m = [U_j]_m = [U_i]_m$. Let $\nu\colon W \to U_j$, $\mu\colon U_i \to W$ be monomorphisms and let $\varrho\colon U_k \to W$ be an epimorphism. Suppose μ is induced by $u \in e_{l_k}Re_{l_i}$, ν is induced by $v \in e_{l_j}Re_{l_k}$ and ϱ is induced by $s \in e_{l_k}Re_{l_k}$. Let s' be the inverse of s in $e_{l_k}Re_{l_k}$. If $x_{k,j} \in Re_{l_k}s_ke_{r_k}R = Re_{l_k}s'se_{l_k}s_ke_{r_k}R \subseteq Re_{l_k}wR$, then $ux_{i,j}v \in e_{l_k}Re_{l_k}wR$. Therefore the image of the monomorphism $\mu p_{i,j}\nu \in \text{End}_R(W)$ is contained in W_e, a contradiction to the non existence of a non quasi-small uniserial pure projective R-module.

For any $a, b \in \mathbb{N}$ let $x_{a,b} \in e_{l_a}Re_{l_b}$ be an element inducing $p_{a,b}$, and let $k_0 \in I'$ be such that $Rx_{k_0,j}$ is the largest element of the set $\{Rx_{k,j} \mid k \in I'\}$. Consider the equality $p_{k_0,j} = \sum_n p_{k_0,n}p_{n,j}$. Observe that $\sum_n x_{k_0,n}x_{n,j}$ is an element inducing $p_{k_0,j}$, therefore there exists $n \in \mathbb{N}$ such that $Rx_{k_0,j} \subseteq Rx_{k_0,n}x_{n,j}$. If $[U_n] \in \mathcal{C}$, then $n \in I$ and, in fact, $n \in I'$. By the choice of k_0, we have $Rx_{k_0,n}x_{n,j} = Rx_{n,j}$ and $p_{k_0,n}$ is an epimorphism. By Lemma 3.7, $[U_{k_0}]_e = [U_n]_e$ and hence $\text{edim}_{U_n}(M) > 0$. It is not possible since we assume $\text{edim}_{U_n}(M) = 0$. Therefore $n \notin I$.

Let $U \in \text{mod-}R$ be indecomposable such that $[U]_e = [U_{k_0}]_e$ and $[U]_m = [U_j]_m$. Let $\nu\colon U \to U_j$, $\mu\colon U_i \to U$ be non-epic monomorphisms and $\varrho\colon U_{k_0} \to U$ an epimorphism. Suppose μ is induced by u, ν is induced by v and ϱ is induced by s. Observe that we may assume that s is an invertible element of $e_{l_{k_0}}Re_{l_{k_0}}$. Then $ux_{i,j}v$ induces $\mu p_{i,j}\nu$ and $sx_{k_0,n}x_{n,j}v$ induces $\varrho p_{k_0,n}p_{n,j}\nu$. Since $Rux_{i,j}v \subseteq Rsx_{k_0,n}x_{n,j}v$, Lemma 5.5 gives the existence of $g \in \text{End}_R(U)$ such that $\mu p_{i,j}\nu = g((\mu p_{i,j}\nu)^2 + \varrho p_{k_0,n}p_{n,j}\nu)$. So we may set $W := U_n$, $\alpha := p_{n,j}\nu$, $\beta := -g\varrho p_{k_0,n}$, $f := \mu p_{i,j}\nu$.

$(c) \Rightarrow (a)$ Having such $U, W, f, g, \alpha, \beta$ it is possible to apply the construction from [**13**, page 201] to get a countably generated module $M \in \mathrm{Add}(U \oplus W)$. The same arguments as in [**13**, Lemma 5.2] give $\mathrm{mdim}_U(M) = 1, \mathrm{edim}_U(M) = 0$, $*\mathrm{dim}_W(M) = \omega$, where $*\mathrm{dim}_W$ is any dimension function related to W, and $*\mathrm{dim}_V(M) = 0$ if $V \in \text{mod-}R$ is indecomposable such that the edge $[V]$ does not intersect neither with $[W]$ nor with $[U]$. □

LEMMA 5.7. *Let R be a serial ring, $U, V \in \text{mod-}R$ be indecomposable modules such that U is of type 2, $U \not\sim V$ and there are homomorphisms $\alpha\colon U \to V$, $\beta\colon V \to U$ with $\beta\alpha(U_e) \neq 0$. If $U = e_iR/e_ire_jR$, $V = e_kR/e_kse_lR$, $a \in e_kRe_i$ induces α and $b \in e_iRe_k$ induces β, then $e_ire_j \notin e_iRe_ibae_ire_jR$.*

PROOF. Observe that U is of type 2, hence U is not projective, so $e_ire_j \neq 0$. If $e_kse_l = 0$, then $ae_ire_j = 0$ and the statement is obvious. Therefore we may assume $e_kse_l \neq 0$.

Since $\alpha(U_e) \neq 0$, $[\mathrm{Im}\ \alpha]_e = [U]_e$. Since $e_kse_l \neq 0$, Im α is finitely presented by Lemma 3.4. Therefore there exists $z \in e_iRe_ire_jR$ such that Ker $\alpha = zR/e_ire_jR$ and Im α is isomorphic to e_iR/zR. There exists an embedding of e_iR/zR to V, therefore, by Lemma 3.5, it is possible to choose $z \in e_iRe_l$, let $z = e_ir'e_l$.

If $e_kte_i \in e_kRe_i$ is an element inducing a monomorphism of $e_iR/e_ir'e_lR$ to $V = e_kR/e_kse_lR$, then $e_kte_ir'e_lR = e_kse_lR$. Observe that we may choose $e_kte_i = a$.

Consider Im β. Again, by Lemma 3.4, it is a finitely presented module, so it is isomorphic to a module of the form e_kR/wR, where $wR \subseteq e_kR$ contains e_kse_lR. There exists a monomorphism from e_kR/wR to U, so we may choose $w \in e_kRe_j$. Let $w = e_ks'e_j$ for some $s' \in R$. Moreover, observe that $[\mathrm{Im}\ \beta]_m = [U]_m$: Of course, Im β is a submodule of U containing Im $\beta\alpha$ and, since $\beta\alpha(U_e) \neq 0$, $[\mathrm{Im}\ \beta\alpha]_e = [U]_e$. In particular, $\mathrm{Hom}_R(\mathrm{Im}\ \beta\alpha, U)$ contains a monomorphism and an epimorphism, therefore Im $\beta\alpha \simeq U$ and also $[U]_m = [\mathrm{Im}\ \beta]_m$.

We assume $V \not\sim U$ and therefore $[V]_e \neq [\mathrm{Im}\ \beta]_e$. Since Im $\beta \simeq e_kR/e_ks'e_jR$, we have $e_ks'e_j \notin e_kRe_kse_lR$.

Let $e_iue_k \in e_iRe_k$ be an element inducing a monomorphism from $e_kR/e_ks'e_jR$ to U. Again we may choose $e_iue_k = b$. Now assume that the statement of the Lemma is not true, that is, $e_ire_j \in e_iRe_ibae_ire_jR$. Since $ae_ire_j \in e_kse_lR$, we have $e_ire_j \in e_iRe_ibe_kse_lR$. Furthermore, by Lemma 3.11, $[U]_m = [e_kR/e_ks'e_jR]_m$ implies $Re_ire_jRe_j = Re_ks'e_jRe_j$. In particular, $e_ks'e_j = ce_ire_jd$ for some $c \in e_kRe_i, d \in e_jRe_j$. Therefore $e_ks'e_j = ce_ire_jd \in (ce_iRe_ibe_k)(se_lRd) \subseteq e_kRe_kse_lR$. This contradiction concludes the proof. □

LEMMA 5.8. *Let R be a serial ring such that there exist $U, V \in \text{mod-}R$ indecomposable modules, $U \not\sim V$, U is of type 2, monomorphisms $g, f \in \mathrm{End}_R(U)$, with f not onto, and homomorphisms $\alpha\colon U \to V, \beta\colon V \to U$ such that $gf^2 - f = \beta\alpha$. Then there exists a pure projective uniserial R-module which is not quasi-small.*

PROOF. Let e_i, e_k be primitive idempotents of R such that $U = e_iR/e_irR$, $V = e_kR/e_ksR$. Consider elements $x, y \in e_iRe_i$ inducing f, g and elements $a \in e_kRe_i, b \in e_iRe_k$ inducing α, β.

First assume that $\beta\alpha(U_e) \neq 0$. Let us choose $r \in e_iRe_j, s \in e_kRe_l$, then, by Lemma 5.7, $e_iRe_ibae_irR$ does not contain e_ir. The relation $gf^2 - f = \beta\alpha$ can be written as $yx^2 - x - ba \in e_irR$. Therefore $(yx - e_i)xe_ir - bae_ir \in e_irRe_ir$. Observe that $xe_irR = e_irR$, since f is a monomorphism. As f is not onto, $yx - e_i$ is invertible in e_iRe_i. If u is its inverse, then $xe_ir - ubae_ir \in e_iRe_irRe_irR$. Note that $e_irR = xe_irR$ and $e_ir \notin e_iRe_ibae_irR$ implies $e_irR = (xe_ir - ubae_ir)R \subseteq e_iRe_irRe_irR$. Therefore $(Re_irR)^2 = Re_irR$ and, by Proposition 5.4, there exists a pure projective uniserial R-module which is not quasi-small.

Now suppose that $\beta\alpha(U_e) = 0$. Recall U_e is an invariant submodule of U, so we may consider restrictions $f_1 = f|_{U_e}$ and $g_1 = g|_{U_e}$ as elements of $\mathrm{End}_R(U_e)$. Then f_1, g_1 are monomorphisms such that $(g_1f_1 - 1)f_1 = 0$. If we prove that f_1 is not onto, then the monogeny class of U_e contains a uniserial module which is not quasi-small (cf Section 2, property (viii)). Then the same arguments as in the proof of Proposition 5.4 show that there exists a pure projective R-module which is uniserial and not quasi-small.

Assume that f_1 is onto, that is, $xe_iRe_irR = e_iRe_irR$. From $\beta\alpha(U_e) = 0$ we have $(yx - e_i)xe_iRe_irR \subseteq e_irR$. We have already observed that $yx - e_i$ is invertible in e_iRe_i and if u is its inverse then $xe_iRe_irR \subseteq ue_irR \subseteq e_iRe_irR$. But if $xe_iRe_irR = e_iRe_irR$ we get $ue_irR = e_iRe_irR$ and this is a contradiction with U_e not finitely generated. □

By [**9**, Lemma 1.33], a serial ring R is right non-singular if and only if $0 \neq r \in e_iRe_j$ and $0 \neq s \in e_jRe_k$ imply $rs \neq 0$. Observe that, in particular, if R is a right non-singular serial ring, then e_iRe_i is a chain domain for every $i \in \{1, \ldots, n\}$. It follows that a serial ring is right non-singular if and only if it is left non-singular. Such a ring is called a *non-singular serial* ring.

LEMMA 5.9. *Let R be a non-singular serial ring such that there is no pure projective R-module which is uniserial and not quasi-small. Then there are no indecomposable $U, V \in \text{mod-}R$, U of type 2, $U \not\sim V$ and homomorphisms $\alpha\colon U \to V$, $\beta\colon V \to U$ such that $\beta\alpha(U_e) \neq 0$.*

PROOF. Suppose that the statement is not true. Let $U = e_iR/e_ire_jR, V = e_kR/e_kse_lR$, $\alpha \in \mathrm{Hom}_R(U,V)$, and $\beta \in \mathrm{Hom}_R(V,U)$ be such that U is of type 2, $U \not\sim V$, and $\beta\alpha(U_e) \neq 0$. Let $a \in e_kRe_i$, $b \in e_iRe_k$ be elements inducing α, β. By Lemma 5.7, $e_ire_j \notin e_iRe_ibae_ire_jR$.

Since R is non-singular, $R_j := e_jRe_j$ is a chain domain. Observe that R_j does not contain nontrivial idempotent ideals of the form R_jxR_j, otherwise there would exist a pure projective R-module which is uniserial and not quasi-small.

Let $X := \{x \in J(R_j) \mid \exists y \in e_iRe_i\ ye_ire_jx = e_ire_j\}$. For every $x \in X$ set $Q_x := \cap_{t\in\mathbb{N}}(R_jxR_j)^t$. Since R_jxR_j is not idempotent, $x \notin Q_x$. By [**4**, Proposition 5.2(d)], Q_x is a completely prime ideal of R_j. In particular, $Q_x = zQ_x = Q_xz$ for every $z \in R_j \setminus Q_x$. Observe that $bae_ire_j \in e_ire_jx^tR_j$ for every $x \in X$ and every $t \in \mathbb{N}$. Otherwise $e_ire_jx^t \in bae_ire_jR_j$ for some $x \in X$ and $t \in \mathbb{N}$, and if $y \in e_iRe_i$ is such that $ye_ire_jx = e_ire_j$, then $e_ire_j = y^te_ire_jx^t \in e_iRe_ibae_ire_jR$, which is not possible by Lemma 5.7. It follows that $bae_ire_j \in e_ire_jQ_x$ for every $x \in X$. Indeed, since $e_ire_j \notin bae_ire_jR$, we have $bae_ire_j \in e_ire_jR_j$. As R is right non-singular, left multiplication by $e_ire_j\colon R_j \to e_ire_jR_j$ is an isomorphism of right R_j-modules by [**9**, Lemma 1.33]. Let $c \in R_j$ be such that $bae_ire_j = e_ire_jc$. We observed $c \in (R_jxR_j)^t$ for every $t \in \mathbb{N}$ and therefore $c \in Q_x$.

The assumption $\beta\alpha(U_e) \neq 0$ can be rewritten as $e_ire_jR \subsetneq bae_iRe_ire_jR$. Let $y \in e_iRe_i$ be such that $baye_ire_j \notin e_ire_jR$. Obviously $ye_ire_j \notin e_ire_jR$ therefore there exists $x \in X$ such that $ye_ire_jx = e_ire_j$. Then $baye_ire_jx = bae_ire_j \in e_ire_jQ_x$. Since $x \notin Q_x$, there is $q \in Q_x$ such that $bae_ire_j = e_ire_jqx$. Then $baye_ire_jx =$

$e_i r e_j q x$ and we may cancel x because R is non-singular (apply [**9**, Lemma 1.33] again). Then we get $baye_i r e_j \in e_i r e_j R$, a contradiction. □

Now we can formulate the main result of the paper.

THEOREM 5.10. *Let R be a non-singular serial ring. Then the following are equivalent*

(a) *There exists a pure projective R-module which is not a direct sum of finitely presented modules.*
(b) *There exists a pure projective R-module which is uniserial and not quasi-small.*
(c) *There exists $0 \neq e_i r e_j \in J(R)$ such that $(Re_i r e_j R)^2 = Re_i r e_j R$.*

PROOF. Suppose (a) holds and there is no pure projective R-module which is uniserial and not quasi-small. By Kaplansky's theorem (see [**9**, Corollary 14.3]), there exists a countably generated pure projective R-module which is not a direct sum of finitely presented modules. Then Lemma 5.1 implies that the equivalent conditions of Proposition 5.2 or Proposition 5.6 are satisfied. Lemma 5.8 imply that the condition (c) of Proposition 5.6 cannot be satisfied. If the equivalent conditions of Proposition 5.2 are true, we get a contradiction with Lemma 5.9. Therefore (a) implies (b).

(b) implies (a) is obvious.

(b) and (c) are equivalent by Proposition 5.4. □

By [**4**, Lemma 7.17], the Jacobson radical of a ring with right Krull dimension does not contain nonzero idempotent ideals. Therefore

COROLLARY 5.11. *Let R be a non-singular serial ring with right Krull dimension. Then every pure projective R-module is a direct sum of finitely presented modules.*

EXAMPLE 5.12. Recall Facchini's example [**4**, Example 9.20]: Let p, q be different primes, and let $\mathbb{Z}_{(p)}, \mathbb{Z}_{(q)}$ be the localizations of $\mathbb{Z}$ at these primes. Let R be a subring of $\mathrm{M}_4(\mathbb{Q})$ consisting of matrices of the form

$$\begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} & 0 & 0 \\ p\mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} & 0 & 0 \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Z}_{(q)} & \mathbb{Z}_{(q)} \\ \mathbb{Q} & \mathbb{Q} & q\mathbb{Z}_{(q)} & \mathbb{Z}_{(q)} \end{pmatrix}$$

Then R is a serial ring such that the Krull-Remak-Schmidt theorem does not hold in mod-R. Observe that R is non-singular and, by [**4**, Proposition 7.29], R has right Krull dimension 1. Therefore every pure projective R-module is a direct sum of finitely presented modules.

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