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# Decidability and modules over Bézout domains

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*This paper is dedicated to Mike Prest and to the memory of Gena Puninski.*

Abstract. I survey recent results on the decidability of first order theories of modules over several noteworthy Bézout domains.

## 1. Introduction

First of all, I would like to thank Mike for four joint papers with him and Annalisa Marcja [**MPT1**], [**MPT2**], [**MPT3**], [**MPT4**], for his constant generous help and encouragement, for uncountably many suggestions, and more generally as a polestar of the model theory of modules [**P1**], [**P2**]. But let me also dedicate a tribute to the memory of Gena (I do not remember how many joint papers...).

This note is devoted to decidability of first order theories of modules over a given ring $R$. This topic may look out of fashion. How effective a decision algorithm can be, provided it is exists? Are there "really" decidable first order theories with equality? It is well known that an answer to this question overlaps computational complexity and the P versus NP problem. In fact, if any first order theory with equality admits a decision procedure running in at most polynomial time, then P = NP.

In the case of Abelian groups, and more generally of modules over Dedekind domains, decidability results date back to the seminal works of W. Szmielew [**Sz**] and then P. Eklof and E. Fischer [**EF**]. However their procedures are far from being effective, see the introduction of [**Lo**]. In fact, M.J. Fischer and M. Rabin proved [**FR**] that the time complexity of the first order theory of the addition of real numbers, i.e. of divisible torsionfree Abelian groups, is at least exponential (see on that also the final part of [**S**]). Thus, even if Lo [**Lo**] provided a double exponential upper bound for the space complexity of the first order theory of all Abelian groups, procedures handling decidability of theories of modules seem far from being feasible.

Moreover, the decision question for modules over a ring $R$ requires strong assumptions on $R$, for instance countability, otherwise the theory of $R$-modules is undecidable, or the decision problem makes no sense (see [**P1**], Chapter 17). In

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detail $R$ should be *effectively given*, which means, when $R$ is for simplicity a commutative domain with unity, that its elements can be listed, possibly with repetitions, as $a_0 = 0$, $a_1 = 1$, $a_2, \ldots, a_n, \ldots$ ($n \in \mathbb{N}$) so that suitable algorithms effectively perform the following, when $m$, $n$ range over non negative integers:

(1) Deciding whether $a_m = a_n$ or not.
(2) Producing $a_m + a_n$ and $a_m \cdot a_n$ (or rather indices of these elements in the list).
(3) Establishing whether $a_m$ divides $a_n$.

Then other familiar procedures can be effectively carried out, such as determining units, calculating additive and (when possible) multiplicative inverses, calculating in the right frameworks greatest common divisor gcd and least common multiple lcm.

On the other hand, for a given domain $R$, and beyond any countability restriction, decidability of $R$-modules deeply overlaps fundamental model theoretic and algebraic questions on $R$-modules, basic steps towards a full comprehension of them, such as

- the analysis of pp-formulae over $R$,
- the description of the Ziegler spectrum of $R$ (both the points and the topology),
- through them, the classification of $R$-modules up to elementary equivalence,

which highly fits with the spirit of [**Sz**] and [**EF**]. Prest's first Bible of model theory of modules [**P1**] devotes a whole chapter to decidability and undecidability, and explains their connection with pp-formulae and Ziegler topology. We will recall its main facts later, over the domains we are going to consider. Also Mike's more recent fundamental book on module theory [**P2**] extensively updates on (un)decidability results and related references.

The conclusion is that decidability of modules is still alive and, as far as I am concerned, *"close to my heart"*. So let me deal with decidability.

In this note I mainly focus on a research program developed in the latest years with Lorna Gregory, Sonia L'Innocente, Francoise Point and Gena Puninski on the decidability of modules on Bézout domains. The next sections recall some necessary basic facts on model theory of modules and then introduce and comment on this program. But I'll devote a short intermezzo to decidability of representations of the Lie algebra $\mathrm{sl}(2, k)$ where $k$ is an algebraically closed field of characteristic 0.

In the remainder of the paper, I tacitly assume some familiarity with model theory of modules, as presented in the initial chapters of [**P1**] (and partly in [**P2**]).

I would like to thank Lorna Gregory and Sonia L'Innocente for their help and their comments, and the referee for her/his valuable suggestions.

## 2. Basic facts on modules

**2.1. Model theory.** Let me fix my notation: for a given ring $R$, $L_R$ is the first order language of (say right) $R$-modules and $T_R$ is the first order theory of $R$-modules in $L_R$.

For $a \in R$, $a \mid x$ denotes the *divisibility formula* $\exists y(ya = x)$ of $L_R$, defining in an $R$-module $M$ the submodule $Ma$. Similarly $xa = 0$ is the *annihilator formula* defining in $M$ the kernel of the multiplication by $a$.

Both exemplify *pp-formulae* in $L_R$. In fact a *pp-formula* $\varphi(x)$ of $L_R$ in one free variable $x$ singles out in any $R$-module $M$ the elements making a fixed finite linear system of equations with coefficients in $R$ in the indeterminates $x$ and possibily more $z_1, \ldots, z_m$ solvable in the module.

Then each pp-formula $\varphi(x)$ defines in a given $R$-module $M$ a subgroup $\varphi(M)$, called a *pp-subgroup*. Sometimes, and when $R$ is commutative, this subgroup is even a submodule.

As said, I will mainly deal below with divisibility and annihilator formulas, including possible variants, like $\exists y(ya = x \wedge yb = 0)$ or $c \mid xd$ with $a, b, c, d \in R$. I will also consider finite sums and conjunctions of pp-formulae, such as $a \mid x + xb = 0 \doteq \exists x_1 \exists x_2 (a \mid x_1 \wedge x_2 b = 0 \wedge x = x_1 + x_2)$, or $c \mid x \wedge xd = 0$, which is clearly equivalent to a pp-formula. The reader will see later that I will mainly be concerned with these simple formulas because of the rings I am working over.

If one factorizes the set of pp-formulae of $L_R$ in one free variable identifying logically equivalent formulae, then one gets a quotient set $L(R)$ which is a lattice with respect to the partial order determined by implication. In this lattice the meet is given by conjunction of formulae, and the join by their sum.

An *invariant statement* is a sentence of $L_R$ saying, for $\varphi(x), \psi(x)$ pp-formulae and $l$ a positive integer, that the index of the subgroup determined by $\varphi(x) \wedge \psi(x)$ in the subgroup determined by $\varphi(x)$ is exactly $l$, or at least $l$. Let us denote by $\mathrm{Inv}(\varphi, \psi) = l$, $\mathrm{Inv}(\varphi, \psi) \geq l$ these statements, respectively. According to a celebrated result by Baur-Monk:

Theorem 2.1. *Every sentence of $L_R$ is equivalent modulo $T_R$ to a Boolean combination of invariant statements, and even to a disjunction of conjunctions of statements* $\mathrm{Inv}(\varphi, \psi) \geq l$, $\mathrm{Inv}(\varphi, \psi) = l$ *with $\varphi(x)$, $\psi(x)$ pp-formulae, $l$ positive integer.*

The Ziegler spectrum, $\mathrm{Zg}_R$, of a ring $R$ is a quasi-compact topological space whose points are (isomorphism classes of) indecomposable pure injective $R$-modules, and a basis of the topology is given by the compact open sets

$$(\varphi(x)/\psi(x)) := \{N \in \mathrm{Zg}(R) : \varphi(N) \supset \psi(N) \cap \varphi(N)\},$$

where $\varphi(x)$, $\psi(x)$ range over the pp-formulae of $L_R$ in the free variable $x$.

Pure injective $R$-modules are those $R$-modules $N$ satisfying the following weak saturation property: every partial pp-type in one variable over $N$, which is finitely satisfiable over $N$, is actually realised in $N$. Recall that a pp-type is a set of pp-formulae.

Every $R$-module admits up to isomorphism a unique minimal pure injective elementary extension, which is called its *pure injective hull*. Moreover every pure injective module decomposes, essentially in a unique way, as the pure injective hull of a direct sum of indecomposable pure injective modules, possibly plus a *superdecomposable* pure injective summand, that is, a pure injective module without indecomposable summands.

When $R$ is commutative, every indecomposable pure injective $R$-module $N$ localizes, in the sense that the set $\mathfrak{p}(N)$ of elements of $R$ which act as nonisomorphisms on $N$ is a prime ideal of $R$ and $N$ has a natural structure of a module (still indecomposable and pure injective) over the localization of $R$ at $\mathfrak{p}(N)$.

The existence of superdecomposable pure injective $R$-modules is closely related to a possible *dimension* function from the lattice $L(R)$ to ordinals or $\infty$, which is called *width*. See [**P1**], Chapter 10, or [**P2**], Chapter 7 for a detailed definition; basically, width progressively eliminates diamonds in $L(R)$. It turns out that, if there is a superdecomposable pure injective module over $R$, then the width of $L(R)$ remains undefined, or equivalently is meant $\infty$. The converse is also true when $L(R)$ is countable, so, in particular, when $R$ is countable, even if in the uncountable case the equivalence between the two conditions is still an open question.

Another important dimension function on $L(R)$ is its *m-dimension*, that is, the Krull-Gabriel dimension of $R$, also treated in [**P1**], Chapter 10, and in [**P2**], Chapter 7. This time, the $m$-dimension – again an ordinal number or $\infty$ – is determined by iterative factoring $L(R)$, and what is obtained from it, by congruence relations that, this time, collapse intervals of finite length. In the next section we will provide an equivalent characterization of it in topological terms, valid in our setting, that is, over Bézout domains.

**2.2. Decidability.** I follow here [**P1**, 17.3]. Assume $R$ is effectively given, as explained in the previous section 1. Then $T_R$, that is, the set of sentences of $L_R$ true in **all** $R$-modules, can be effectively listed, just as the set of logical consequences of the axioms of $L_R$-modules. Hence, in order to deduce that $T_R$ is decidable, it suffices to produce a similar list of sentences of $L_R$ true in **some** $R$-module. The previous considerations on model theory of $L_R$-modules, both on pp-formulae and on indecomposable pure injective modules, easily lead to restrict the attention to

- sentences that are finite conjunctions of invariament statements $\mathrm{Inv}(\varphi, \psi) \geq l$, $\mathrm{Inv}(\varphi, \psi) = l$.
- modules that are finite direct sums of indecomposable pure injectives.

A general machinery deciding the existence of a module $M$ of this kind realizing a given sentence as above consists of:

- a recursive enumeration of all points $N$ in $\mathrm{Zg}_R$,
- a recursive enumeration of all the open sets $(\varphi/\psi)$ of a basis for the topology of $\mathrm{Zg}_R$,
- an algorithm that, for every choice of $N$, $\varphi$, $\psi$ and of a positive integer $l$, effectively answers whether $|\varphi(N) : \varphi(N) \cap \psi(N)| = l$ or not.

Warning: in order to get decidability one does not necessarily need, for instance, a recursive presentation of $\mathrm{Zg}_R$, but this is often helpful, as it accompanies and supports the decidability analysis.

When the only occurring indexes of pp-subgroups for a ring $R$ are 1 or $\infty$, hence $\infty$ when $> 1$ (we will see soon rings allowing this simplification), things get easier. In fact we are given

(1) finitely many pairs of pp-formulae $(\varphi(x), \psi(x))$ for which we require $\mathrm{Inv}(\varphi, \psi) > 1$,
(2) finitely many pairs of pp-formulae $(\varphi'(x), \psi'(x))$ for which we expect $\mathrm{Inv}(\varphi', \psi') = 1$,

and we look for a finite direct sum $M$ of indecomposable pure injective $R$-modules making the conjunction of all these formulae true. Assume there exists such an indecomposable pure injective module for every statement in (1), so satisfying that particular sentence and all the ones in (2). Build the direct sum of the modules resulting in this way, and get a module respecting all the previous requirements. Hence the general question reduces itself to check, for every choice of $\varphi$ and $\psi$ in (1), the existence of a module lying in $(\varphi/\psi)$ but out of all the open sets $(\varphi'/\psi')$, equivalently to exclude, for every $\varphi$ and $\psi$, that the open set $(\varphi/\psi)$ is contained in the finite union of the neighborhoods $(\varphi'/\psi')$ - a topological question.

Let me also say that undecidability results of modules in general rely on the interpretation of the word problem for groups (see [**P1**, 17.2]). A typical ring with an undecidable theory of modules is that $k\langle X, Y\rangle$ of polynomials in two non commuting indeterminates $X$ and $Y$ over a field $k$.

## 3. Intermezzo

Let me spend here some words on an intriguing decidability project on modules, different from the main one in this paper. It regards representations of the Lie algebra $\mathrm{sl}(2,k)$ of $2\times 2$ traceless matrices over a field of characteristic 0. They can be viewed as modules over the universal enveloping algebra of $\mathrm{sl}(2,k)$, $U(\mathrm{sl}(2,k))$.

The general theory of these representations over any field $k$ of characteristic 0 is undecidable, as it interprets $k\langle X, Y\rangle$-modules (as proved by Prest and Puninski in [**PrP**, Corollary 6.3]).

On the other hand Herzog developed in [**Her**] the model theory of the so called *pseudo-finite dimensional* representations of $\mathrm{sl}(2,k)$ where $k$ is an algebraically closed field. These are the possibly infinite dimensional (over $k$) models of the theory of finite dimensional representations. The latter are completely understood in the setting of $\mathrm{sl}(2,k)$, so of $U(\mathrm{sl}(2,k))$. Among other things, Herzog equipped them with an explicit and nice axiomatization [**Her**, Theorem 63].

L'Innocente and Macintyre [**LM**] started from this analysis with the aim to approach the decidability problem of pseudo-finite representations. Their paper [**LM**] contains interesting positive partial results, based on some widely accepted conjectures of Diophantine geometry.

A beautiful theorem in this direction is that obtained by Herzog and L'Innocente in [**HL**]. For every positive integer $n$, let $L(n)\subseteq k[X,Y]$ be the representation of $\mathrm{sl}(2,k)$ of homogeneous polynomials of total degree $n$. Thus [**HL**] proves that, if $\psi(x)\to\varphi(x)$ is an implication of pp-formulae of $L_{U(\mathrm{sl}(2,k))}$ true in every module, then the function mapping every $n$ into the dimension of $\varphi(L(n))/\psi(L(n))$ over $k$ is primitive recursive. In particular, when $M$ is a finitely generated representation of $U(\mathrm{sl}(2,k))$, then the function mapping any $n$ into the dimension over $k$ of $\mathrm{Hom}_{U(\mathrm{sl}(2,k))}(M, L(n))$ is primitive recursive. As a consequence the set of positive integers $n$ for which $\varphi(L(n))$ properly includes $\psi(L(n))$, so associated with a basic open subset of the Ziegler spectrum of $U(\mathrm{sl}(2,k))$, is computable, and therefore Diophantine. This answers positively a question set in [**LM**].

## 4. Bézout domains

I focus now on Bézout domains, more generally on Prüfer domains.

A commutative domain $R$ with identity is *Bézout* if every 2-generated ideal, and consequently every finitely generated ideal, is principal. A Bézout domain is *coherent*: the intersection of 2 principal ideals is also principal. Then one can determine, for every $a, b\in R$ a greatest common divisor $\gcd(a,b)$ and a least common multiple $\mathrm{lcm}(a,b)$, both defined up to invertible factors, and satisfying the Bézout identities: for some suitable $u, v, g, h\in R$,

$$au + bv = \gcd(a,b), \quad a = g\gcd(a,b), \quad b = h\gcd(a,b).$$

Bézout domains include several noteworthy examples, such as

- principal ideal domains,
- the ring of algebraic integers (not a principal ideal domain, but a directed union of Dedekind domains),
- the ring of entire (complex or real) functions in 1 variable,
- $\mathbb{Z}+X\mathbb{Q}[X]$, and more generally the rings coming from the so-called $D+M$-construction, namely $D+XQ[X]$ where $D$ is a principal ideal domain that is not a field and $Q$ is its field of fractions (one gets in this way a Bézout domain, which is neither Noetherian nor a unique factorization domain),
- (commutative) valuation domains.

Prüfer domains are a larger setting. In fact a domain is Prüfer if all its localizations at maximal ideals, and consequently at non zero prime ideals, are commutative valuation domains.

Notably over a Prüfer domain (in particular over a Bézout domain) $R$ the Cantor-Bendixson rank of the Ziegler spectrum equals the $m$-dimension of $L(R)$: this follows from [**P2**, Corollary 5.3.29 and Corollary 5.3.60]. Recall that the Cantor-Bendixson rank of a topological space is again either an ordinal or $\infty$. At any (successor) step isolated points are forgotten, so producing the so called *derivative* of the preceding space. Limit steps are handled forming intersections.

The model theory of modules over Bézout domains is considered in [**PT1**]. That paper is largely devoted to nonexistence of width and existence of superdecomposable pure injective modules. But at the beginning it recalls that

- over a Prüfer domain $R$ every pp-formula is equivalent to a finite sum of formulas $\exists y(ya = x \wedge yb = 0)$, or also to a finite conjunction of formulas $c \mid xd$;
- over a Bézout domain $R$ every pp-formula is equivalent to a finite sum of formulas $c \mid x \wedge xd = 0$, or also to a finite conjunction of formulas $a \mid x + xb = 0$.

The former statement dates back to old results by Warfield, while the latter uses gcd. As a consequence, over a Bézout domain $R$ a basis for $\mathrm{Zg}_R$ is given by the open sets $(c \mid x \wedge xd = 0 \,/\, a \mid x + xb = 0)$, with $a, b, c, d \in R$.

In the next subsection I recall and comment on some decidability results regarding some of the Bézout domains in the previous list. The case of Dedekind domains is classical. It was considered by Eklof and Fischer, and is illustrated in detail in [**P1**, Section 2.$\mathbb{Z}$]. Actually [**P1**, Corollary 2.$\mathbb{Z}$.11] provides a description of the Ziegler spectrum of a Dedekind domain $R$. Let $\mathfrak{p}$ range over maximal ideals of $R$. Then the indecomposable pure injective $R$-modules are

- $R/\mathfrak{p}^n$ with $n$ a positive integer,
- adic modules, that is, $\mathfrak{p}$-adic completions of localizations of $R$ at $\mathfrak{p}$,
- Prüfer modules $Q/R_{\mathfrak{p}}$,
- and the generic module $Q$ (the field of fractions of $R$ and of each localization).

It turns out that the Cantor-Bendixson rank of $\mathrm{Zg}_R$ is 2, with $Q$ being the unique point of maximal rank. Decidability easily follows for an effectively given $R$ [**P1**, Theorem 17.13].

On the other hand there exist countable Dedekind domains, and indeed countable principal ideal domains, with an undecidable theory of modules. Hence these domains cannot be effectively given. For instance, let $K$ be any undecidable subset of the set of all primes. Let $\mathcal{F}_K$ be the family of finite fields $\mathbb{F}_p$ with $p \in K$. By [**He1**, Theorem A] there is a countable principal ideal domain $R_K$ whose collection of residue fields is just $\mathcal{F}_K$. Following [**PoPr**, Chapter 3], construct for every prime $p$ the sentence of $L_{R_K}$ saying *"there exists an $R_K$-module of size $p$"*. Clearly for every prime $p$ this sentence is in $T_{R_K}$ if and only is $p \in K$. Consequently, if $T_{R_K}$ is decidable, then $K$ is – a contradiction. It follows that $R_K$ cannot be effectively given. So undecidability depends on the lack of an effective presentation.

**4.1. Valuation domains.** In 2015 Lorna Gregory [**G**] extended over valuation domains previous partial results by Gena Puninski, Vera Puninskaya and myself of 2007 [**PPT**] and, answering in the most general form a conjecture in that paper, proved the following beautiful result.

THEOREM 4.1. *The theory $T_V$ of modules over a(n effectively given) valuation domain $V$ is decidable if and only if there is an algorithm which decides the prime radical relation in $V$, namely, for every $a, b \in V$, decides whether $a \in \mathrm{rad}(b)$ (equivalently whether the prime ideals of $V$ containing $b$ also include $a$).*

Note that the direction from left to right is easy to prove: for every effectively given commutative ring R with unity, if $T_R$ is decidable, then there is an algorithm deciding $a \in \mathrm{rad}(bR)$ for $a, b \in R$. This is because, for every $a$ and $b$,

$$a \in \mathrm{rad}(bR)$$

if and only if $T_R$ contains the following sentence of $L_R$

$$\exists u(u \neq 0 \wedge ub = 0) \rightarrow \exists v(v \neq 0 \wedge va = 0).$$

Hence a decision algorithm for $T_R$ implies an algorithm for the radical relation.

The proof of the previous claim is simple. Let $a^n \in bR$ for some positive integer $n$. Take $u \neq 0$ such that $ub = 0$, then $ua^n = 0$. Let $m \leq n$ be the minimal positive integer such that $ua^m = 0$. Put $v = ua^{m-1}$, clearly $v \neq 0$ but $va = 0$.

Conversely we can assume without loss of generality that $b$ is not a unit of $R$. Let $\mathfrak{p}$ be a prime ideal of $R$ including $b$. Then $1 + \mathfrak{p}$ is annihilated by $b$ in $R/\mathfrak{p}$. Then there is some element of $R/\mathfrak{p}$ annihilated by $a$, in other words some $v \in R$, $v \notin \mathfrak{p}$ such that $va \in \mathfrak{p}$. As $\mathfrak{p}$ is prime, $a \in \mathfrak{p}$. Thus $a \in \mathrm{rad}(bR)$.

Of course the opposite direction of the Gregory Theorem is by far the most difficult. We will provide more details on that when discussing the general case, over arbitrary Bézout domains. Actually that approach will focus on inclusions of basic open sets in the Ziegler topology

$$(\star) \qquad (\varphi \,/\, \psi) \subseteq \cup_{i=1}^{n} (\varphi_i \,/\, \psi_i)$$

corresponding to the case when reciprocal indexes of pp-subgroups are 1 or $\infty$. Some highly non trivial work has to be added over valuation domains in the general case, when finite $\neq 1$ invariants arise.

**4.2. The $D + M$-construction.** Let $D$ be a principal ideal domain that is not a field, and let $Q$ denote its field of fraction. Let $R = D + XQ[X]$ be the Bézout domain that $D$ and $Q$ generate via the $D + M$-construction. Thus $R$ is consisting of the polynomials of $Q[X]$ having their constant term in $D$.

The decidability of the theory of $R$-modules is considered in [**PT2**]. It is based on a complete analysis of the Ziegler spectrum $\mathrm{Zg}_R$.

The first step towards this description is the following classification of nonzero prime ideals of $R$:

- $pR$ where $p$ ranges over the prime elements of $D$,
- $f(X)R$ where $f(X)$ ranges over the irreducible polynomials of $Q[X]$ with constant term 1,
- finally $XQ[X]$.

The only non principal ideal is the last one, which is also the intersection of the various ideals $pR$.

Next recall that every indecomposable pure injective $R$-module can be viewed as a module over a localization of $R$ at some prime ideal. This implies that $\mathrm{Zg}_R$ can be viewed as the union of two subspaces,

- the union of the closed spaces $\mathrm{Zg}_{R_{f(X)R}}$ where $f(X)$ ranges over the irreducible polynomials of $Q[X]$ having 1 as a constant term, and
- the union of the closed spaces $\mathrm{Zg}_{R_{pR}}$ where $p$ ranges over the prime elements of $D$.

Tha latter subspaces contain $\mathrm{Zg}_{R_{XQ[X]}}$, which is their intersection. Moreover the intersection of two different subspaces $\mathrm{Zg}_{R_{f(X)R}}$, or of a subspace $\mathrm{Zg}_{R_{f(X)R}}$ and a subspace $\mathrm{Zg}_{R_{pR}}$, is just the point $Q(X)$ - the field of fractions of the polynomial ring $Q[X]$. Of course in all these subspaces one has to consider isolated, adic and Prüfer points in the relative topology.

On this ground one gets the main theorem about the spectrum.

THEOREM 4.2. [**PT2**] *The Cantor-Bendixson rank of* $\mathrm{Zg}_R$*, that is, the m-dimension of* $L(R)$*, is 4, with* $Q(X)$ *being the unique point of maximal rank.*

Now assume $R$ effectively given and turn to decidability. Theorem 4.2 ensures that $\mathrm{Zg}_R$ is countable and provides two explicit lists, of points and basic open sets of the topology. The expected effective list of sentences true in some finite direct sum of indecomposable pure injective $R$-modules is easier to find passing to localizations at prime ideals of $R$, so to valuation domains. In order to control their behaviour, some further effectiveness conditions are necessary on $R$, requiring algorithms

(1)′ listing all the prime elements of $D$,
(2)′ listing all the irreducible polynomials of $Q[X]$ (with constant term 1),
(3)′ calculating for every prime $p$ of D the size of the residue field $D/pD$.

We say that $D$ is *strongly effectively given* when it satisfies these further assumptions. For instance the ring $\mathbb{Z}$ of integers is strongly effectively given, by a procedure going back to Kronecker for checking irreducibility of rational polynomials.

THEOREM 4.3. [**PT2**] *Let $D$ be a strongly effectively given principal ideal domain and let $R = D + XQ[X]$ be the corresponding Bézout domain. Then $T_R$ is decidable.*

In particular the theory of modules over $\mathbb{Z} + X\mathbb{Q}[X]$ is decidable.

**4.3. Entire complex valued functions.** The case of entire functions with complex values is also worth mentioning, despite its uncountable power, because a nice description of the Ziegler spectrum of this ring is provided in [**LPPT**].

Recall that a function $f : \mathbb{C} \to \mathbb{C}$ is said to be *entire*, if it is given by an everywhere convergent power series $\sum_{n=0}^{\infty} a_n z^n$ with complex coefficients $a_n$, i.e. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$.

For instance, the exponential function $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is entire, as is the sine function $\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$. More examples and explanations can be found in any complex analysis textbook, such as [**A**]. For instance, each entire function is differentiable, and its derivative is of the same kind.

Entire functions are preserved under pointwise addition and multiplication, which equips them with a structure of a ring, and indeed of a Bézout domain $\mathbb{E}$. Moreover the residue fields of this domain are infinite. As a consequence the indexes of pp-subgroups are either 1 or $\infty$. This is also because $\mathbb{E}$ is a $\mathbb{C}$-algebra over the constant functions. Let us confirm that the cardinality of $\mathbb{E}$ is the continuum $2^{\aleph_0}$.

The first result on $\mathrm{Zg}_{\mathbb{E}}$ regards its isolated points.

THEOREM 4.4. [**LPPT**] *The isolated points of* $\mathrm{Zg}_{\mathbb{E}}$ *are just the finite length points* $\mathbb{E}/(z-t)^k\mathbb{E}$*,* $t \in \mathbb{C}$*,* $k$ *a positive integer, and are dense in* $\mathrm{Zg}_{\mathbb{E}}$*.*

A full classification of closed points can also be obtained. They are:

(1) The modules $\mathbb{E}/(z-t)^k\mathbb{E}$, $t \in \mathbb{C}$, $k \geq 2$;
(2) The modules $\mathbb{E}/M^k$, still with $k \geq 2$, where $M$ ranges over the maximal ideals of $\mathbb{E}$ that are called *free* (meaning that the intersection of the zero sets of their functions is empty);
(3) finally the generic modules $Q(\mathbb{E}/P)$ – the fields of fractions of $\mathbb{E}/P$ where $P$ runs over prime ideals of $\mathbb{E}$; in particular, when $P = 0$, one obtains in this way the field of meromorphic functions.

The next step is to consider the first Cantor-Bendixson derivative of $\mathrm{Zg}_{\mathbb{E}}$, which is obtained by dropping from $\mathrm{Zg}_{\mathbb{E}}$ the isolated points. It turns out that the remaining modules are exactly those on $\mathbb{E}_S$, where $S$ is the multiplicatively closed set consisting of nonzero polynomials over $\mathbb{C}$. Moreover:

THEOREM 4.5. [**LPPT**] *The first Cantor-Bendixson derivative of* $\mathrm{Zg}_{\mathbb{E}}$ *is perfect, that is, has no isolated point.*

Incidentally: the pure injective envelope of $\mathbb{E}_S$ is a superdecomposable module, over $\mathbb{E}_S$ itself and hence over $\mathbb{E}$.

Another example of superdecomposable pure injective module over $\mathbb{E}$ was already given in [**PT1**, Proposition 6.2 and Example 6.3]. It is based on the following sufficient criterion: *let $R$ be a Bézout domain with a non trivial idempotent ideal $I$ (so $I \neq 0, R$ and $I = I^2$), then $R$ possesses superdecomposable pure injective modules* (see [**Pu**, Theorem 12.12]).

It is easy to build in $R = \mathbb{E}$ an ideal $I$ as required. For every sequence $c = \{c_n :$ $n$ a positive integer $\}$ of natural numbers $c_n$ such that $c_n \leq n$ for all $n$ and $c_n \to \infty$, consider the complex entire function $f_c$ such that $f_c(z) = \prod_{n=1}^{\infty}(1 + z/n^3)^{c_n}$ for all $z$. Let $I$ be the ideal generated by these entire functions. Clearly $I$ is not trivial. On the other hand every function $f_c$ can be viewed as a multiple of $f_{c'}^2$, where $c'_n$ is the integer part of $c_n/2$ for all $n$. Hence $I$ is idempotent.

**4.4. Algebraic integers.** Another crucial example of a Bézout domain, this time countable, is the ring $\mathbb{A}$ of algebraic integers. The decision problem of $\mathbb{A}$-modules is discussed in [**LPT**]. The final statement is short and clear.

THEOREM 4.6. [**LPT**] *The theory of modules over the ring* $\mathbb{A}$ *of algebraic integers is decidable.*

On the other hand, the proof applies to the larger framework of the effectively given Bézout domains $R$ that, like $\mathbb{A}$, satisfy the following conditions:

- Every non zero prime ideal $\mathfrak{p}$ is maximal (namely, $R$ has Krull dimension 1),
- The residue field $R/\mathfrak{p}$ is infinite,
- The maximal ideal of the localization $R_{\mathfrak{p}}$ is not finitely generated.

Hence the positive result for algebraic integers opens the road to possible generalizations. This time the decidability proof avoids any analysis of $\mathrm{Zg}_{\mathbb{A}}$. Previous results by van den Dries and Macintyre ([**D**], [**DM**]) and Prestel and Schmid ([**PrS**]) on $\mathbb{A}$ inspire the approach, and localization at prime ideals is also helpful. The *prime radical relation*, $a \in \mathrm{rad}(bR)$, can be first order represented over Bézout domains such that each nonzero prime ideal is maximal, by the statement $\forall v(\gcd(a, v) = 1 \to \gcd(b, v) = 1)$ (in the language of the ring $R$). In fact I claim that $a \in \mathrm{rad}(bR)$ if and only if this sentence is in the first order theory of $R$:

In fact, assume that this sentence is true in $R$, but $a \notin \mathrm{rad}(bR)$, that is, $a \notin \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ containing $b$. As $\mathfrak{p}$ is maximal, there are $c \in \mathfrak{p}$ and $r \in R$ such that $ar + c = 1$. Then $a, c$ are coprime, while $b, c$ are not. The converse is proved by similar arguments.

Once again, decidability matches with existence of superdecomposable pure injective objects over $\mathbb{A}$. This can be deduced, as used for $\mathbb{E}$, from the existence of non trivial idempotent ideals. In fact $\mathbb{A}$ contains such ideals, for instance, the ideal generated by all roots $\sqrt[2^n]{2}$ when $n$ runs over positive integers.

## 5. The main theorem

My aim is now the enlarge the decidability analysis over general Bézout (or even Prüfer) domains with infinite residue fields. Here is the main result, by Gregory, L'Innocente, Puninski and me.

THEOREM 5.1. [**GLPT**] *Let $R$ be a(n effectively given) Bézout domain with an infinite residue field for every maximal ideal. Then $T_R$ is decidable if and only if there is an algorithm that answers a double prime radical relation, that is, decides, given $a, b, c, d \in R$, whether, for all prime ideals $\mathfrak{p}, \mathfrak{q}$ of $R$ with $\mathfrak{p} + \mathfrak{q} \neq R$, $b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $d \in \mathfrak{q}$ implies $c \in \mathfrak{q}$.*

Generalizations to Prüfer domains are also obtained.

The hypothesis on infinite residue fields involves a large class of noteworthy algebraic examples. We already met among them algebraic integers and complex valued entire functions, but further interesting domains can be added, see [**GLPT**].

Its benefit is that, as already underlined, in order to prove decidability of modules it suffices to check effectively the inclusions of basic open sets of the Ziegler topology

$$(\star) \qquad (\varphi/\psi) \subseteq \cup_{i=1}^{n}(\varphi_i/\psi_i).$$

where $\varphi$, $\psi$, $\varphi_i$, $\psi_i$ $(i = 1, \ldots, n)$ range over pp-formulae in one free variable.

Here is how [**GLPT**] proceeds in this direction, moving from [**G**].

**Step 1.** Reduce the open sets in $(\star)$ into a simple form, very elementary as in $(x = x \,/\, xd = 0)$, $(xb = 0 \,/\, x = 0)$ and $(x = x \,/\, c \mid x)$, or a bit more complicated as $(xb = 0 \,/\, c \mid x)$. This is a quite long and intricate reduction procedure, using the following tools:

- Over Bézout domains, gcd;
- In the Prüfer setting, a result by Tuganbaev [**T**, Lemma 1.3], saying that, if $R$ is a Prüfer domain, then for all $a, b \in R$ there exist $\alpha, r, s \in R$ such that $b\alpha = as$ and $a(\alpha - 1) = br$;
- Over a Prüfer domain $R$, any indecomposable pure injective $R$-module $N$ is *pp-uniserial*, i.e. its lattice of pp-definable submodules is a chain.

Notably, all the required reductions can be performed effectively when $R$ is effectively given. This is also true in the remainder of this section.

**Step 2.** Localize at prime ideals $\mathfrak{r}$ of a Prüfer domain $R$ and follow Gregory's approach over valuation domains. Here the following order relation play a key role, for any relevant $\mathfrak{r}$: for $a, b \in R \setminus \{0\}$, $a \leq_{\mathfrak{r}} b$ if and only $bR_{\mathfrak{r}} \subseteq aR_{\mathfrak{r}}$.

One can express this relation in equivalent ways by using Tuganbaev's result or, over a Bézout domain, gcd. Here are these alternative characterizations, expressed in terms of membership conditions to $\mathfrak{r}$.

- (Over any Prüfer domain) Let $a, b$ be non zero elements of $R$, $\alpha, r, s \in R$, $b\alpha = as$ and $a(\alpha - 1) = br$. Then $a \leq_{\mathfrak{r}} b$ if and only if either $\alpha \notin \mathfrak{r}$ or $r \notin \mathfrak{r}$.
- (Over a Bézout domain $R$) Let $a, b \in R \setminus \{0\}$. Then $a \leq_{\mathfrak{r}} b$ if and only if $a/\gcd(a, b) \notin \mathfrak{r}$.

**Step 3.** With every indecomposable pure injective module $N$ over a Prüfer domain $R$ two ideals of $R$ are associated,

- Ass $N := \{r \in R : \text{there exists } m \in N \setminus \{0\} \text{ with } mr = 0\}$ (so $b \in$ Ass $N$ if and only if the annihilator condition $xb = 0$ is realized by some non zero element of $N$), and
- Div $N := R \setminus \{r \in R : r \mid m \text{ for all } m \in N\}$ (so $a \in$ Div $N$ if and only if there is some element of $N$ that does not realize the divisibility condition $a \mid x$).

Actually both Ass $N$ and Div $N$ are proper prime ideals of $R$, and so is their union Ass $N \cup$ Div $N$, as the set of elements of $R$ that do not act as automorphisms on $N$. Over $R$ as before, it turns out that either Ass $N \subseteq$ Div $N$ or Div $N \subseteq$ Ass $N$, so the intersection of these ideals is either Ass $N$ or Div $N$ respectively.

For every choice of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ of $R$, look at the indecomposable pure injective $R$-modules having $\mathfrak{p}$, $\mathfrak{q}$ as Ass $N$, Div $N$ respectively. Consider the prime ideal $\mathfrak{p} \cap \mathfrak{q}$.

**Step 4.** Now translate ($\star$), as resulting after Step 1, in the local setting through order relations of the elements $a, b, c, d, \ldots$ with respect to the relevant $\leq_{\mathfrak{p}\cap\mathfrak{q}}$, or to equivalent membership conditions to the corresponding $\mathfrak{p} \cap \mathfrak{q}$.

Next let us recall the *double prime radical relation* DPR, which handles the membership conditions of the elements of $R$ to the involved prime ideals: for every Bézout domain $R$ (actually for every commutative ring $R$) and $a, b, c, d \in R$,

$$(a, b, c, d) \in \mathrm{DPR}(R)$$

if and only if, for all prime ideals $\mathfrak{p}, \mathfrak{q}$, if $\mathfrak{p} + \mathfrak{q} \neq R$ then $a \in \mathfrak{p}$ or $b \notin \mathfrak{p}$ or $c \in \mathfrak{q}$ or $d \notin \mathfrak{q}$.

Here are some equivalent characterizations.

- (*Algebraic, local*) Let $R$ be any commutative domain. For $a, b, c, d \in R$, $(a, b, c, d) \notin \mathrm{DPR}(R)$ if and only if there is some maximal ideal $\mathfrak{m}$ of $R$ such that $a \notin \mathrm{rad}(bR_{\mathfrak{m}})$ and $c \notin \mathrm{rad}(dR_{\mathfrak{m}})$.
- (*Algebraic, global*) Let $R$ be a Prüfer domain. For $a, b, c, d \in R$, $(a, b, c, d) \notin \mathrm{DPR}(R)$ if and only if $(\mathrm{rad}(b) : a) + (\mathrm{rad}(d) : c)$ is a proper ideal of $R$.
- (*Topological, the most important for us*) Let $R$ be a Prüfer domain. For $a, b, c, d \in R$, $(a, b, c, d) \in \mathrm{DPR}(R)$ if and only if $(xb = 0 \,/\, d \mid x) \subseteq (xa = 0 \,/\, x = 0) \cup (x = x \,/\, c \mid x)$.

Now let us sketch the proof of the main theorem.

($\Rightarrow$) follows from the topological characterization of DPR.

($\Leftarrow$) The previous analysis already emphasizes the role of pairs of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of $R$, and in this way suggests the notion of *Boolean combination of conditions on a pair of prime ideals*, meaning a Boolean combination $\Delta$ of conditions of the form $a \in P$, $b \notin P$, $c \in Q$ and $d \notin Q$ where $a, b, c, d \in R$ and $P$, $Q$ are variables for prime ideals. We say that a pair of prime ideals $(\mathfrak{p}, \mathfrak{q})$ of $R$ satisfies $\Delta$ if when we replace all instances of $P$ by $\mathfrak{p}$ and all instances of $Q$ by $\mathfrak{q}$ the statement of $\Delta$ becomes true in $R$ (so the various $a, b, c, d$ are meant as constants of the language, taken from $R$).

Now let $R$ be any effectively given Bézout domain, and $\Delta$ be a Boolean combination of conditions on a pair of prime ideals. I claim that, if $\mathrm{DPR}(R) \subseteq R^4$ is recursive, then there is an algorithm which answers whether, for all prime ideals $\mathfrak{p}, \mathfrak{q}$, $\mathfrak{p} + \mathfrak{q} \neq R$ implies that $(\mathfrak{p}, \mathfrak{q})$ satisfies $\Delta$.

In fact one can assume that $\Delta$ is a conjunction of disjunctions, and each disjunct consists of conditions of the form

$$a_l \in P,\ b_j \notin P,\ c_h \in Q,\ d_k \notin Q,$$

where the various $l, j, h, k$ range over suitable finite sets of indexes.

Clearly one can check whether for all primes $\mathfrak{p}, \mathfrak{q}$, $\mathfrak{p} + \mathfrak{q} \neq R$ implies that $(\mathfrak{p}, \mathfrak{q})$ satisfies $\Delta$ (that is, all the disjuncts in the conjunction of $\Delta$) if one can decide whether, for all prime ideals $\mathfrak{p}, \mathfrak{q}$, $\mathfrak{p} + \mathfrak{q} \neq R$ implies that one of the following holds: for some $l, j, h, k$,

$$a_l \in \mathfrak{p},\ b_j \notin \mathfrak{p},\ c_h \in \mathfrak{q} \text{ or } d_k \notin \mathfrak{q}.$$

But this is equivalent to saying that for all prime ideals $\mathfrak{p}, \mathfrak{q}$, $\mathfrak{p} + \mathfrak{q} \neq R$ implies that

$$\text{either } \prod_l a_l \in \mathfrak{p}, \text{ or } \gcd(b_j)_j \notin \mathfrak{p}, \text{ or } \prod_h c_h \in \mathfrak{q}, \text{ or } \gcd(d_k)_k \notin \mathfrak{q},$$

that is, the 4-uple $(\prod_l a_l,\ \gcd(b_j)_j,\ \prod_h c_h, \gcd(d_k)_k$ is in $\mathrm{DPR}(R)$.

This concludes the sketch of the proof.

Over valuation domains $V$ (the setting of [**G**]) one can restrict the analysis to the case when the ideals $\mathfrak{p}$ and $\mathfrak{q}$ are radical, say $\mathrm{rad}(tV)$, $\mathrm{rad}(sV)$ respectively. Consequently the double prime radical relation can be reduced to the single prime radical relation. For instance, a proper inclusion like $\mathfrak{p} \subset \mathfrak{q}$ holds if and only if $s \notin \mathrm{rad}(tV)$.

The case of algebraic integers is now a consequence of the main theorem.

COROLLARY 5.2. *Let $R$ be an effectively given Bézout domain of Krull dimension 1 all of whose residue fields are infinite. Then the theory of $R$-modules is decidable.*

In fact, under the Krull dimension 1 hypothesis, the double prime radical relation can be easily reduced to the classical "one-dimensional" prime radical relation, and the latter can be effectively decided.

## 6. From Bézout to Prüfer

Let me enlarge now the scenery to Prüfer domains. I introduce a larger family of prime radical relations over any Prüfer domain $R$: for every positive integer $n$ we introduce a $(2n+2)$-ary relation $\mathrm{DPR}_n$ in $R$ by putting, for $a, c, b_i, d_i \in R$ $(1 \leq i \leq n)$,

$$(a, c, b_1, \dots, b_n, d_1, \dots, d_n) \in \mathrm{DPR}_n(R)$$

if and only if for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of $R$ with $\mathfrak{p} + \mathfrak{q} \neq R$, either $a \in \mathfrak{p}$ or $c \in \mathfrak{q}$ or some $b_i$ is not in $\mathfrak{p}$ or some $d_i$ is not in $\mathfrak{q}$. Hence DPR, as defined in the last section, is just $\mathrm{DPR}_1$.

The previous approach over Bézout domains now implies, without gcd:

THEOREM 6.1. *Let $R$ be an effectively given Prüfer domain with an infinite residue field for every maximal ideal. If there are algorithms deciding in $R$ the membership to* $\mathrm{DPR}_n$ *for every positive integer $n$ uniformly in $n$, then the theory $T_R$ of all $R$-modules is decidable.*

Two questions arise in a natural way from this theorem:

- First, is the condition on the $\mathrm{DPR}_n$ not only sufficient but also necessary to guarantee that $T_R$ is decidable?
- Secondly, can one bound the $n$'s to check, just as in the Bézout case where $n = 1$ is enough?

Let me recall an old result by Heitmann [**He2**], showing that if a Prüfer domain has Krull dimension $d$ then every finitely generated ideal can be generated by $d+1$ elements. As a consequence the following can be proved.

THEOREM 6.2. *Let $R$ be a Prüfer domain all of whose residue fields are infinite and $N$ be a positive integer such that any finitely generated ideal of $R$ can be generated by $N$ elements (in particular, this is the case when $R$ has Krull dimension $N-1$). If there are algorithms deciding membership of* $\mathrm{DPR}_N(R)$ *then $T_R$ is decidable.*

Here is a partial converse to this result.

THEOREM 6.3. *Let $R$ be a Prüfer domain of Krull dimension* 1 *all of whose residue fields are infinite. Then the following are equivalent:*

(1) *$T_R$ is decidable.*
(2) $\mathrm{DPR}_2(R)$ *is recursive.*
(3) *There is an algorithm which given $a, b_1, b_2 \in R$ answers whether $a \in$* $\mathrm{rad}(b_1 R + b_2 R)$.

But work on Prüfer domains and Bézout domains possibly with finite residue fields is still in progress; see [**GLT**]. This also raises the following intriguing question: do there exist Prüfer domains, or even Bézout domains, whose theory of modules is undecidable because it interprets the word problem of groups? I do not know any such ring. In other words, does decidability/undecidability in this setting only depend on the choice and on the possibility of an effective presentation? Note that the case of principal ideal domains was already discussed in Chapter 4, just before 4.1.

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