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Total positivity, Grassmannian and modified Bessel functions

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Abstract. A rectangular matrix is called *totally positive,* (according to F. R. Gantmacher and M. G. Krein) if all its minors are positive. A point of a real Grassmannian manifold $G_{l,m}$ of l-dimensional subspaces in $\mathbb{R}^m$ is called *strictly totally positive* (according to A. E. Postnikov) if one can normalize its Plücker coordinates to make all of them positive. The totally positive matrices and the strictly totally positive Grassmannians, that is, the subsets of strictly totally positive points in Grassmannian manifolds arise in many areas: in classical mechanics (see the book of F. R. Gantmacher and M. G. Krein); in a wide context of analysis, differential equations and probability theory (see the book of S. Karlin); in physics, for example, in construction of solutions of the Kadomtsev-Petviashvili (KP) partial differential equation (see a paper by T. M. Malanyuk, a paper by M. Boiti, F. Pemperini, A. Pogrebkov, a paper of Y. Kodama, L. Williams). Different problems of mathematics, mechanics and physics led to constructions of totally positive matrices by many mathematicians, including F. R. Gantmacher, M. G. Krein, I. J. Schoenberg, S. Karlin, A. E. Postnikov and ourselves. One-dimensional families of totally positive matrices whose entries are modified Bessel functions of the first kind have arisen in our study (in collaboration with S. I. Tertychnyi) of model of the overdamped Josephson effect in superconductivity and double confluent Heun equations related to it.

In the present paper we give a new construction of multidimensional families of totally positive matrices different from the above-mentioned families. Their entries are again formed by values of modified Bessel functions of the first kind, but now with non-negative integer indices. Their columns are numerated by the indices of the modified Bessel functions, and their rows are numerated by their arguments. This yields new multidimensional families of strictly totally positive points in all the Grassmannian manifolds. These families represent images of explicit injective mappings of the convex open simplex $\{x = (x_1, \dots, x_l) \in \mathbb{R}^l \mid 0 < x_1 < \cdots < x_l\} \subset \mathbb{R}^l$ to the Grassmannian manifolds $G_{l,m}$, $l < m$.

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1. Introduction

1.1. Brief survey on totally positive matrices. Main result. The following notion was introduced in the classical books [**16**, **17**] in the context of the classical mechanics.

Definition 1.1. [**1**, **16**, **28**], [**17**, p. 289 of the Russian edition] A rectangular $l \times m$-matrix is called *totally positive* (nonnegative), if its minors of all sizes are positive (nonnegative).

Example 1.2. It is known that every generalized Vandermonde matrix

$$(f(x_i, y_j))_{i=1,\dots,m;\ j=1,\dots,n},\ f(x,y) = x^y,$$
$$0 < x_1 < \dots < x_m,\ 0 \leq y_1 < y_2 < \dots < y_n$$

is totally positive [**13**, chapter XIII, section 8].

The study of $n \times n$ matrices with positive elements goes back to Perron [**27**] who had shown that for such a matrix the eigenvalue that is largest in the modulus is simple, real and positive, and the corresponding eigenvector can be normalized to have all the components positive (1907).

In 1908 this result was generalized by G. Frobenius [**13**, chapter 13, section 2] to those matrices with non-negative coefficients that are block-non-decomposable. For each of these matrices he proved that its complex eigenvalues of maximal modulus are roots of a polynomial $P(\lambda) = \lambda^h - r^h$, all of them are simple and one of them is real and positive.

In 1935–1937 F. R. Gantmacher and M. G. Krein [**14**, **15**] observed that if the matrix satisfies a stronger condition of total positivity (in fact a weaker, oscillation property is sufficient), then *all* its eigenvalues are simple, real and positive.

Earlier in 1930 I. Schoenberg [**30**] studied a more general class of matrices including totally positive ones: namely, the $m \times n$-matrices such that for every $k \leq \min\{m, n\}$ all the non-zero minors of order k have the same sign (either all positive, or all negative). He proved important results relating the latter property to variation-diminishing property of the corresponding linear transformations $\mathbb{R}^n \to \mathbb{R}^m$. (Recall that a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ is called *variation-diminishing,* if it does not increase the number of sign changes in the sequence of coordinates of any vector.) Further results in this direction were obtained by Motzkin in his 1933 dissertation [**26**] and most complete results were obtained by Gantmacher and Krein [**17**].

A two-sided sequence $(a_j)_{j\in\mathbb{Z}}$ of real numbers is called *totally nonnegative (positive),* if the infinite matrix $(a_{j-i})_{i,j\in\mathbb{Z}}$ is totally nonnegative (positive). There is a remarkable result on characterization of totally nonnegative sequences. It says that for each totally nonnegative sequence distinct from a geometric progression the corresponding generating function $\sum_j a_j z^j$ converges as a Laurent series in some annulus, extends meromorphically to all of $\mathbb{C}^*$ and is a product of $\exp(q_1 z + q_2 z^{-1})$ and an infinite product of fractions of appropriate linear functions; here $q_1, q_2 \geq 0$. In 1948 I. Schoenberg [**31**] proved sufficiency: total nonnegativity of Laurent series of the latter functions. The converse (necessity) was proved by A. Edrei in 1953 [**8**]. See also [**20**, Theorem 8.9.5] and references in this book and in [**8**, **31**].

The characterization of all totally nonnegative two-sides sequences is a basic fundamental result used in the description of the characters of representations of the infinite unitary group $U(\infty) = \varinjlim U(n)$ and the infinite symmetric group $S(\infty) =$

$\varinjlim S(n)$. See papers by E. Thoma [**33**], D. Voiculescu [**36**], joint papers by A. M. Vershik and S. V. Kerov [**34, 35**] and references therein.

Many results on characterization and properties of totally positive matrices and their relations to other areas of mathematics (e.g., combinatorics, dynamical systems, geometry and topology, probability theory, Fourier analysis, representation theory), mechanics and physics are given in [**1**–**4**, **8**–**24**, **28**, **29**, **31**, **34**, **35**] (see also references therein). F. R. Gantmacher and M. G. Krein [**16, 17**] considered totally positive matrices in the context of applications to mechanical problems. S. Karlin [**20**] considered them in a wide context of analysis, differential equations and probability theory. In 2008 G. Lusztig suggested an analogue of the theory of total positivity in the Lie group context [**22**].

Total positivity was used to construct solutions of the Kadomtsev-Petviashvili (KP) differential equation in a paper by T. M. Malanyuk [**25**], a paper of M. Boiti, F. Pempinelli, A. Pogrebkov [**4**, section II], and a paper of Y. Kodama, L. Williams [**21**]. S. Fomin's talk at the ICM-2010 [**11**] was devoted to deep relations between total positivity and cluster algebras. There exist several approaches to construction of totally positive matrices, see [**5, 16, 20, 29**], [**17**, p. 290 of Russian edition]. In the previous paper [**5**] we have constructed a class of explicit one-dimensional families of totally positive matrices given by a finite collection of double-sided infinite vector functions, whose components are modified Bessel functions of the first kind[1]. Matrices of such kind arise in a paper of V. M. Buchstaber and S. I. Tertychnyi [**6**] in the construction of solutions of the non-linear differential equations in a model of overdamped Josephson junction in superconductivity. It was shown in [**5**] that the nature of the modified Bessel functions as coefficients of appropriate generating function implies that the infinite vector formed by appropriate minors of the above-mentioned matrices satisfies the *differential-difference heat equation* with positive constant potential in the Hilbert space l_2.

In the present paper we provide a new construction of multidimensional family of totally positive matrices formed by a finite collection of one-sided infinite vector functions. This family is parameterized by a domain in $\mathbb{R}^l$. The rows of the matrix correspond to coordinates x_i in $\mathbb{R}^l$, and their entries are modified Bessel functions of the corresponding coordinates.

Definition 1.3. (see [**20**, chapter 2, Definition 1.1, p. 46]). A function $K(x,y)$ on a product of two totally ordered sets $X \times Y$ is called a *totally positive (strictly totally positive) kernel* of order $r \in \mathbb{N}$, if for every $1 \le m \le r$, $x_1 < \cdots < x_m$ in X and $y_1 < \cdots < y_m$ in Y the determinant of the matrix $(K(x_i, y_j))_{i,j=1}^m$ is nonnegative (respectively, positive).

We recall that the modified Bessel functions $I_j(y)$ of the first kind are Laurent series coefficients for the family of analytic functions

$$g_y(z) := e^{\frac{y}{2}(z+\frac{1}{z})} = \sum_{j=-\infty}^{+\infty} I_j(y) z^j.$$

[1] After our paper [**5**] was published, we have found that total positivity of the matrices considered there, i.e., [**5**, Theorem 1.3], was not new, and it follows from [**20**, Theorem 10.1 (a), p.428]. At the same time, in [**5**] we have elaborated a new method of proof of their total positivity. This method is again used in the present paper.

Equivalently, they are defined by the integral formulas

$$I_j(y) = \frac{1}{\pi}\int_0^{\pi} e^{y\cos\phi}\cos(j\phi)d\phi,\ j \in \mathbb{Z}.$$

REMARK 1.4. It is known that the infinite matrix $(A_{km})_{k,m\in\mathbb{Z}}$ with $A_{km} = I_{m-k}(x)$ is totally positive for every $x > 0$, see [**20**, Theorem 10.1 (a), p. 428], [**5**, Theorem 1.3] and Footnote 1.

THEOREM 1.5. *For every $r \in \mathbb{N}$ the function*

$$K(x,j) = I_j(x), \quad j \in \mathbb{Z}_{\geq 0},\ x > 0$$

is a strictly totally positive kernel of order r with $X = \mathbb{R}_+$ and $Y = \mathbb{Z}_{\geq 0}$.

Let us reformulate Theorem 1.5 in a more explicit way. To do this, set

$$X_l = \{x = (x_1, \dots, x_l) \in \mathbb{R}_+^l \mid x_1 < x_2 < \dots < x_l\};$$
$$K_m = \{k = (k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m \mid k_1 < k_2 < \dots < k_m\}.$$

For every $x \in X_l$ and $k \in K_m$ set

$$A_{k,x} = (a_{ij})_{i=1,\dots,l;\ j=1,\dots m},\ a_{ij} = I_{k_j}(x_i). \tag{1.1}$$

In the special case, when $l = m$, set

$$f_k(x) = \det A_{k,x}. \tag{1.2}$$

THEOREM 1.6. *For every $m \in \mathbb{N}$, $k \in K_m$ and $x \in X_m$ one has $f_k(x) > 0$.*

Theorem 1.6 is equivalent to Theorem 1.5 and will be proved in Section 2.
The following statement is an immediate consequence of the theorem.

COROLLARY 1.7. *For every $x = (x_1, \dots, x_l) \in X_l$ the one-sided infinite matrix formed by the values $a_{ij} = I_j(x_i)$, $i = 1, \dots, l$, $j = 0, 1, 2, \dots$ is totally positive.*

REMARK 1.8. Various necessary and sufficient conditions on a kernel K to be (strictly) totally positive were stated and proved in S. Karlin's book [**20**, chapter 2]. If $K(x,y)$ is defined on the product of two intervals and is smooth enough, a sufficient condition for its strict total positivity is that a suitable matrix of partial derivatives of K (which is a matrix function of (x,y)) is strictly totally positive for all (x,y) [**20**, chapter 2, Theorem 2.6, p. 55]. (The same condition written in the form of non-strict inequality is necessary for total positivity, see [**20**, chapter 2, Theorem 2.2, p. 51].) In [**20**, chapter 3, p. 109] S. Karlin presented a construction of totally positive kernel coming from a single modified Bessel function of the first kind. Namely, set

$$\kappa_\alpha(x;\lambda) = \begin{cases} e^{-(x+\lambda)}(\frac{x}{\lambda})^{\frac{\alpha}{2}} I_\alpha(2\sqrt{x\lambda}) \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases},$$

$$K_\alpha(x,y) = \kappa_\alpha(x-y;\lambda).$$

It was shown in [**20**, chapter 3, p. 109] that for every $\alpha > 1$ and every $r < \alpha + 2$ the function $K_\alpha(x,y)$ is a totally positive kernel of order r. S. Karlin also presented constructions of totally positive matrices coming from the Green function of a linear differential operator presented as a product of n first order linear differential operators of appropriate type. All his operators act on functions of a variable x. Karlin's constructions dealt with the fundamental solution $\phi(x,t)$ of the corresponding linear differential equation: $\phi(x,t) = 0$ for all $x \leq t$, $\phi_x^{(j)}(t,t) = 0$ for all

$j < n-1$, the derivative $\phi_x^{(n-1)}(x,t)$ has a discontinuity at $x = t$. Karlin showed [**20**, p. 503] that appropriate class of matrices of type $\phi(x_i, t_j)$ have positive determinants. A similar construction of totally positive matrices was associated to any given classical Sturm–Liouville differential operator [**20**, pp. 535–538].

Let us emphasize that each totally positive matrix given by our construction includes values of modified Bessel functions with different indices, which are solutions of Sturm–Liouville equations with different parameters. On the other hand, each totally positive matrix from the above-mentioned Karlin's construction is expressed via either just one given modified Bessel function, or via fundamental solutions of just one given linear equation. Therefore, our result is not covered by Karlin's constructions.

It is known that the modified Bessel functions $I_\nu(x)$ of the first kind are given by the series

$$I_\nu(x) = (\frac{1}{2}x)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}x^2)^k}{k!\Gamma(\nu+k+1)},$$

and the latter series extends them to all the real values of the index ν.

Thus, the modified Bessel functions of the first kind yield examples of totally positive kernels of two following different kinds. For every $r \in \mathbb{N}$, Karlin's example, see Remark 1.8, yields a totally positive kernel $K_\alpha(x,y) = \kappa_\alpha(x-y;\lambda)$ of order r constructed from just one modified Bessel function I_α with arbitrary real index $\alpha > 1$ such that $r < \alpha + 2$. Our main result (Theorem 1.5) gives other, strictly totally positive kernel $K(y,s) = I_s(y)$ depending on $y \in \mathbb{R}_+$ and $s \in \mathbb{Z}_{\geq 0}$.

Open Question. *Is it true that the determinants $f_k(x)$ in (1.2) with $x \in X_m$ are all positive for every $m \in \mathbb{N}$ and every $k = (k_1, \dots, k_m)$ with* **(may be non-integer)** k_j and $k_1 \in \mathbb{R}_{\geq 0}$*, $k_1 < \dots < k_m$? In other words, is it true that the kernel $K(y,s) = I_s(y)$ is strictly totally positive as a function in $(y,s) \in \mathbb{R}_+ \times \mathbb{R}_{\geq 0}$?*

1.2. A brief survey on total positivity in Grassmannian manifolds and Lie groups. A point L of Grassmannian manifold $G_{l,m}$ of l-subspaces in $\mathbb{R}^m$, $m > l$ is represented by an $l \times m$-matrix, whose rows form a basis of the subspace represented by the point L. Recall that the *Plücker coordinates* of the point L are the l-minors of this matrix. The Plücker coordinates of the point are well-defined up to multiplication by a common factor, and they are considered as homogeneous coordinates representing a point of the projective space $\mathbb{RP}^N$, $N = \binom{m}{l} - 1$. The Plücker coordinates induce the Plücker embedding of the Grassmannian manifold to $\mathbb{RP}^N$.

Definition 1.9. A point $L \in G_{l,m}$ is called *strictly totally positive,* if it can be represented by a matrix with all the maximal minors positive.

A. E. Postnikov's paper [**29**] deals with the matrices $l \times m$, $m \geq l$ of rank l satisfying the condition of nonnegativity of its minors of the maximal size. One of its main results provides an explicit combinatorial cell decomposition of the corresponding subset in the Grassmannian $G_{l,m}$, called the *totally nonnegative Grassmannian.* The cells are coded by combinatorial types of appropriate planar networks. K. Talaska [**32**] developed further and generalized Postnikov's result. In particular, for a given point of the totally nonnegative Grassmannian the results of [**32**] allow to decide what is its ambient cell and what are its affine coordinates in the cell. S. Fomin and A. Zelevinsky [**12**] studied a more general notion of total positivity (nonnegativity) for elements of a semisimple complex Lie group with a given double Bruhat cell decomposition. They have proved that the totally positive parts of the double Bruhat cells are bijectively parameterized by the product of the positive quadrant $\mathbb{R}_+^m$ and the positive subgroup of the maximal torus. For other results on totally positive (nonnegative) Grassmannians see [**19**].

Theorem 1.6 of the present paper implies the following corollary.

Corollary 1.10. *For every $l, m \in \mathbb{N}$, $l < m$, and every $k \in K_m$ the mapping $H_k : X_l \to G_{l,m}$ sending x to the l-subspace in $\mathbb{R}^m$ generated by the vectors*

$$v_k(x_i) = (I_{k_1}(x_i) \dots I_{k_m}(x_i)),\ i = 1, \dots, l$$

is well-defined and injective. Its image is contained in the open subset of strictly totally positive points.

Proof. The well-definedness and positivity of Plücker coordinates are obvious, since the l-minors of the matrix $A_{k,x}$ are positive, by Theorem 1.6. Let us prove injectivity. First let us fix two distinct points $x, y \in X_l$. Let us show that $H_k(x) \neq H_k(y)$. Fix a component y_i that is different from every component x_j of the vector x. Then the vectors $v_k(x_1), \dots, v_k(x_l), v_k(y_i)$ are linearly independent, since every $(l+1)$-minor of the matrix formed by them is non-zero, by Theorem 1.6. Hence, $v_k(y_i)$ is not contained in the l-subspace $H_k(x)$, which is generated by the vectors $v_k(x_1), \dots, v_k(x_l)$. Thus, $H_k(y) \neq H_k(x)$. □

Example 1.11. Consider the infinite matrix with elements

$$A_{ms} = I_{s-m}(x), \quad m, s \in \mathbb{Z}.$$

It is known, see [**20**, Theorem 10.1 (a), p. 428], [**5**, Theorem 1.3], that this matrix is totally positive for every $x > 0$. Therefore, the subspace generated by any of its l rows is l-dimensional, and it represents a strictly totally positive point of the infinite-dimensional Grassmannian manifold of l-subspaces in the infinite-dimensional vector space. Every submatrix in A_{ms} given by its l rows and a finite number $r > l$ of columns represents a strictly totally positive point of the Grassmannian manifold $G_{l,r}$.

2. Positivity. Proof of Theorem 1.6

We prove Theorem 1.6 by induction in m. In its proof we will use the following classical properties of the modified Bessel functions I_j of the first kind [**37**, section 3.7].

$$I_j = I_{-j}; \tag{2.1}$$

$$I_j|_{\{y>0\}} > 0;\ I_j(0) = 0 \text{ for } j \neq 0;\ I_0(0) > 0; \tag{2.2}$$

$$I_0' = I_1;\ I_j' = \frac{1}{2}(I_{j-1} + I_{j+1}). \tag{2.3}$$

For $m = 1$ the statement of the theorem follows obviously from (2.2).

Let the statement of the theorem be proved for $m = m_0$. Let us prove it for $m = m_0 + 1$. To do this, consider the sequence of the determinants $f_k(x)$ for all $k \in K_m$ as an infinite-dimensional vector function

$$(f_k(x_1, w))_{k \in K_m} \tag{2.4}$$

in the new variables

$$(x_1, w), \quad w = (w_2, \dots, w_m), \ w_j = x_j - x_1; \quad w \in X_{m-1}.$$

The proof of the induction step is analogous to the arguments from [**5**, section 2]. We show (the next two propositions and corollary) that the vector function (2.4) with fixed w and variable x_1 satisfies a linear bounded autonomous ordinary differential equation on the Hilbert space l_2 with coordinates f_k, $k \in K_m$. We show that the positive quadrant $\{f_k \geq 0 \mid k \in K_m\} \subset l_2$ is invariant under the positive flow of the vector field defining this differential equation, and that the initial value $(f_k(0, w))_{k \in K_m}$ lies there. This implies that $f_k(x_1, w) \geq 0$ for all $x_1 \geq 0$, and then we easily deduce that the latter inequality is strict for $x_1 > 0$.

Let us recall the notion of the discrete Laplacian Δ_{discr} acting on functions on the Cayley graph of the additive group $\mathbb{Z}^m$. Namely, for every $j = 1, \dots, m$ let T_j denote the corresponding shift operator:

$$(T_j f)(k) = f(k_1, \dots, k_{j-1}, k_j - 1, k_{j+1}, \dots, k_l).$$

Then

$$\Delta_{discr} = \frac{1}{2} \sum_{j=1}^{m} (T_j + T_j^{-1} - 2). \tag{2.5}$$

Thus, one has

$$(\Delta_{discr} f)(p) = \frac{1}{2} \sum_{s=1}^{m} (f(p_1, \dots, p_{s-1}, p_s - 1, p_{s+1}, \dots, p_m)$$

$$+ f(p_1, \dots, p_{s-1}, p_s + 1, p_{s+1}, \dots, p_m)) - m f(p). \tag{2.6}$$

Remark 2.1. We will deal with the class of sequences $f(k)$ with the following properties:

(i) $f(k) = 0$, whenever $k_i = k_j$ for some $i \neq j$;

(ii) $f(k)$ is an even function in each component k_i.

This class includes $f(k) = f_k(x_1, w)$: statement (i) is obvious; statement (ii) follows from equality (2.1). In this case the discrete Laplacian is well-defined by the above formulas (2.5), (2.6) on the restrictions of the latter sequences $f(k)$ to $k \in K_m$, as in [**5**, Remark 2.1]. (Each sequence $(f(k))_{k \in K_m}$ can be extended to a sequence $(f(k))_{k \in \mathbb{Z}^m}$ satisfying (ii) and antisymmetric with respect to permutation of components, hence satisfying (i).) In more detail, it suffices to check well-definedness of formula (2.6) for $p = (p_1, \dots, p_m) \in K_m$ with $p_1 = 0$. All the terms of the sum in (2.6) except for the first one are well-defined, since they are numerated by indices $k \in K_m$ and maybe some indices k with equal components $k_j = k_{j+1}$, for which $f(k) = 0$, by (i). The first term equals $f(-1, p_2, \dots, p_m) + f(1, p_2, \dots, p_m) = 2f(1, p_2, \dots, p_m)$, by (ii), and hence, is well-defined.

PROPOSITION 2.2. *(analogous to* [**5**, Proposition 2.2]*). For every $m \geq 1$ and $w \in \mathbb{R}^{m-1}$ the vector function $(f(x_1, k) = f_k(x_1, w))_{k \in K_m}$ satisfies the following linear differential-difference equation*[2]*:*

$$\frac{\partial f}{\partial x_1} = \Delta_{discr} f + mf. \tag{2.7}$$

PROOF. Each function $f_k(x_1, w)$ is the determinant of the matrix whose columns are the vector functions

$$V_{k_j}(x_1, w) = (I_{k_j}(x_1), I_{k_j}(x_1 + w_2), \dots . I_{k_j}(x_1 + w_m)), \quad j = 1, \dots, m.$$

The derivative $\frac{\partial f_k(x_1,w)}{\partial x_1}$ thus equals the sum over $j = 1, \dots, m$ of the same determinants, where the column V_{k_j} is replaced by its derivative $\frac{\partial V_{k_j}}{\partial x_1}$. According to (2.3), the latter derivative equals $\frac{1}{2}(V_{k_j-1} + V_{k_j+1})$. Therefore,

$$\frac{\partial f}{\partial x_1} = \frac{1}{2} \sum_{j=1}^{m} (T_j + T_j^{-1}) f = (\Delta_{discr} f + mf). \tag{2.8}$$

□

REMARK 2.3. (analogous to [**5**, Remark 23]). For every $k \in K_m$ the k-th component of the right-hand side in (2.7) is a linear combination with strictly positive coefficients of the components $f(x_1, k')$ with $k' \in K_m$ obtained from $k = (k_1, \dots, k_m)$ by adding ± 1 to some k_i. This follows from (2.8).

PROPOSITION 2.4. [**5**, Proposition 2.4]. *For every constant $R > 1$ and every $j \geq R^2$ one has*

$$|I_j(z)| < \frac{R^j}{j!} \text{ for every } 0 \leq z \leq R. \tag{2.9}$$

COROLLARY 2.5. *(analogous to* [**5**, Corollary 2.6]*). For every $w \in \mathbb{R}^{m-1}$ and $x_1 \in \mathbb{R}$ one has $(f_k(x_1, w))_{k \in K_m} \in l_2$. Moreover, there exists a function $C(R) > 0$ in $R > 1$ such that*

$$\sum_{k \in K_m} |f_k(x_1, w)|^2 < C(R) \text{ whenever } |x_1| + |w| \leq R, \tag{2.10}$$

here $|w| = |w_2| + \dots + |w_m|$.

PROOF. The proof of Corollary 2.5 repeats the proof of [**5**, Corollary 2.6] with minor changes. Namely, let us fix a number $R > 1$ and set

$$M := \max_{j \in \mathbb{Z},\ 0 \leq z \leq R} |I_j(z)|.$$

By (2.9) and [**5**, Remark 2.5], M is finite. Recall that $0 \leq k_1 < \dots < k_m$ for every $k = (k_1, \dots, k_m) \in K_m$. For every $(x_1, w) \in \mathbb{R}^m$ with $|x_1| + |w| \leq R$ one has

$$|f_k(x_1, w)| \leq m! M^m \text{ for every } k \in K_m, \tag{2.11}$$

$$|f_k(x_1, w)| < m! \frac{R^{k_m}}{k_m!} M^{m-1} \text{ whenever } k_m \geq R^2. \tag{2.12}$$

Indeed, the modulus of each element of the $m \times m$-matrix $A_{k,x}$ is no greater than M, whenever $|x_1| + |w| \leq R$. Therefore, the modulus $|f_{k,n}(x)|$ of its determinant

[2] In similar formula (2.7) in [**5**] there is a misprint: the right-hand side should be multiplied by $\frac{1}{2}$.

defined as the sum of $m!$ products of its elements is no greater than $m!M^m$. This proves (2.11). The last column of the matrix $A_{k,x}$ consists of the values $I_{k_m}(x_i) = I_{k_m}(x_1+w_i)$. If $k_m \geq R^2$ and $|x_1|+|w| \leq R$, then their moduli are no greater than $\frac{R^{k_m}}{k_m!}$, by inequality (2.9). Therefore, each one of the $m!$ above-mentioned products of matrix elements in the expression of determinant has module no greater than $M^{m-1}\frac{R^{k_m}}{k_m!}$. This implies (2.12). The number of those $k \in K_m$ for which $k_m < R^2$ is less than R^{2m}. Therefore, the sum in (2.10) through $k \in K_m$ does not exceed[3]

$$C(R) := R^{2m}(m!M^m)^2 + (m!M^{m-1})^2 \sum_{k\in K_m} \frac{R^{2k_m}}{(k_m!)^2} < +\infty.$$

□

Definition 2.6. [**5**, Definition 2.7]. Let Ω be the closure of an open convex subset in a Banach space. For every $x \in \partial\Omega$ consider the union of all the rays issued from x that intersect Ω in at least two distinct points (including x). The closure of the latter union of rays is a convex cone, which will be here referred to as the *generating cone* $K(x)$.

Proposition 2.7. [**5**, Proposition 2.8]. *Let H be a Banach space, $\Omega \subset H$ be the closure of an open convex subset. Let v be a C^1 vector field on a neighborhood of the set Ω in H such that $v(x) \in K(x)$ for every $x \in \partial\Omega$. Then the set Ω is invariant under the flow of the field v: each positive semitrajectory starting at Ω is contained in Ω.*

Now the proof of the induction step in Theorem 1.6 is analogous to the argument in [**5**, end of section 2]. The right-hand side of differential equation (2.7) is a bounded linear vector field on the Hilbert space l_2 of sequences $(f_k)_{k\in K_m}$. We will denote the latter vector field by v. Let $\Omega \subset l_2$ denote the "positive quadrant" defined by the inequalities $f_k \geq 0$. For every point $y \in \partial\Omega$ the vector $v(y)$ lies in its generating cone $K(y)$: the components of the field v are non-negative on Ω, by Remark 2.3. The vector function $(f_k(x_1) = f_k(x_1, w))_{k\in K_m}$ in $x_1 \geq 0$ is an l_2-valued solution of the corresponding differential equation, by Corollary 2.5. One has $(f_k(0))_{k\in K_m} \in \Omega$:

$$f_k(0) = 0 \text{ whenever } k_1 > 0;$$

$$f_{(0,k_2,\dots,k_m)}(0) = I_0(0)f_{(k_2,\dots,k_m)}(w_2,\dots w_m) > 0. \tag{2.13}$$

The latter equality and inequality follow from definition, the properties (2.2) and the induction hypothesis. This together with Proposition 2.7 implies that

$$f_k(x_1, w) \geq 0 \text{ for every } k \in K_m \text{ and } x_1 \geq 0. \tag{2.14}$$

Now let us prove that the inequality is strict for all $k \in K_m$ and $x_1 > 0$. Indeed, let $f_p(x_0) = 0$ for some $p = (p_1,\dots,p_m) \in K_m$ and $x_0 > 0$. All the derivatives of the function f_p are non-negative, by (2.7), Remark 2.3 and (2.14). Therefore, $f_p \equiv 0$ on the segment $[0, x_0]$. This together with (2.7), Remark 2.3 and (2.14) implies that $f_{p'} \equiv 0$ on $[0, x_0]$ for every $p' \in K_m$ obtained from p by adding ± 1 to some component. We then get by induction that $f_{(0,k_2,\dots,k_m)}(0) = 0$, – a contradiction to (2.13). The proof of Theorem 1.6 is complete.

[3]In paper [**5**] there is a misprint in a similar formula for the constant $C(R)$ on p. 3863: in its right-hand side one should replace the factor $l!M^{l-1}$ and the term $\frac{R^{|k|_{n,\max}}}{(|k|_{n,\max})!}$ by their squares.

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