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Surfaces with big automorphism groups

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In memory of Professor Selim Krein, whose Voronezh winter mathematical schools were of immense benefit to the author

Abstract. We describe a classification of $\mathbb{G}_m$-actions on quasi-homogeneous (Gizatullin) surfaces and recent results about $\mathbb{A}^1$-fibrations on these surfaces and their algebraic automorphism groups. We present also a classification of generalized quasi-homogeneous surfaces, where a normal complex affine surface V is generalized quasi-homogeneous if the group of holomorphic automorphisms of V acts transitively on the complement to a finite subset in V.

Introduction

The aim of this survey is to present recent developments in the theory of affine algebraic surfaces with big automorphism groups, i.e. automorphism groups that possess open orbits with finite complements. Unless it is stated otherwise, we consider these surfaces over the field of complex numbers $\mathbb{C}$.

The classical example of such a surface is the plane $\mathbb{C}^2$ which we equip with a coordinate system (x, y). Let us recall some facts about the group of algebraic automorphisms $\mathrm{Aut}(\mathbb{C}^2)$ of $\mathbb{C}^2$. It contains the subgroup of affine transformations $\mathcal{A}_2 = \{\varphi = (q_1, q_2) \in \mathrm{Aut}(\mathbb{C}^2) | \deg q_1 = \deg q_2 = 1\}$ and the subgroup $\mathcal{E}_2$ of elementary automorphisms that are of the form $(x, y) \to (a_1x + b, a_2y + p(x))$, where $q_1, q_2 \in \mathbb{C}[x, y]$, $p \in \mathbb{C}[x]$, $a_1, a_2 \in \mathbb{C}^*$, and $b \in \mathbb{C}$. The Jung-van der Kulk theorem states that $\mathrm{Aut}(\mathbb{C}^2)$ is the amalgamated product of $\mathcal{A}_2$ and $\mathcal{E}_2$, i.e. every element α of $\mathrm{Aut}(\mathbb{C}^2)$ can be presented as a composition $\alpha = \varphi_1 \circ \psi_1 \circ \ldots \circ \varphi_m \circ \psi_m$, where $\varphi_i \in \mathcal{A}_2$ and $\psi_i \in \mathcal{E}_2$ are determined uniquely modulo the subgroup $\mathcal{A}_2 \cap \mathcal{E}_2$.[1] The group $\mathrm{Aut}(\mathbb{C}^2)$ itself is not algebraic and the theorem of Wright [**26**] states that every algebraic subgroup G of $\mathrm{Aut}(\mathbb{C}^2)$ is of bounded length. That is, there exists $m_0 > 0$ such that every $\alpha \in G$ can be presented as a composition $\varphi_1 \circ \psi_1 \circ \ldots \circ \varphi_m \circ \psi_m$ with $m \leq m_0$. By Serre's theorem [**22**], such a subgroup is isomorphic to a subgroup of either $\mathcal{A}_2$ or $\mathcal{E}_2$. This fact yields the earlier results of Gutwirth [**16, 17**] and Renschler [**21**] that state that in suitable polynomial coordinate systems on $\mathbb{C}^2$

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[1] This theorem is valid over any field (not necessarily algebraically closed or of characteristic zero).

every effective $\mathbb{G}_m$-action[2] is given by

(1) $$(x,y) \to (\lambda^k x, \lambda^l y), \text{ where } \lambda \in \mathbb{C}^* \text{ and } k,l \in \mathbb{Z} \text{ are coprime,}$$

and every $\mathbb{G}_a$-action is given by

(2) $$(x,y) \to (x, y+tp(x)), \text{ where } t \in \mathbb{C}_+.$$

Actually, Gutwirth formulated his result (which implies (2)) in terms of $\mathbb{A}^1$-fibrations. Recall that an $\mathbb{A}^1$-fibration is a dominant morphism $\varrho : X \to B$ of algebraic varieties such that general fibers of ϱ are isomorphic to $\mathbb{C}$. If X and B are affine then there is a $\mathbb{G}_a$-action whose general orbits are fibers of ϱ. Vice versa, every nontrivial $\mathbb{G}_a$-action on an affine algebraic variety X of dimension at most 3 generates an $\mathbb{A}^1$-fibration into an affine base B (the so-called categorical quotient morphism [**11**]). In this terminology the Gurwirth's result says that every polynomial on $\mathbb{C}^2$ which yields an $\mathbb{A}^1$-fibration is a variable (more precisely, a variable in a suitable coordinate system). A much more delicate fact is the Abhyankar-Moh-Suzuki theorem that says that an irreducible polynomial whose zero locus is isomorphic to $\mathbb{C}$ is a variable or, equivalently, every plane curve isomorphic to $\mathbb{C}$ can be sent to a coordinate line by an automorphism of $\mathbb{C}^2$. It is worth mentioning that the analytic analogue of this statement does not hold - there are proper holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ such that their images cannot be sent to a coordinate line by holomorphic automorphisms of $\mathbb{C}^2$ [**10**]. However, there is a positive result in the analytic setting - M. Suzuki proved in [**25**] that every holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^2$ can be presented in a suitable holomorphic coordinate system by Formula (1). Nothing like this can be true in the case of the holomorphic $\mathbb{G}_a$-actions.

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1. Danielewski's Surfaces

In this section we consider one of the simplest nontrivial examples of a surface with a big automorphism group - a Danielewski's surface $S_{1,p}$. Recall that a Danielewski's surface is a hypersurface in $\mathbb{C}^3_{x,y,z}$ [3] such that it is given by $S_{n,p} = \{(x,y,z) \in \mathbb{C}^3 : \ x^n y = p(z)\}$ where p is a polynomial of degree at least 2 with simple roots. The remarkable fact established by Danielewski is that $S_{n,p} \times \mathbb{C}$ is isomorphic to $S_{k,p} \times \mathbb{C}$ while for different n and $k \in \mathbb{N}$ the surfaces $S_{n,p}$ and $S_{k,p}$ are not isomorphic (i.e. these surfaces yield a counterexample to the general version of the Zariski cancellation conjecture). Actually, in the original approach of Danielewski and Fieseler a stronger fact was established: the fundamental groups of $S_{n,p}$ and $S_{k,p}$ at infinity are different which implies that they are not even homeomorphic. However, there are other methods to prove the absence of isomorphism between $S_{1,p}$ and $S_{n,p}$ for $n \geq 2$, which are more interesting for the purpose of this survey. Namely, one can see that $S_{1,p}$ admits two obvious $\mathbb{A}^1$-fibrations $x : S_{1,p} \to \mathbb{C}$ and $y : S_{1,p} \to \mathbb{C}$ and, therefore, it admits two $\mathbb{G}_a$-actions with different general orbits. On the other hand, for $n \geq 2$ there is no $\mathbb{A}^1$-fibration on $S_{n,p}$ but the obvious one $x : S_{n,p} \to \mathbb{C}$. It is worth studying underlying reasons for this fact.

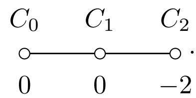
Consider $\mathbb{C}^2_{x,z}$ as the complement in $S' = \mathbb{P}^1 \times \mathbb{P}^1$ to the lines $C_0' = \{x = \infty\}$ and $C_1' = \{z = \infty\}$. Let $C_2' = \{x = 0\}$. Suppose for simplicity that $p(z) = z(z-1)$. Let $\pi : \tilde{S} \to S'$ be the blowing up of S' at the two points in C_2' given by $x = z = 0$

[2]Recall that a $\mathbb{G}_m$-action (resp. $\mathbb{G}_a$-action) on an algebraic variety X is an algebraic action of the group $\mathbb{C}^*$ (resp. $\mathbb{C}_+$).

[3]The lower indices mean that $\mathbb{C}^3$ is equipped with a coordinate system (x,y,z).

and $x = z - 1 = 0$ and let C_i be the proper transform of C'_i. Then one can see that the Danielewski surface $S_{1,p}$ is obtained from $\tilde{S}$ by removing the divisor $D = C_0 + C_1 + C_2$, i.e. $\tilde{S}$ is a simple normal crossing (SNC) completion of $S_{1,p}$ with D as the boundary divisor.

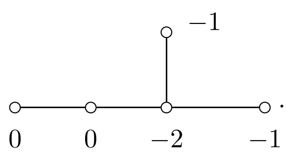
Recall that the weighted dual graph Γ of a closed SNC curve in a complete surface is the graph whose vertices are the irreducible components of the curve, the edges between the vertices are the intersection points of these components, and the weight of every vertex C is the selfintersection number $C \cdot C$ of this component C in the surface. Since after blowing the surface up at a smooth point of C one has to reduce the selfintersection number of the proper transform of C by 1 and $C'_i \cdot C'_i = 0$ (for C'_i in S'), we have the following linear dual graph of D:



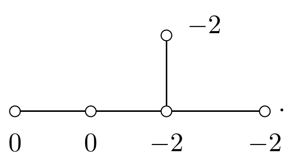
It is a classical fact [**2**] that any smooth rational curve C in a smooth complete surface Y with selfintersection $C \cdot C = 0$ is a fiber of a $\mathbb{P}^1$-fibration $Y \to B$ onto a curve B. In particular, C_0 induces a fibration $\tau : \tilde{S} \to \mathbb{P}^1$. Note that C_1 is a section of this fibration since it meets C_0 transversely. Therefore, the restriction of τ to $S_{1,p}$ is an $\mathbb{A}^1$-fibration (that is nothing but the one provided by the function x). The only fiber of τ different from $\mathbb{P}^1$ is the fiber over $0 \in \mathbb{P}^1$ containing C_2. It contains also two exceptional (-1)-curves E_0 and E_1 of π, where each E_i meets C_2 transversely and $E_i \cap S_{1,p}$ corresponds to the line $x = z - i = 0$ in $S_{1,p}$.

One can see that the dual graph of the SNC-curve $D + E_0 + E_1$ has the following form:

(3)



In order to obtain $S_{2,p}$, one has to consider the blowing up of $\tilde{S}$ at the two points $x = z = y = 0$ and $x = z - 1 = y = 0$ and remove the proper transforms of D and also of E_0 and E_1 (whose weights in the resulting surface are equal to -2). This leads to the following dual graph for the boundary divisor of the SNC-completion of $S_{2,p}$

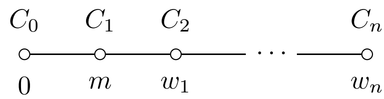


We see that this graph is not linear and this is the crucial difference.

2. Gizatullin surfaces

THEOREM 2.1. ([**12**]) *Let V be a normal affine surface different from $\mathbb{C}^* \times \mathbb{C}$ or the torus $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$. Then the following are equivalent:*
(i) *V admits two different $\mathbb{A}^1$-fibrations with affine bases;*
(ii) $\mathrm{Aut}(V)$ *has an open orbit whose complement is at most finite;*

(iii) *V admits an SNC-completion $\tilde{V}$ such that every component of $D = \tilde{V} \setminus V$ is rational and the dual graph $\Gamma(D)$ of D is*



where $n \geq 0$, $m \in \mathbb{Z}$, and every $w_i \leq -2$.

Furthermore, for the given surface V the ordered sequence of weights $[[w_1, w_2, \ldots, w_n]]$ is unique up to the reversion $[[w_n, \ldots, w_2, w_1]]$, while m can be chosen arbitrarily.

Remark 2.2. One can easily see that the automorphism group $\mathrm{Aut}(V)$ acts transitively on V when $V = \mathbb{C}^* \times \mathbb{C}$ or $V = \mathbb{T}$, i.e. statement (ii) is valid for each of these surfaces. However, they do not satisfy (i) and (iii).

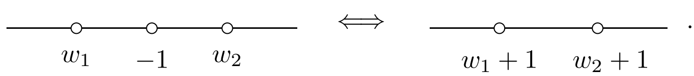
Definition 2.3. (1) A surface with such an SNC-completion is called a Gizatullin (or quasi-homogeneous) surface.

(2) The graph in Theorem 2.1 (iii) is called a zigzag. In the case of $m = 0$ we call such a zigzag standard.

(3) The $\mathbb{P}^1$-fibration defined by C_0 on $\tilde{V}$ has only one non-general fiber and it contains $C_2, \ldots, C_n$ and some other irreducible curves called feathers. The union of D and feathers is an SNC-curve and its dual graph is called the extended graph of the zigzag.

Remark 2.4. In a smooth Gizatullin surface the complement to the open orbit of $\mathrm{Aut}(V)$ may be non-empty (Danilov-Gizatullin and Kovalenko [**19**]).

Recall that given a (-1)-vertex in a dual graph with at most two neighbors (a so-called linear vertex), we can contract it, which leads to a change of the graph as below:



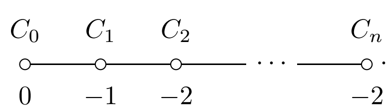
Using these transformations, we can expand the Gizatullin theorem further.

Theorem 2.5. ([**12**]) *A normal affine surface is a Gizatullin surface or $\mathbb{C}^* \times \mathbb{C}$ if and only if any SNC-completion of this surface has rational components and its dual graph is contractible to a linear one.*

Hence one can see that the surface $S_{2,p}$ in the previous section is not a Gizatullin surface and thus it has only one $\mathbb{A}^1$-fibration.

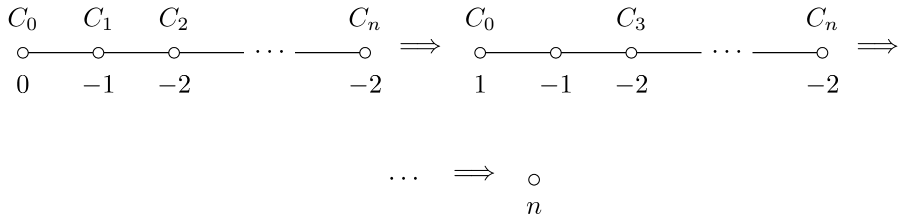
3. Danilov-Gizatullin surfaces

Definition 3.1. Let a smooth surface V with the Picard group different from $\mathbb{Z}_2$ have an SNC-completion with rational components and a dual graph of the form



Then it is called a Danilov-Gizatulin surface (e.g. the Danielewski surface $S = \{xy = z(z-1)\}$ is a Danilov-Gizatullin surface).

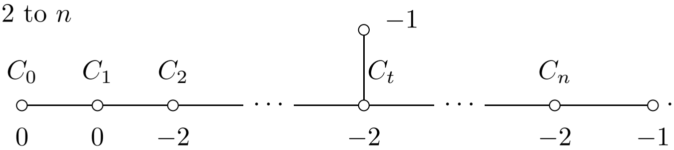
Using transformations



we get an SNC-completion of this surface by an irreducible curve of selfintersection index n. One can check that this completion is a Hirzebruch surface Σ_d over $\mathbb{P}^1$ and the curve is a section L with $L \cdot L = n$, where $d \leq n$.

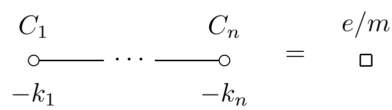
THEOREM 3.2. ([**5**]) *The isomorphism class of a Danilov-Gizatullin surface is completely determined by the number $L \cdot L$ (which is called the index of the surface).*

REMARK 3.3. Though the standard zigzag of such a surface is unique, the extended graph is not. Here is its general form in which the index t can take any value from 2 to n

(4) 

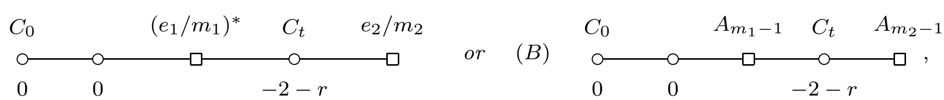
4. $\mathbb{G}_m$-actions on normal surfaces

NOTATION 4.1. We abbreviate by $A_k = [[(-2)_k]]$ the weighted linear graph with k vertices of weight -2 each and by a box □ with a rational weight e/m the weighted linear graph



with $k_1, \ldots, k_n \geq 2$, where m/e coincides with the continuous fraction $[k_1, \ldots, k_n]$[4], $0 < e < m$ and $\gcd(m, e) = 1$. A box with the weight $(e/m)^*$ will denote the similar linear chain of vertices with the reverse order of the weights.

THEOREM 4.2. ([**6**]) *A standard zigzag occurs as the boundary divisor D of a smooth Gizatullin surface $V \neq \mathbb{C}^2$ or $\mathbb{C} \times \mathbb{C}^*$ with a nontrivial $\mathbb{G}_m$-action if and only if it can be written in one of the forms*



where $r \geq 0$, $m_i \geq 1$, $\gcd(e_i, m_i) = 1$ for $i = 1, 2$, and either

$$\frac{e_1}{m_1} + \frac{e_2}{m_2} = 1, \quad \textit{or} \quad \frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 - \frac{1}{m_1 m_2}.$$

Furthermore, the extended graph $\Gamma_{\text{ext}}(D)$ has only one branch vertex C_t.

[4] For a sequence of real numbers $k_0, \ldots, k_n$ with $k_0, \ldots, k_{n-1} \geq 2$ and $k_n \geq 1$ we let $[k_0, \ldots, k_n]$ be the continued fraction defined inductively via $[k_0] = k_0$ and $[k_0, \ldots, k_n] = k_0 - \frac{1}{[k_1, \ldots, k_n]}$.

A few words about the proof of this result. By the Sumihiro theorem [**23**], there exists a $\mathbb{G}_m$-equivariant completion $\tilde{V}$ of V. One can show that such a completion can be chosen as a zigzag. Hence the $\mathbb{G}_m$-action preserves the 0-vertex C_0 and the $\mathbb{P}^1$-fibration associated with it. In particular, it preserves the feathers (since they are components of the singular fiber of this fibration) and the points where the feathers meet the components of the boundary. Therefore, if C_i is a branch point of $\Gamma_{\rm ext}(D)$, then the restriction of the $\mathbb{G}_m$-action to it is trivial, since it has at least three fixed points on $C_i \simeq \mathbb{P}^1$. Another easy claim is that when this $\mathbb{G}_m$-action is trivial on two components of the boundary, then it is trivial on V. This yields the last claim of the theorem, from which it is not difficult to extract the rest.

Remark 4.3. When $r = 0$ in class (B) from Theorem 4.2, then V is just a Danilov-Gizatullin surface with the extended graph as in (4). When $r \geq 1$ and $t \neq n$, then there are $r + 1$ feathers $F, F_1, \ldots, F_r$ that are neighbors of C_t (in the case of $t = n$ there is one extra feather neighboring C_t). Furthermore, $F_1, \ldots, F_r$ are (-1)-vertices in $\Gamma_{\rm ext}(D)$, while the weight of F is $1 - t$.

Definition 4.4. (1) We call a Gizatullin surface special if it has a dual graph as in class (B) from Theorem 4.2 with $r \geq 1$. It is special of type I if either $t \in \{2, n\}$ or $r = 1$. Otherwise it is special of type II.

(2) We say that two $\mathbb{G}_m$-actions (resp. $\mathbb{A}^1$-fibrations) on an algebraic surface V are equivalent if there exists an automorphism of V that transforms one into the other.

Theorem 4.5. ([**7**, **9**]) *Let V be a smooth affine surface with a nontrivial $\mathbb{G}_m$-action. Then this action is unique up to equivalence and taking the inverse action $\lambda \to \lambda^{-1}$ if and only if V is not a toric surface* [5]*, a Danilov-Gizatullin surface of index $n \geq 4$, or a special Gizatullin surface.*

In the case of a Danilov-Gizatullin surface V with an extended graph $\Gamma_{\rm ext}(D)$ as in (4), the multiplicities of the two feathers in the singular fiber of the $\mathbb{P}^1$-fibration $\psi_t : \tilde{V} \to \mathbb{P}^1$ associated with C_0 are 1 and $t - 1$ respectively. As we mentioned, when $\tilde{V}$ is a $\mathbb{G}_m$-equivariant completion, then the $\mathbb{G}_m$-action preserves ψ_t and, in particular, the $\mathbb{A}^1$-fibration $\psi_t|_V : V \to \mathbb{C}$. This enabled P. Russell to conclude that for different values of t such actions are not equivalent, i.e. there are at least $n - 1$ non-equivalent $\mathbb{G}_m$-actions on V. In fact, there are no more [**6**]. In the case of special smooth Gizatullin surfaces, one has the following.

Theorem 4.6. [**8**]) *Let V be a special smooth Gizatullin surface. If V is of type I, then the equivalence classes of $\mathbb{G}_m$-actions on V form in a natural way a 1-parameter family, while in case of type II they form a 2-parameter family.*

The proof of this result is based on a difficult generalization of Theorem 3.2 [**8**, Corollary 6.1.4] that can be formulated in the following form:
Up to an affine transformation of $C_t \setminus C_{t-1} \simeq \mathbb{C}$, the isomorphism class of the special Gizatullin surface V from Theorem 4.2 (B) is uniquely determined by the set of points $\{F_i \cap C_t\}_{i=1}^r$.

[5]Recall that a surface is called toric if it admits an effective algebraic action of the torus $\mathbb{T}$. The list of smooth affine toric surfaces consists of $\mathbb{T}$, $\mathbb{C}^* \times \mathbb{C}$ and $\mathbb{C}^2$ while every other affine normal toric surface can be obtained as the quotient of $\mathbb{C}^2$ with respect to a linear action of the group of d-roots of unity for some $d \in \mathbb{N}$.

5. $\mathbb{A}^1$-fibrations

There is an analogue of Theorem 4.5 for $\mathbb{A}^1$-fibrations.

THEOREM 5.1. ([**7**]) *Let V be a smooth affine surface with a nontrivial $\mathbb{G}_m$-action such that V is not a Danilov-Gizatullin surface of index $n \geq 4$ or a special Gizatullin surface. Then V admits at most two non-equivalent $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$.*

In the case of a Danilov-Gizatullin surface, the number of non-equivalent $\mathbb{A}^1$-fibrations is finite when the index $n \leq 5$. If $n \geq 6$, then there exist $\mathbb{A}^1$-fibrations that are not invariant under any $\mathbb{G}_m$-action (unlike the ones discussed in the previous section). It is unknown if the number of such fibrations is finite for $n = 6$.

THEOREM 5.2. ([**8**, Example 6.3.21]) *If V is a Danilov-Gizatullin surface of index $n \geq 7$, then V carries continuous families of pairwise non-equivalent $\mathbb{A}^1$-fibrations with the number of parameters being an increasing function of n.*

The similar fact holds for special Gizatullin surfaces.

THEOREM 5.3. ([**8**, Corollary 6.3.22]) *Let V be a special Gizatullin surface with a zigzag of length n. Then there are families of pairwise non-equivalent $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$ depending on $r(n) \geq 1$ parameters with $\lim_{n\to\infty} r(n) = \infty$.*

There is no analogue of the Abhyankar-Moh-Suzuki Theorem for Gizatullin surfaces. Indeed, consider the Danielewski surface $S = \{xy = z(z-1)\}$ and the curves L_1 and L_2 in it given by $x = 1$ and by $x = z = 0$ respectively. Both of them are isomorphic to $\mathbb{C}$, but there is no automorphism of S that transforms one into the other, since L_1 is a principal divisor while L_2 is not.

A more delicate question was studied in [**15**]. It turns out that, if the Picard group $\mathrm{Pic}(V)$ of a Gizatullin surface V is not a torsion group, then there may exist an affine line $L \simeq \mathbb{C}$ in V that is not a component of a fiber of any $\mathbb{A}^1$-fibration $V \to \mathbb{C}$. Actually, the authors found such an L for which the logarithmic Kodaira dimension of $V \setminus L$ is nonnegative, while for any component of a fiber of an $\mathbb{A}^1$-fibration the similar dimension is $-\infty$.

6. Amalgams

Recall that a tree is a connected (non-weighted) graph without cycles. Consider a pair $(T, \mathcal{G})$ that consists of a tree T and a collection $\mathcal{G}$ of vertex groups $(G_P)_{P\in \mathrm{vert}T}$ and edge groups $(G_\nu)_{\nu\in \mathrm{edge}T}$ such that for every edge $\nu = [P, Q]$ of T, there are monomorphisms $G_\nu \to G_P$ and $G_\nu \to G_Q$ identifying G_ν with proper subgroups of the vertex groups G_P and G_Q. Then one can construct a unique group $G = \lim_{\to}(T, \mathcal{G})$ called the free amalgamated product of the vertex groups, where G is freely generated by the subgroups (G_P) and (G_ν) with unified subgroups $G_P \cap G_Q = G_\nu$ for each $\nu = [P, Q] \in \mathrm{edge}T$ [**22**, Ch. I, Sections 4,5].

The following analogue of the Jung-Van der Kulk theorem was proved in [**5**].

THEOREM 6.1. *Let V be a Danilov -Gizatullin surface of index $n \leq 5$ and $\mathrm{Aut}_o(V)$ be the connected component of identity in $\mathrm{Aut}(V)$. Then $\mathrm{Aut}_o(V)$ is an amalgamated product of at most three algebraic groups.*

On the other hand, the following result holds [**20**].

PROPOSITION 6.2. *Let Y be a normal affine variety that does not admit a unipotent group action with a general orbit of dimension ≥ 2. Suppose that the connected component $\mathrm{Aut}_o(Y)$ of identity in $\mathrm{Aut}(Y)$ is an amalgamated product of a countable number of algebraic groups. Then the set of non-equivalent $\mathbb{A}^1$-fibrations on Y over affine bases is countable.*

In combination with Theorems 5.2 and 5.3, this Proposition yields the next fact.

THEOREM 6.3. ([**20**]) *Let V be either a Danilov-Gizatullin surface of index $n \geq 7$ or a special Gizatullin surface. Then $\mathrm{Aut}_o(V)$ is not an amalgamated product of a countable number of algebraic groups.*

Another characterization of the complexity of the automorphism groups of Gizatullin surfaces was found by Blanc and Dubouloz [**3**]. Namely, there exists a surface V such that for the normal subgroup $N(V) \subset \mathrm{Aut}(V)$ generated by all algebraic subgroups of $\mathrm{Aut}(V)$ the quotient $\mathrm{Aut}(V)/N(V)$ contains a free group on an uncountable set of generators.

7. Generalized Gizatullin surfaces

Recall that a holomorphic vector field ν on a complex space X is called complete if the ODE $\frac{d}{dt}\Phi(x,t) = \nu(\Phi(x,t))$ with initial data $\Phi(x,0) = x$ has a solution $\Phi(x,t)$ defined for all values of complex time $t \in \mathbb{C}$ and every starting point $x \in X$, i.e. the flow $\Phi(*,t)$ yields a complex one-parameter subgroup of the holomorphic automorphism group $\mathrm{Aut}_{\mathrm{hol}}(X)$.

Every $\mathbb{G}_a$-action (resp. $\mathbb{G}_m$-action) on an algebraic variety X is the flow of a complete algebraic vector field which is called locally nilpotent (resp. semi-simple). However, there are complete algebraic vector fields that are neither locally nilpotent nor semi-simple and their flows are not algebraic but only holomorphic actions of $\mathbb{C}_+$. Up to conjugation by elements of $\mathrm{Aut}(\mathbb{C}^2)$, such complete algebraic fields on $\mathbb{C}^2$ were classified by Brunella [**4**]. Consider some examples of these fields.

EXAMPLE 7.1. The vector field $yx\frac{\partial}{\partial x}$ is complete, but its flow $(x,y) \to (e^{ty}x, y)$ is not algebraic, i.e. elements of this flow are only holomorphic automorphisms. Of course, the canonical forms of complete algebraic vector fields found by Brunella are more complicated and here is one of them: $ax\frac{\partial}{\partial x} + p(x^n y^m)[ny\frac{\partial}{\partial y} - mx\frac{\partial}{\partial x}]$ where $a \in \mathbb{C}$, $n, m \geq 1$ are coprime and $p(z) \in \mathbb{C}[z]$.

DEFINITION 7.2. Denote by $\mathrm{AAut}(X)$ the subgroup of the group $\mathrm{Aut}_{\mathrm{hol}}(X)$ of holomorphic automorphisms of X generated by the elements of the flows of complete algebraic vector fields on X. We call a normal affine surface V a generalized Gizatullin surface if $\mathrm{AAut}(V)$ admits an open orbit whose complement is at most finite.[6]

The technique of Brunella and some fundamental results of McQuillan were originally used in the proof of Theorem 7.4 below that describes generalized Gizatullin surfaces. It was later extracted in a much easier way from the next very general result of Guillot and Rebelo on semi-complete vector fields [**14**].

[6]It is worth mentioning that in all Kovalenko's examples [**19**] the group $\mathrm{AAut}(V)$ acts transitively on V while $\mathrm{Aut}(V)$ does not.

THEOREM 7.3. *Let V be a normal affine algebraic surface that admits a nonzero complete algebraic vector field. Then either:*

(1) all complete algebraic fields share the same rational first integral (i.e. there is a rational map $f : V \dashrightarrow B$ such that all complete algebraic vector fields on V are tangent to the fibers of f), or

(2) V is a rational surface with an open orbit of $\mathrm{AAut}(V)$ and, furthermore, for every complete algebraic vector field ν on V there is a regular function $f : V \to \mathbb{C}$ (depending on ν) with general fibers isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$ such that the flow of ν sends fibers of f to fibers of f.

As we have mentioned, Theorem 7.3 is an essential ingredient in the proof of the following main result of [**18**].

THEOREM 7.4. *A normal complex affine algebraic surface V is generalized Gizatullin if and only if it admits an SNC-completion $\bar{V}$ for which the boundary $\bar{V} \setminus V$ is connected, consists of rational curves, and has a dual graph that belongs to one of the following types*

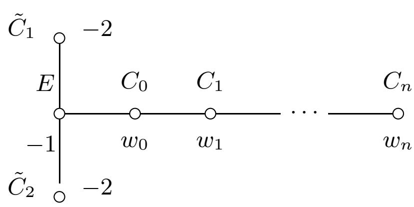
(1) *a standard zigzag or a linear chain of three 0-vertices (i.e. Gizatullin surfaces and $\mathbb{C} \times \mathbb{C}^*$),*

(2) *a circular graph with the following possibilities for weights*

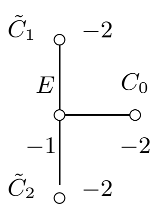
(2a) $((0, 0, w_1, \dots, w_n))$ *where* $n \geq 0$ *and* $w_i \leq -2$,

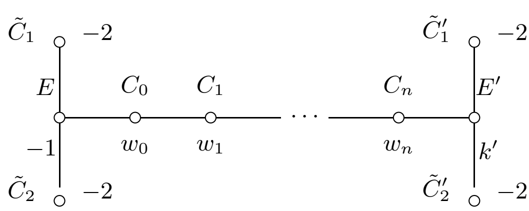
(2b) $((0, 0, w))$ *with* $-1 \leq w \leq 0$ *or* $((0, 0, 0, w))$ *with* $w \leq 0$,

(2c) $((0, 0, -1, -1))$;

(3)


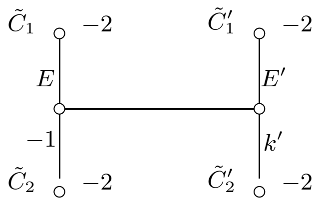
where $n \geq 0$, $w_0 \geq 0$ *and* $w_i \leq -2$ *for* $i \geq 1$,

(4)


(5)


where $n \geq 0$, $w_0 \geq 0$ *and* $w_i \leq -2$ *for* $i \geq 1$; *moreover* $k' \leq -1$ *if* $n = 0$ *while for* $n > 0$ *one has* $k' \leq -2$,

(6)



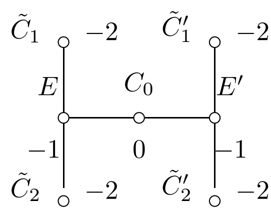
where $k' \geq -1$.

REMARK 7.5. With the exception of (2a), every graph in Theorem 7.4 can appear as the dual graph of an SNC-curve in a rational surface. In order to guarantee the same in (2a), one has to impose the following additional necessary and sufficient assumption: there is a linear chain $C_1 + \ldots + C_n$ of rational curves with weights $[[u_1, \ldots, u_n]]$ such that $u_i \geq w_i$, the curve $C_2 + \ldots + C_{n-1}$ is contractible and the weights of the proper transforms of C_1 and C_n become zero after this contraction.

Without going into a rather technical description of the proof of this result, let us consider some interesting examples.

EXAMPLE 7.6. (a) Consider hypersurfaces $\{xp(x) + yq(y) + xyz = 1\} \subset \mathbb{C}^3_{x,y,z}$, where the polynomials $1 - xp(x)$ and $1 - yq(y)$ have simple roots only (say, $x + y + xyz = 1$). None of them admits nontrivial algebraic $\mathbb{G}_a$ or $\mathbb{G}_m$-actions, but they are generalized Gizatullin surfaces and as in the case of the torus, for each such a surface V the dual graph of an SNC-completion $\bar{V}$ can be chosen as a cycle.

(b) The following special case of (5)



yields a surface which is a locally trivial twisted $\mathbb{C}^*$-fibration over $\mathbb{C}^*$ (i.e. it is nothing but a complexification of the Klein bottle).

We conclude the paper with a theorem describing some singular generalized Gizatullin surfaces, which include a unique non-toric surface of this type.

THEOREM 7.7. ([**18**]) *Let V_0 be a normal generalized Gizatullin surface such that for a complete algebraic vector field ν_0 on V_0 there is a surjective rational first integral $f_0 : V_0 \dashrightarrow B$ into a complete curve B. Then*

(1) *either V_0 is toric (and, in particular, quasi-homogeneous), or V_0 is isomorphic to the hypersurface $y(x^2 + y^2) + z^2 = 0$ and, in particular, it has $(-D_4)$-singularity*[7]*;*

[7] A singularity of type $-D_{n+1}$ is locally isomorphic to the hypersurface $yx^2 + y^n + z^2 = 0$ in $\mathbb{C}^3_{x,y,z}$.

(2) *up to a constant nonzero factor* ν_0 *is semi-simple.*

References

[1] S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148–166. MR0379502

[2] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004. MR2030225

[3] J. Blanc and A. Dubouloz, *Affine surfaces with a huge group of automorphisms*, Int. Math. Res. Not. IMRN **2** (2015), 422–459, DOI 10.1093/imrn/rnt202. MR3340326

[4] M. Brunella, *Complete polynomial vector fields on the complex plane*, Topology **43** (2004), no. 2, 433–445, DOI 10.1016/S0040-9383(03)00050-8. MR2052971

[5] V. I. Danilov, M. H. Gizatullin, *Automorphisms of affine surfaces*. I. Math. USSR Izv. 9 (1975), 493–534; II. *ibid.* **11** (1977), 51–98.

[6] H. Flenner, S. Kaliman, and M. Zaidenberg, *Completions of* $\mathbb{C}^*$*-surfaces*, Affine algebraic geometry, Osaka Univ. Press, Osaka, 2007, pp. 149–201. MR2327238

[7] H. Flenner, S. Kaliman, and M. Zaidenberg, *Uniqueness of* $\mathbb{C}^*$*- and* $\mathbb{C}_+$*-actions on Gizatullin surfaces*, Transform. Groups **13** (2008), no. 2, 305–354, DOI 10.1007/s00031-008-9014-0. MR2426134

[8] H. Flenner, S. Kaliman, and M. Zaidenberg, *Smooth affine surfaces with non-unique* $\mathbb{C}^*$*-actions*, J. Algebraic Geom. **20** (2011), no. 2, 329–398, DOI 10.1090/S1056-3911-2010-00533-4. MR2762994

[9] H. Flenner and M. Zaidenberg, *On the uniqueness of* $\mathbb{C}^*$*-actions on affine surfaces*, Affine algebraic geometry, Contemp. Math., vol. 369, Amer. Math. Soc., Providence, RI, 2005, pp. 97–111, DOI 10.1090/conm/369/06807. MR2126657

[10] F. Forstneric, J. Globevnik, and J.-P. Rosay, *Nonstraightenable complex lines in* $\mathbf{C}^2$, Ark. Mat. **34** (1996), no. 1, 97–101, DOI 10.1007/BF02559509. MR1396625

[11] G. Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopaedia of Mathematical Sciences, vol. 136, Springer-Verlag, Berlin, 2006. Invariant Theory and Algebraic Transformation Groups, VII. MR2259515

[12] M. H. Gizatullin, I. *Affine surfaces that are quasihomogeneous with respect to an algebraic group*, Math. USSR Izv. 5 (1971), 754–769; II. *Quasihomogeneous affine surfaces*, *ibid.* 1057–1081.

[13] M. H. Gizatullin, *Quasihomogeneous affine surfaces* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1047–1071. MR0286791

[14] A. Guillot and J. Rebelo, *Semicomplete meromorphic vector fields on complex surfaces*, J. Reine Angew. Math. **667** (2012), 27–65. MR2929671

[15] R. V. Gurjar, K. Masuda, M. Miyanishi, and P. Russell, *Affine lines on affine surfaces and the Makar-Limanov invariant*, Canad. J. Math. **60** (2008), no. 1, 109–139, DOI 10.4153/CJM-2008-005-8. MR2381169

[16] A. Gutwirth, *An inequality for certain pencils of plane curves*, Proc. Amer. Math. Soc. **12** (1961), 631–638, DOI 10.2307/2034258. MR0126759

[17] A. Gutwirth, *The action of an algebraic torus on the affine plane*, Trans. Amer. Math. Soc. **105** (1962), 407–414, DOI 10.2307/1993728. MR0141664

[18] S. Kaliman, F. Kutzschebauch, M. Leuenberger, *Complete algebraic vector fields on affine surfaces*, preprint, 42 p., arXiv:1411.5484.

[19] S. Kovalenko, *Transitivity of automorphism groups of Gizatullin surfaces*, Int. Math. Res. Not. IMRN **21** (2015), 11433–11484, DOI 10.1093/imrn/rnv025. MR3456050

[20] S. Kovalenko, A. Perepechko, and M. Zaidenberg, *On automorphism groups of affine surfaces*, Algebraic varieties and automorphism groups, Adv. Stud. Pure Math., vol. 75, Math. Soc. Japan, Tokyo, 2017, pp. 207–286. MR3793368

[21] R. Rentschler, *Opérations du groupe additif sur le plan affine* (French), C. R. Acad. Sci. Paris Sér. A-B **267** (1968), A384–A387. MR0232770

[22] J.-P. Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell. MR607504

[23] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28, DOI 10.1215/kjm/1250523277. MR0337963

[24] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace* $\mathbf{C}^2$ (French), J. Math. Soc. Japan **26** (1974), 241–257, DOI 10.2969/jmsj/02620241. MR0338423

[25] M. Suzuki, *Sur les opérations holomorphes du groupe additif complexe sur l'espace de deux variables complexes* (French), Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 517–546. MR0590938

[26] D. Wright, *Abelian subgroups of* $\mathrm{Aut}_k(k[X, Y])$ *and applications to actions on the affine plane*, Illinois J. Math. **23** (1979), no. 4, 579–634. MR540400

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