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Pluriharmonics in general potential theories

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Abstract. The general purpose of this paper is to investigate the notion of "pluriharmonics" for the general potential theory associated to a convex cone subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$. For such F there exists a maximal linear subspace $E \subset F$, called the *edge*, and F decomposes as $F = E \oplus F_0$. The *pluriharmonics* or *edge functions* are u's with $D^2u \in E$. Many subequations F have the same edge E, but there is a unique smallest such subequation. These are the focus of this investigation. Structural results are given. Many examples are described, and a classification of highly symmetric cases is given. Finally, the relevance of edge functions to the solutions of the Dirichlet problem is established.

Table of Contents

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1. Introduction.

This paper is concerned with the edge E of a convex cone subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$, obtained from the decomposition

$$F \ = \ E \oplus F_0 \tag{1.1}$$

into a vector subspace $E \subset \mathrm{Sym}^2(\mathbb{R}^n)$ and a cone $F_0 \subset E^\perp$, called the *reduced constraint set*, which contains no lines. (See §2 for definition of *subequation*.) The interest in the edge E is that it gives us a notion of *pluriharmonics*, or *edge functions*, for the potential theory associated to the subequation F. These edge functions u are defined by $D^2u \in E$.

The edge E of a subequation is, in a sense, a crude invariant, since many subequations F have the same edge E. However, there is a canonical choice for F completely determined by E, namely $E + \mathcal{P}$, where $\mathcal{P} \equiv \{A : A \geq 0\}$. This is a subequation with edge E, and it must be contained in all other subequations with edge E (since by definition we always have $\mathcal{P} \subset F$). A large part of this paper is devoted to studying and classifying these subequations $E+\mathcal{P}$ for which the edge E is a determinative invariant. It is for these **minimal subequations** with edge E (Def. 5.2) that the pluriharmonics or edge functions play the most important role.

Let's look at some examples. The simplest is just $\mathcal{P}$ itself. Here the edge $E = \{0\}$ and the $\mathcal{P}$-subharmonics are the convex functions. The edge functions are those u with $D^2u \equiv 0$, that is, the affine functions.

The other orthogonally invariant edge is $E = \{A : \mathrm{tr}\, A = 0\}$, and we have $\mathcal{P} + E = \{A : \mathrm{tr}\, A \geq 0\} \equiv \Delta$. Here the subsolutions are the classical subharmonics, and the edge functions are just the harmonics.

The examples become more interesting if we look at $\mathrm{U}(n)$-invariant edges in $\mathbb{C}^n = (\mathbb{R}^{2n}, J)$. The reduction into irreducibles is

$$\mathrm{Sym}^2(\mathbb{R}^{2n}) \ = \ (\mathbb{R}\cdot \mathrm{Id}) \oplus \mathrm{Herm}_0^{\mathrm{sym}} \oplus \mathrm{Herm}^{\mathrm{skew}},$$

$\mathrm{Herm}_0^{\mathrm{sym}} = \{A : AJ = JA \text{ and } \mathrm{tr}\, A = 0\}$ and $\mathrm{Herm}^{\mathrm{skew}} = \{A : AJ = -JA\}$. Here there are two new edges.

The first is where we set $E = \mathrm{Herm}^{\mathrm{skew}}$. This gives the complex Monge-Ampère subequation:

$$\mathcal{P}_{\mathbb{C}} \ = \ E + \mathcal{P} \ = \ \{A : A_{\mathbb{C}} = A - JAJ \geq 0\}.$$

The subsolutions are the plurisubharmonics, and the edge functions (or pluriharmonics) are the classical pluriharmonic functions in complex analysis.

The other is where $E = \mathrm{Herm}_0^{\mathrm{sym}}$. This subequation is rather new.

$$\mathcal{P}(\mathrm{Lag}) \ = \ E + \mathcal{P} \ = \ \left\{A : \mathrm{tr}\left(A\big|_W\right) \geq 0 \ \text{ for all Lagrangian planes } W\right\}.$$

The subsolutions are the *Lagrangian plurisubharmonics* which were studied in [**10**]. In this case the edge functions are certain quadratic functions.

If one now looks for $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$-invariant edges, there are many interesting examples. In fact, in Chapter 7 a wide variety of subequations are given. The reader might enjoy this section.

One interesting result concerning these minimal subequations is that they can be used to characterize the dual subharmonics (negatives of the superharmonics). (In this paper, degree-2 means degree ≤ 2.)

THEOREM 6.4. *Let $\mathcal{P}^+ = E + \mathcal{P}$ be a minimal subequation. The following conditions on an upper semi-continuous function u are equivalent.*

(1) u is dually $\mathcal{P}^+$-subharmonic.

(2) u is "sub" the edge functions.

(3) u is locally "sub" the edge functions.

(4) u is locally "sub" the degree-2 polynomial edge functions.

Here the notion of subharmonic is taken in the viscosity sense (see below). We note that u is "sub" a function v if for each compact set $\overline{\Omega}$, one has $u \leq v$ on $\partial\Omega \Rightarrow u \leq v$ on $\overline{\Omega}$.

For example, when $F = \mathcal{P}$ (the first example above), this theorem says that the $\widetilde{\mathcal{P}}$-subharmonic functions (the negatives of supersolutions for the real Monge-Ampère equation) are characterized by being "sub" the affine functions.

When $F = \mathcal{P}_{\mathbb{C}}$, the $\widetilde{\mathcal{P}}_{\mathbb{C}}$-subharmonic functions (the negatives of supersolutions for the complex Monge-Ampère equation) are characterized by being "sub" the standard pluriharmonics in complex analysis. In fact the degree-2 polynomial pluriharmonics will do.

When $F = \mathcal{P}(\mathrm{Lag})$, the $\widetilde{\mathcal{P}}(\mathrm{Lag})$-subharmonic functions (the negatives of supersolutions for the Lagrangian Monge-Ampère equation [**10**]) are characterized by being "sub" an explicit family of quadratic polynomials.

Chapter 2 of this paper lays down the fundamental notions of the edge E, the span S and the reduced constraint set F_0, which appears in (1.1) above. By definition S is the linear span of F_0, but it could also be defined as the orthogonal complement of E. This chapter then looks at geometrically defined equations, which give many interesting examples.

Chapter 3 introduces a further refinement of the structure of convex cone subequations. These equations are partitioned into two classes. The first consists of subequations which are *extremely degenerate* (see Definition 3.1). A basic example is the Laplacian on $\mathbb{R}^k$, considered as a subequation on $\mathbb{R}^n$ where $n > k$. The second class consists of those subequations which are *dimensionally complete* (see Definition 3.1*). This means that all the variables in $\mathbb{R}^n$ are required to define the subequation. These complementary concepts are determined entirely by properties of the edge, or equivalently the span, of F. For example,

$$F \text{ is complete} \quad \iff \quad E \cap \mathcal{P} = \{0\} \quad \iff \quad S \cap (\mathrm{Int}\,\mathcal{P}) \neq \emptyset.$$

There are several other equivalent criteria; see Propositions 3.5 and 3.10. A pair of orthogonal subspaces E and S of $\mathrm{Sym}^2(\mathbb{R}^n)$, which satisfy these criteria for E and S, is called an **edge-span pair**.

In Chapter 4 the structure of the subequation F is further illuminated by proving that there exists a unique subspace $W \subset \mathbb{R}^n$, called the **support of** F, with the property that

$$F \;=\; \mathrm{Sym}^2(W)^{\perp} \oplus F_1 \qquad \text{where } \; F_1 \subset \mathrm{Sym}^2(W) \tag{1.2}$$

is a complete convex cone subequation in W. This F_1 is called the **the supporting subequation of** F.

Combining (1.1) and (1.2) give the decomposition

$$F \;=\; \mathrm{Sym}^2(W)^{\perp} \oplus E_1 \oplus F_0 \tag{1.3}$$

where the edge of F is $E = \mathrm{Sym}^2(W)^\perp \oplus E_1$ and E_1 is the edge of the supporting subequation F_1.

In Chapter 5 our minimal subequations are defined and discussed. We start with any basic edge $E \subset \mathrm{Sym}^2(\mathbb{R}^n)$, i.e, one which satisfies $E \cap \mathcal{P} = \{0\}$. Then (Lemma 5.1) the sum

$$\mathcal{P}^+ \equiv E + \mathcal{P} \quad \text{is a subequation, and it has edge } E.$$

(By the edge criteria $\mathcal{P}^+$ is then complete.) Such subequations are called **minimal** and are the main focus of this paper. These subequations have many special properties. Theorem 5.4 mentions eight of them, while Theorem 5.5 claims that any one of these eight properties implies the subequation is minimal.

In Chapter 7 many examples of minimal subequations are given.

In Chapter 8 we classify all the minimal subequations which are invariant under the compact group G where $G = \mathrm{O}_n, \mathrm{U}_n, \mathrm{Sp}_n \cdot \mathrm{Sp}_1, \mathrm{Sp}_n$ and $\mathrm{Sp}_n \cdot \mathrm{S}^1$ (all acting on their fundamental representation spaces).

We note that the general definition of F-plurisubharmonics is based on viscosity theory [**3**], [**2**]. The reader is referred to [**5**], [**7**] or [**8**] for definitions and properties.

In Chapter 6 the generalized pluriharmonics (or edge functions) are introduced from a viscosity point of view, and basic properties are discussed.

One might speculate, in light of Theorem 6.4 above, that for minimal subequations the Dirichlet Problem can be solved by replacing the standard Perron family with the subfamily consisting only of $\mathcal{P}^+$-pluriharmonics. For the two extreme subequations – the convexity and the Laplacian subequations – this is in fact true. In §8 we prove something close for all minimal subequations.

THEOREM 9.3. *Let $\mathcal{P}^+$ be a minimal subequation, and $\Omega \subset \mathbb{R}^n$ a domain with smooth strictly convex boundary. Then the standard solution to the Dirichlet problem for any $\varphi \in C(\partial\Omega)$ is the upper envelope of functions in the Perron subfamily of functions which can be written locally as the maximum of a finite number of pluriharmonics.*

2. Preliminaries.

In this section we review the basic properties of convex cone subequations and define many of the associated objects (cf. [**6**]).

We start with a closed convex cone $\mathcal{P}^+$ in $\mathrm{Sym}^2(\mathbb{R}^n)$, which we always assume is a non-empty proper subset. Using the natural inner product $\langle A, B\rangle = \mathrm{tr}(AB)$ we have

$$(\textbf{Polar Cone}) \qquad \mathcal{P}_+ \equiv \{A : \langle A, B\rangle \geq 0 \ \forall\, B \in \mathcal{P}^+\}. \tag{2.1}$$

If, in addition, $\mathcal{P}^+$ satisfies the following positivity condition, then $\mathcal{P}^+$ is referred to as a subequation. (We will frequently evoke the bipolar theorem, which says that the polar of the polar is the original convex cone.)

Definition 2.1. $\mathcal{P}^+$ is a **subequation** if satisfies the **positivity condition**

$$\mathcal{P}^+ + \mathcal{P} = \mathcal{P}^+, \qquad \text{i.e. } \mathcal{P} \subset \mathcal{P}^+ \tag{P}$$

or equivalently if $\mathcal{P}_+ \subset \mathcal{P}$. The equivalence follows since $\mathcal{P}$ is self-polar.

Subequations have the important **topological property**

$$\mathcal{P}^+ = \overline{\mathrm{Int}\,\mathcal{P}^+} \tag{T}$$

since $\mathcal{P}^+ + \epsilon I \subset \mathcal{P}^+ + \mathrm{Int}\,\mathcal{P} \subset \mathrm{Int}\,\mathcal{P}^+$ for all $\epsilon > 0$.

The following is an important class of examples.

Example 2.2. (The Geometric Case). Given a closed subset $\mathbb{G} \subset G(p, \mathbb{R}^n)$ of the Grassmannian of unoriented p-planes in $\mathbb{R}^n$, let

$$\mathcal{P}(\mathbb{G}) \ \equiv \ \{A : \langle A, P_W \rangle = \operatorname{tr}(A\big|_W) \geq 0 \ \forall W \in \mathbb{G}\}$$

where $W \in \mathbb{G}$ is identified with P_W, orthogonal projection onto W. Each such $\mathcal{P}^+ = \mathcal{P}(\mathbb{G})$ is a convex cone subequation with polar $\mathcal{P}_+ =$ Convex Cone Hull $(\mathbb{G})$ $\equiv CCH(\mathbb{G})$.

In addition to the polar cone $\mathcal{P}_+$ we associate two vector spaces E and S with $\mathcal{P}^+$ which form an orthogonal decomposition $\mathrm{Sym}^2(\mathbb{R}^n) = E \oplus S$.

The Edge and the Span

$$\textbf{(The Edge } E\textbf{)} \qquad E \ \equiv \ \mathcal{P}^+ \cap (-\mathcal{P}^+) \ = \ \{A : A + \mathcal{P}^+ = \mathcal{P}^+\} \tag{2.2}$$

$$\textbf{(The Dual Span } S\textbf{)} \qquad S \ \equiv \ \operatorname{span} \mathcal{P}_+ \tag{2.3}$$

Note that the edge E is the unique maximal vector space contained in $\mathcal{P}^+$. To verify the equality in (2.2), use the fact that $A + \mathcal{P}^+ = \mathcal{P}^+ \iff -A + \mathcal{P}^+ = \mathcal{P}^+$, along with the fact that $\mathcal{P}^+$ is a convex cone with vertex 0. Note that S is by definition a vector space, whereas E is obviously closed under addition and scalar multiplication.

Lemma 2.3. *The Edge E of a subequation enjoys the properties:*

(2.4) (Orthogonality) E and S are orthogonal compliments in $\mathrm{Sym}^2(\mathbb{R}^n)$.

(2.5) $E \cap (\operatorname{Int} \mathcal{P}^+) \ = \ \emptyset$. In particular, $E \cap (\operatorname{Int} \mathcal{P}) = \emptyset$.

Proof of (2.4). It is easy to see that $E \perp S$. Then since $\mathcal{P}_+ \subset S \Rightarrow S^\perp \subset \mathcal{P}^+$, and since $S^\perp$ is a vector subspace, this implies that $S^\perp \subset E$. Therefore, $\mathrm{Sym}^2(\mathbb{R}^n) = S^\perp + S \subset E + S$ thereby proving that $E + S = \mathrm{Sym}^2(\mathbb{R}^n)$ is an orthogonal decomposition. ■

Proof of (2.5). If this does not hold, we can pick $A \in E \cap \operatorname{Int} \mathcal{P}^+$. Given $B \in \mathrm{Sym}^2(\mathbb{R}^n)$, we have $A + \epsilon B \in \mathcal{P}^+$ if $\epsilon > 0$ is sufficiently small. Therefore, $B = -\frac{1}{\epsilon} A + (\frac{1}{\epsilon} A + B) \in E + \mathcal{P}^+ = \mathcal{P}^+$. This contradicts the assumption that $\mathcal{P}^+$ is a proper subset of $\mathrm{Sym}^2(\mathbb{R}^n)$. ■

$$\text{Let} \ \ \pi : \mathrm{Sym}^2(\mathbb{R}^n) \ \longrightarrow \ S \ \ \text{denote orthogonal projection.} \tag{2.6}$$

As a constraint on the second derivative, the important part of $\mathcal{P}^+$ is

$$\textbf{(The Reduced Constraint Set)} \qquad \mathcal{P}_0^+ \ \equiv \ \pi(\mathcal{P}^+). \tag{2.7}$$

$$\mathcal{P}^+ \ = \ E \oplus \mathcal{P}_0^+, \qquad \text{i.e.,} \qquad A \in \mathcal{P}^+ \quad \iff \quad \pi(A) \in \mathcal{P}_0^+. \tag{2.8a}$$

$$\operatorname{Int} \mathcal{P}^+ \ = \ E \oplus \operatorname{Int} \mathcal{P}_0^+, \quad \text{i.e.,} \quad A \in \operatorname{Int} \mathcal{P}^+ \quad \iff \quad \pi(A) \in \operatorname{Int} \mathcal{P}_0^+. \tag{2.8b}$$

Since $\pi(A)$ captures the important part of $A \equiv D^2 u$, $\pi(D^2 u)$ is called **the reduced hessian of u for $\mathcal{P}^+$.**

Note that

$$\mathcal{P}_0^+ \text{ is the polar of } \mathcal{P}_+ \text{ in its span } S. \tag{2.9}$$

The closed convex cone $\mathcal{P}_0^+ \subset S$ is not a subequation unless $S \equiv \mathrm{Sym}^2(\mathbb{R}^n)$, i.e., $E = \{0\}$, in which case $\mathcal{P}_0^+ = \mathcal{P}^+$.

We say that the subequation $\mathcal{P}^+$ is **self polar** if

$$\mathcal{P}_0^+ \ = \ \mathcal{P}_+ \tag{2.10}$$

3. Extremely Degenerate Versus Complete

Our subequations divide into two kinds. The first is that of extreme degeneracy. These subequations on n-dimensional euclidean space $\mathbb{R}^n$ are better understood as subequations on a lower dimensional subspace (see Prop. 3.4 below and the Support Theorem 4.3).

Definition 3.1*. A convex cone subequation $\mathcal{P}^+$ is said to be **extremely degenerate** if there exists a proper subspace $W \subset \mathbb{R}^n$ such that the following equivalent conditions are satisfied:

(1*) The reduced constraint set $\mathcal{P}_0^+ \subset \mathrm{Sym}^2(W)$,

(2*) The polar $\mathcal{P}_+ \subset \mathrm{Sym}^2(W)$, or equivalently the dual span $S \subset \mathrm{Sym}^2(W)$.

Proof that (1*) $\iff$ (2*). Note that: $\mathcal{P}_+ \subset \mathrm{Sym}^2(W) \iff S \equiv \mathrm{span}\, \mathcal{P}_+ \subset \mathrm{Sym}^2(W) \iff \mathcal{P}_0^+ \subset \mathrm{Sym}^2(W)$ by (2.10). ∎

Remark. In [**6**] we said for (2*) that "$\mathcal{P}_+$ only involves the variables in W", and for (1*) that "$\mathcal{P}^+$ can be defined using the variables in W".

The remaining subequations are defined by taking the negations of (1*) and (2*).

Definition 3.1. A subequation which is not extremely degenerate will be called **dimensionally complete**, or just **complete**. In other words

(1) $\mathcal{P}_0^+ \not\subset \mathrm{Sym}^2(W)$ for any proper subspace $W \subset \mathbb{R}^n$, or

(2) $\mathcal{P}_+ \not\subset \mathrm{Sym}^2(W)$ for any proper subspace $W \subset \mathbb{R}^n$.

See [**6**] for many interesting results for complete convex cone subequations. The purpose of this paper is to investigate a special class of such subequations described in Section 5.

Extreme Degeneracy

Let $A\big|_W$ denote the restriction of A to the subspace $W \subset \mathbb{R}^n$ as a quadratic form. In terms of the 2×2-blocking of $\mathrm{Sym}^2(\mathbb{R}^n)$ induced by $\mathbb{R}^n = W \oplus W^\perp$, $A\big|_W$ is the $(1,1)$-component of A.

The subequation $\mathcal{P}^+$ can be restricted to a subequation $\mathcal{P}_W^+$ on W by defining

$$\mathcal{P}_W^+ \;\equiv\; \left\{ A\big|_W : A \in \mathcal{P}^+ \right\}. \tag{3.1}$$

Note that $\mathcal{P}_W^+ \subset \mathrm{Sym}^2(W)$ satisfies positivity, since if $P \in \mathrm{Sym}^2(W)$, with $P \geq 0$, then $A\big|_W + P = (A+Q)\big|_W$, where $Q \in \mathrm{Sym}^2(\mathbb{R}^n)$ restricts to P on W and has all other components 0, and therefore $A + Q \in \mathcal{P}^+$. Thus $\mathcal{P}_W^+$ is a subequation on W.

We have proved the following. Let $\mathrm{Sym}^2(W)^\perp$ denote the orthogonal complement of $\mathrm{Sym}^2(W)$ in $\mathrm{Sym}^2(\mathbb{R}^n)$.

Proposition 3.2. *If $\mathcal{P}^+$ is extremely degenerate, i.e., if $\mathcal{P}_W^+$ and W satisfy the equivalent conditions (1*) and (2*) above, then*

$$\mathcal{P}^+ \;=\; \mathcal{P}_W^+ \oplus \mathrm{Sym}^2(W)^\perp. \tag{3.2}$$

Moreover, $\mathcal{P}^+$ and $\mathcal{P}_W^+$ have the same reduced constraint set $\mathcal{P}_0^+$ since $\mathrm{Sym}^2(W)^\perp \subset E$, i.e., $S \subset \mathrm{Sym}^2(W)$.

Definition 3.3. If (3.2) is satisfied, we say that $\mathcal{P}^+$ **reduces to** $\mathcal{P}_W^+$, and that $\mathcal{P}^+$ **is the trivial extension of $\mathcal{P}_W^+$ from W to $\mathbb{R}^n$.**

Proposition 3.4. [**5**, Thm. A.4]**.** *Suppose that $\mathcal{P}^+$ reduces to a subequation $\mathcal{P}_W^+$ on W. If $z = (x, y) \in W \oplus W^\perp = \mathbb{R}^n$ denotes coordinates, then $u(x, y)$ is*

$\mathcal{P}^+$-subharmonic if and only if for each y, $u(x,y)$ is $\mathcal{P}^+_W$-subharmonic in x, but otherwise u is just upper semi-continuous in (x,y), i.e., there is no constraint on u with respect to the y-variable.

The Edge Criteria

Extreme degeneracy and completeness can be described in a very simple way in terms of edges.

Proposition 3.5. (The Edge Criteria). *The following conditions on a convex cone subequation $\mathcal{P}^+$ are equivalent.*

(1)≅(2) $\mathcal{P}^+$ is complete.

(3) $E \cap \mathcal{P} = \{0\}$.

(4a) $P_e \notin E$ for all $|e| = 1$.

(4b) $-P_e \notin \mathcal{P}^+$ for all $|e| = 1$.

Stated as the edge criteria for extreme degeneracy, we have that the following are equivalent.

(1)≅(2*) $\mathcal{P}^+$ is extremely degenerate.*

(3) $E \cap \mathcal{P} \neq \{0\}$.*

*(4*a) $P_e \in E$ for some $|e| = 1$.*

*(4*b) $-P_e \in \mathcal{P}^+$ for some $|e| = 1$.*

Proof. We will prove the extreme degeneracy version. First we note that (4*a) and (4*b) are equivalent. One key to the proof is the following Lemma taken from [**6**].

Lemma 3.6. *Suppose that W is a hyperplane in $\mathbb{R}^n$ with unit normal e. Then*

$$P_e \in E \qquad \Longleftrightarrow \qquad \mathcal{P}_+ \subset \mathrm{Sym}^2(W).$$

Proof.

Corollary 3.7. *We have that (2*) $\Longleftrightarrow$ (4*a).*

Proof. The only thing to note is that if condition (2*), that $\mathcal{P}_+ \subset \mathrm{Sym}^2(W)$ for some proper subspace $W \subset \mathbb{R}^n$, holds, then $\mathcal{P}_+ \subset \mathrm{Sym}^2(W')$ for any hyperplane $W' \supset W$. ∎

Since (4*a) implies (3*) is trivial, the only thing left to prove is that (3*) implies (4*a).

Lemma 3.8. *Suppose $P \geq 0$ has null space $N \subset \mathbb{R}^n$. Then*

$$P \in E \qquad \Rightarrow \qquad \mathrm{Sym}^2(N^\perp) \ \subset \ E \tag{3.3}$$

Proof. The proof is modeled on the proof of (2.5). It suffices to show that $\mathrm{Sym}^2(N^\perp) \subset \mathcal{P}^+$. Given $A \in \mathrm{Sym}^2(N^\perp)$, we write $A = -tP + (A + tP)$ and note that since $P \in E$, we have $-tP \in E$ for all $t \geq 0$. Now $A + tP \in \mathcal{P}$ if $t >> 0$ is sufficiently large since $P\big|_{N^\perp}$ is positive definite. This proves that $A \in E + \mathcal{P} \subset \mathcal{P}^+$. ∎

Corollary 3.9. *We have that (3*) ⇒ (4*a).*

Proof. If (3*) holds, choose $P \in E \cap \mathcal{P}$ with $P \neq 0$. Since $P \neq 0$, the subspace $N^\perp \neq \{0\}$. Pick $e \in N^\perp$ with $|e| = 1$. Then $P_e \in E$. ∎

The Span Criteria

The span criteria for completeness also provides an easy check in examples for completeness, and has important consequences for the subequation.

Proposition 3.10. *The following conditions on a convex cone subequation $\mathcal{P}^+$ are equivalent.*

(1)≅(2) $\mathcal{P}^+$ *is complete.*

(5) $S \cap (\operatorname{Int} \mathcal{P}) \neq \emptyset$.

(6) $\operatorname{Int}_S \mathcal{P}_+ \subset \operatorname{Int} \mathcal{P}$.

Proof. First we show that for a pair of vector spaces E and S which are orthogonal complements in $\operatorname{Sym}^2(\mathbb{R}^n)$, (3) and (5) are equivalent.

Lemma 3.11. *Suppose subspaces E and S of* $\operatorname{Sym}^2(\mathbb{R}^n)$ *are orthogonal complements. Then*

E satisfies the Edge Criteria (3) $\iff$ *S satisfies the Span Criteria (5).*

Proof. (3) $\Rightarrow$ (5). If (5) is false, i.e., $S \cap (\operatorname{Int} \mathcal{P}) = \emptyset$, then by the Hahn-Banach Theorem there exists an open half-space U with $S \subset \partial U$ and $\operatorname{Int} \mathcal{P} \subset U$. Let $N \in U$ denote the unit normal to the hyperplane ∂U. Then $S \subset \partial U \;\Rightarrow\; N \in E = S^\perp$, while $\operatorname{Int} \mathcal{P} \subset U \;\Rightarrow\; \langle N, P\rangle > 0\ \forall\, P > 0$, which implies $N \in \mathcal{P}$. Thus we have $N \in E \cap \mathcal{P}$, but $N \neq 0$ so that (3) is false.

(5) $\Rightarrow$ (3). By (5) we can pick $P \in S \cap (\operatorname{Int} \mathcal{P})$. If $A \in E \cap \mathcal{P}$, then $\langle A, P\rangle = 0$ since $A \in E$ and $P \in S$. However, since $A \geq 0$ and $P > 0$, this implies $A = 0$. ■

Proof that (5) ⇒ (6). By (5) we can choose $P \in S \cap (\operatorname{Int} \mathcal{P})$. Given $A \in \operatorname{Int}_S \mathcal{P}_+$, for $\epsilon > 0$ sufficiently small we have $A - \epsilon P \in \mathcal{P}_+$. Thus for all non-zero $Q \in \mathcal{P} \subset \mathcal{P}^+$ we have $0 \leq \langle A - \epsilon I, Q\rangle = \langle A, Q\rangle - \epsilon \langle P, Q\rangle$. Since $P > 0$, one has $\langle P, Q\rangle > 0$, which proves that $\langle A, Q\rangle > 0$ for all non-zero $Q \geq 0$. Thus $A > 0$, which proves (6). ■

Proof that (6) ⇒ (5). Now $\mathcal{P}_+$ is a closed convex cone in S. Hence $\operatorname{Int}_S \mathcal{P}_+ \neq \emptyset$ is equivalent to S equaling the span of $\mathcal{P}_+$, which it does by the definition of S. Now pick $P \in \operatorname{Int}_S \mathcal{P}_+$. Then $P \in S$ and by (6) we have $P > 0$, which proves (5).■

This completes the proof of Proposition 3.9. ■

The edge and span criteria (3) and (5) for completeness motivates the following definition, which will be used in the next section.

Definition 3.12.

(a) A subspace $E \subset \operatorname{Sym}^2(\mathbb{R}^n)$ is called a **basic edge subspace** if

(3) $E \cap \mathcal{P} \;=\; \{0\}$.

(b) A subspace $S \subset \operatorname{Sym}^2(\mathbb{R}^n)$ is called a **basic span subspace** if

(5) $S \cap (\operatorname{Int} \mathcal{P}) \;\neq\; \emptyset$.

(c) If in addition E and S are orthogonal complements, then E, S well be referred to as a **basic edge-span pair**.

4. The Supporting Subequation

For each subequation there is a smallest subspace W of $\mathbb{R}^n$ to which the subequation reduces.

Definition 4.1. (Support). Given a convex cone subequation $\mathcal{P}^+$ we define the **support of** $\mathcal{P}^+$ to be the subspace $W \subset \mathbb{R}^n$ which is the intersection of all subspaces $W' \subset \mathbb{R}^n$ which that

$$\mathcal{P}^+ \;=\; \mathcal{P}^+_{W'} \oplus \operatorname{Sym}^2(W')^\perp. \tag{4.1}$$

Lemma 4.2. *The orthogonal complement of the support W of $\mathcal{P}^+$ equals:*

$$V \;\equiv\; \operatorname{span}\{e \in \mathbb{R}^n : P_e \in E, |e| = 1\}. \tag{4.2}$$

Proof. Note that (4.1) holds $\iff$ $\mathrm{Sym}^2(W')^\perp \subset E$ $\iff$ $S \subset \mathrm{Sym}^2(W')$ $\iff$ $\mathcal{P}_+ \subset \mathrm{Sym}^2(W')$ $\iff$ $P_e \in E$ for all $e \perp W'$ with $|e| = 1$. ∎

The support illuminates the structure of the subequation.

THEOREM 4.3. (Structure Theorem). *Suppose $\mathcal{P}^+ \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a convex cone subequation with support $W \subset \mathbb{R}^n$. Then*

$$\mathcal{P}^+ \;=\; \mathcal{P}^+_W \oplus \mathrm{Sym}^2(W)^\perp \qquad \text{and} \tag{4.3}$$

$$\mathcal{P}^+_W \;\subset\; \mathrm{Sym}^2(W) \;\; \text{is a complete subequation.} \tag{4.4}$$

Proof. To be done later.

Definition 4.4. If W is the support of $\mathcal{P}^+$, the subequation $\mathcal{P}^+_W$ will be called the **supporting subequation of** $\mathcal{P}^+$, and its edge E_W will be called the **supporting edge of** $\mathcal{P}^+$

Note that the edge of $\mathcal{P}^+$,

$$E \;=\; E_W \oplus \mathrm{Sym}^2(W)^\perp, \tag{4.5}$$

is larger than its supporting edge E_W unless $\mathcal{P}^+$ is complete.

Note also that the original subequation $\mathcal{P}^+$ and the supporting subequation $\mathcal{P}^+_W$ have the same span S and the same reduced constraint set $\mathcal{P}^+_0$.

5. Minimal Subequations

These subequations are the focus of this paper. They are all constructed as follows, starting with a basic edge-span pair.

Lemma 5.1. *Suppose $E, S \subset \mathrm{Sym}^2(\mathbb{R}^n)$ are orthogonal complements with $E \cap \mathcal{P} = \{0\}$, or equivalently $S \cap (\mathrm{Int}\,\mathcal{P}) \neq \emptyset$. That is, E, S is a basic edge-span pair. Then*

$$\mathcal{P}^+ \;\equiv\; E + \mathcal{P} \;\; \textit{is a subequation, and it has edge } E \textit{ and span } S. \tag{5.1}$$

Moreover, if Q^+ is any subequation with edge E, then $\mathcal{P}^+ \subset Q^+$.

Proof. Obviously $\mathcal{P}^+$ satisfies positivity. It remains to show that $\mathcal{P}^+ \equiv E + \mathcal{P}$ is closed. Let $\pi : \mathrm{Sym}^2(\mathbb{R}^n) \to S$ denote orthogonal projection as in (2.5). Since $E + \mathcal{P} = E \oplus \pi(\mathcal{P})$,

$$\mathcal{P}^+ \text{ is closed if and only if } \pi(\mathcal{P}) \text{ is closed.} \tag{5.2}$$

Now we prove that:

$$\pi(\mathcal{P}) \text{ is closed.}$$

Let $K \equiv \mathcal{P} \cap \{\mathrm{tr} = 1\}$, a compact base for $\mathcal{P}$. The image $\pi(K)$ is a compact subset of S. The basic edge condition $E \cap \mathcal{P} = \{0\}$ is equivalent to $0 \notin \pi(K)$. This is enough to conclude that the cone on the compact convex set $\pi(K)$ is closed. Thus, $\mathcal{P}^+ \equiv E + \mathcal{P}$ is a subequation. ∎

To prove that $\mathcal{P}^+$ has edge E we must show that

$$\mathcal{P}^+ \cap (-\mathcal{P}^+) \;=\; E \qquad \text{or equivalently} \qquad \pi(\mathcal{P}) \cap (-\pi(\mathcal{P})) \;=\; \{0\}.$$

Suppose $A \in \pi(\mathcal{P}) \cap (-\pi(\mathcal{P}))$, i.e., $A = \pi(P_1) = -\pi(P_2)$ with $P_1, P_2 \in \mathcal{P}$. Then $\pi(P_1 + P_2) = 0$, i.e., $P_1 + P_2 \in E$. Since $E \cap \mathcal{P} = \{0\}$, $P_1 + P_2 = 0$. But this implies $P_1 = P_2 = 0$ and hence $A = 0$. Since $\mathcal{P}^+$ has edge E, it has span $S = E^\perp$. Finally, $\mathcal{P}^+ \subset Q^+$, since $E \subset Q^+$ and positivity for Q^+ implies $\mathcal{P}^+ \equiv E + \mathcal{P} \subset Q^+$. ∎

Definition 5.2. The subequation $\mathcal{P}^+ = E + \mathcal{P}$ constructed in Lemma 5.1 will be referred to as a **minimal subequation**, or the **minimal subequation with edge** E.

Corollary 5.3. *Suppose $\mathcal{P}^+$ is a minimal subequation with edge-span E, S. Then*

$$(a)\ \ E \cap \mathcal{P} = \{0\}, \qquad (b)\ \ S \cap (\operatorname{Int} \mathcal{P}) \neq \emptyset, \qquad (c)\ \ \mathcal{P}^+ \ \text{ is complete.}$$

Proof. By definition of minimal we have $\mathcal{P}^+ = E' + \mathcal{P}$ where E' satisfies (a). By Lemma 5.1 the edge E of $\mathcal{P}^+$ equals E'. Lemma 3.10 says that (a) and)b) are equivalent. Either the edge criteria (a) $\Rightarrow$ (c), or the span criteria (b) $\Rightarrow$ (c), completes the proof. ∎

There are many additional interesting properties of minimal subequations, besides the various completeness criteria in Section 3.

THEOREM 5.4. (Minimality Properties). *Suppose $\mathcal{P}^+ \equiv E + \mathcal{P}$ is the minimal subequation with edge E and span S. Then*
(1) $\mathcal{P}^+ = E + \mathcal{P}$, (1$a$) $\mathcal{P}_0^+ = \pi(\mathcal{P})$, (1$b$) $\mathcal{P}^+ = E \oplus \pi(\mathcal{P})$
(2) $\operatorname{Int} \mathcal{P}^+ = E + \operatorname{Int} \mathcal{P}$, (2$a$) $\operatorname{Int} \mathcal{P}_0^+ = \pi(\operatorname{Int} \mathcal{P})$, (2$b$) $\operatorname{Int} \mathcal{P}^+ = E \oplus \operatorname{Int} \pi(\mathcal{P})$
(3) $\mathcal{P}_+ = S \cap \mathcal{P}$, and (3*) $\operatorname{Int}_S \mathcal{P}_+ = S \cap (\operatorname{Int} \mathcal{P})$.

In fact, for complete subequations each of these eight properties characterizes minimality.

THEOREM 5.5. (Minimality Criteria). *Suppose $\mathcal{P}^+ \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a complete convex cone subequation, with edge E span S, reduced constraint set $\mathcal{P}_0$, and polar cone $\mathcal{P}_+$. Then $\mathcal{P}^+$ is the minimal subequation with edge E if and only if any one of the eight equivalent conditions in Theorem 5.4 hold.*

Proof of Theorem 5.4. Assertion (1) is by Definition 4.2. Next we show the following.

$$\text{(1), (1a) and (1b) are equivalent for any subequation } \mathcal{P}^+ \text{ with edge } E. \tag{5.3}$$

(1) $\Rightarrow$ (1a): By definition $\mathcal{P}_0^+ = \pi(\mathcal{P}^+)$. Since $\pi(E) = \{0\}$, (1) implies that $\pi(\mathcal{P}^+) = \pi(\mathcal{P})$.
(1a) $\Rightarrow$ (1b): This follows because $\mathcal{P}^+ = E \oplus \pi(\mathcal{P}^+)$.
(1b) $\Rightarrow$ (1): This is obvious.

Proof of (2). Obviously the open set $E + \operatorname{Int} \mathcal{P} \subset \operatorname{Int} \mathcal{P}^+$. If $A \in \operatorname{Int} \mathcal{P}^+$, then for small $\epsilon > 0$, $A - \epsilon I \in \operatorname{Int} \mathcal{P}^+ \subset \mathcal{P}^+$. Hence there exist $B_0 \in E$ and $P \geq 0$ such that $A - \epsilon I = B_0 + P$. Therefore, $A = B_0 + (P + \epsilon I) \in E + \operatorname{Int} \mathcal{P}$, proving that $\operatorname{Int} \mathcal{P}^+ = E + \operatorname{Int} \mathcal{P}$.

Just as in (5.3), we have

$$\text{(2), (2a) and (2b) are equivalent for any subequation } \mathcal{P}^+ \text{ with edge } E. \tag{5.4}$$

Proof of (3). Since $0 \in \mathcal{P}^+$ and $\mathcal{P}^+$ is $\mathcal{P}$-monotone, we have $\mathcal{P} \subset \mathcal{P}^+$. Since $\mathcal{P}$ is self polar, taking polars implies that $\mathcal{P}_+ \subset \mathcal{P}$ and therefore $\mathcal{P}_+ \subset S \cap \mathcal{P}$.

Suppose $B \in S \cap \mathcal{P}$. To show $B \in \mathcal{P}_+$ it suffices to show that $\langle A, B\rangle \geq 0$ for all $A \in \mathcal{P}^+$. By minimality, if $A \in \mathcal{P}^+$, then $A = A_0 + P$ with $A_0 \in E$ and $P \in \mathcal{P}$. Now $\langle A, B\rangle = \langle P, B\rangle \geq 0$ since $\langle A_0, B\rangle = 0$. ∎

Proof of (3*). Note that $S \cap (\operatorname{Int} \mathcal{P})$ is an open set in S, and it is contained in $S \cap \mathcal{P}$, which is a subset of $\mathcal{P}_+$ by (3). Hence, $S \cap (\operatorname{Int} \mathcal{P}) \subset \operatorname{Int}_S \mathcal{P}_+$.

The only non-trivial part (and the most important part) of showing $\operatorname{Int}_S \mathcal{P}_+ = S \cap (\operatorname{Int} \mathcal{P})$ is to show that:

$$\operatorname{Int}_S \mathcal{P}_+ \ \subset\ \operatorname{Int} \mathcal{P}. \tag{5.5}$$

Since S is a basic span subspace, $S \cap (\mathrm{Int}\,\mathcal{P}) \neq 0$. Choose $P \in S \cap (\mathrm{Int}\,\mathcal{P})$. Given $A \in \mathrm{Int}_S\,\mathcal{P}_+$, for $\epsilon > 0$ sufficiently small we have $A - \epsilon P \in \mathcal{P}_+$. Thus for all non-zero $Q \in \mathcal{P} \subset \mathcal{P}^+$ we have $0 \leq \langle A - \epsilon I, Q\rangle = \langle A, Q\rangle - \epsilon\langle P, Q\rangle$. Since $P > 0$, one has $\langle P, Q\rangle > 0$, which proves that $\langle A, Q\rangle > 0$ for all non-zero $Q \geq 0$. Thus $A > 0$. ∎

Proof of Theorem 5.5. By Theorem 5.4, if $\mathcal{P}^+$ is minimal, then $\mathcal{P}^+$ satisfies each of the eight conditions. For the converses we use the hypothesis that $\mathcal{P}^+$ is complete. By the edge criteria, Proposition 3.5(3), the edge E of $\mathcal{P}^+$ satisfies $E \cap \mathcal{P} = \{0\}$. Therefore we can apply the construction in Lemma 4.1 to yield a minimal subequation $Q^+ \equiv E + \mathcal{P}$ satisfying all the eight conditions. If $\mathcal{P}^+$ satisfies (1) then $\mathcal{P}^+ = Q^+$ and so it is minimal. Similarly, if $\mathcal{P}^+$ satisfies (2), then $\mathrm{Int}\,\mathcal{P}^+ = \mathrm{Int}\,Q^+$, so that $\mathcal{P} = Q^+$ is minimal.

By (5.3) we have that (1), (1a) and (1b) are equivalent.

By (5.4) we have that (2), (2a) and (2b) are equivalent.

Finally, if $\mathcal{P}^+$ satisfies (3*), then since $Q_+ = S \cap \mathcal{P}$ also, we have $\mathcal{P}_+ = Q_+$ and hence $\mathcal{P}^+ = Q^+$ is minimal. As noted above, (3) ⇒ (3*). ∎

Remark 5.6. The property (5.5) is extremely important and useful. See [**6**] for more details of the following.

Given $A \geq 0$ define $\Delta_A u \equiv \langle D^2 u, A\rangle$, or equivalently, from the subequation point of view,

$$\Delta_A \;\equiv\; \{B \in \mathrm{Sym}^2(\mathbb{R}^n) : \langle B, A\rangle \geq 0\}.$$

Then u is $\mathcal{P}^+$-subharmonic if and only if u is Δ_A-subharmonic for all $A \in \mathrm{Int}_{\mathrm{rel}}\,\mathcal{P}_+$. If (5.5) is true, then each such operator Δ_A is just a linear coordinate change of the standard Laplacian on $\mathbb{R}^n$ (or said differently, it is the Laplacian on $\mathbb{R}^n$ with a different metric). Thus results of standard potential theory, such as $u \in L^1_{\mathrm{loc}}$, are valid for $\mathcal{P}^+$-subharmonic functions.

One final property of minimal subequation is the following.

Proposition 5.7. *Suppose $\mathcal{P}^+$ is a minimal subequation. Then $\mathcal{P}^+$ is contained in its dual subequation*

$$\widetilde{\mathcal{P}^+} \;\equiv \sim(-\,\mathrm{Int}\,\mathcal{P}^+) \;=\; -(\sim \mathrm{Int}\,\mathcal{P}^+). \tag{5.6}$$

Proof. Since $\mathcal{P}^+ = E + \mathcal{P}$ and $\widetilde{\mathcal{P}}^+ + \mathcal{P} = \widetilde{\mathcal{P}}^+$, it suffices to show that $E \equiv \mathcal{P}^+ \cap (-\mathcal{P}^+) \subset \widetilde{\mathcal{P}}^+$. Suppose $A \notin \widetilde{\mathcal{P}}^+$, i.e., $-A \in \mathrm{Int}\,\mathcal{P}^+$. Then by 5.4(2) we have $-A = B_1 + P$ with $B_1 \in E$ and $P > 0$. If $A \in \mathcal{P}^+$ also, then $A = B_2 + Q$ with $B_2 \in E$ and $Q \geq 0$. Therefore, $P + Q = -B_1 - B_2 \in E$. However, $P + Q > 0$ contradicting Corollary 5.3(a). ∎

6. Edge Functions – Pluriharmonics

Suppose as before that $\mathcal{P}^+$ is a complete convex cone subequation with edge E.

Definition 6.1. An **edge function**, or **$\mathcal{P}^+$-pluriharmonic function** is a function u such that both

$$u \text{ and } -u \text{ are } \mathcal{P}^+\text{-subharmonic.} \tag{6.1}$$

Thus, by definition, u is continuous.

Definition 6.2. An upper semi-continuous function u is **"sub"** the edge functions on an open set $X \subset \mathbb{R}^n$ if for all domains $\Omega \subset\subset \mathbb{R}^n$ and all edge functions h on Ω

which are continuous on $\overline{\Omega}$,

$$u \ \le\ h \quad \text{on } \partial\Omega \qquad \Rightarrow \qquad u \ \le\ h \quad \text{on } \overline{\Omega}. \tag{6.2}$$

Proposition 6.3. *If u is dually $\mathcal{P}^+$-subharmonic on X, i.e., u is $\widetilde{\mathcal{P}}^+$-subharmonic for the dual subequation $\widetilde{\mathcal{P}}^+$ (see (5.6)), then u is "sub" the edge functions on X.*

Proof. Suppose u is $\widetilde{\mathcal{P}}^+$-subharmonic and h is an edge function. Then $-h$ is $\mathcal{P}^+$-subharmonic and (6.2) follows from comparison (see Thm. 6.2 in [**9**]). ∎

Now if a subequation becomes smaller, it dual subequation becomes larger. Consequently, the only subequation $\mathcal{P}^+$, with a given edge E, for which Proposition 6.3 might have a converse is the minimal subequation with edge E (see Definition 5.2).

THEOREM 6.4. *Suppose that $E \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a basic vector subspace, so that $\mathcal{P}^+ \equiv E + \mathcal{P}$ is the minimal subequation with edge E. Then the following conditions on a function u are equivalent.*

(1) u is dually $\mathcal{P}^+$-subharmonic.

(2) u is "sub" the edge functions.

(3) u is locally "sub" the edge functions.

(4) u is locally "sub" the degree-2 polynomial edge functions.

Proof. Because of Proposition 6.3 we need only prove that if u is locally "sub" the degree-2 polynomial edge functions, then u is dually $\mathcal{P}^+$-subharmonic. For this suppose that u is not $\widetilde{\mathcal{P}}^+$-subharmonic on X. Then (see Lemma 2.4 in [**7**]) there exists $z_0 \in X$, a quadratic polynomial test function φ, and $\alpha > 0$ such that

$$u(z) \ \le\ \varphi(z) - \alpha|z - z_0|^2 \quad \text{near } z_0 \text{ with equality at } z_0, \tag{6.3}$$

but

$$D^2_{z_0}\varphi \notin \widetilde{\mathcal{P}}^+, \quad \text{i.e.,} \quad -D^2_{z_0}\varphi \in \mathrm{Int}\,\mathcal{P}^+. \tag{6.4}$$

By Theorem 4.4(2) we have $\mathrm{Int}\,\mathcal{P}^+ = \mathrm{Int}\,\mathcal{P} + E$. Thus

$$-D^2_{z_0}\varphi \ =\ P + B \quad \text{with} \quad P > 0 \quad \text{and} \quad B \in E. \tag{6.5}$$

Consider the degree-2 edge polynomial

$$\begin{aligned} h(z) \ &\equiv\ \varphi(z_0) + \langle D_{z_0}\varphi, z - z_0\rangle - \tfrac{1}{2}\langle B(z - z_0), z - z_0\rangle \\ &=\ \varphi(z) - \tfrac{1}{2}\langle D^2_{z_0}\varphi, z - z_0\rangle - \tfrac{1}{2}\langle B(z - z_0), z - z_0\rangle \\ &=\ \varphi(z) + \tfrac{1}{2}\langle P(z - z_0), z - z_0\rangle. \end{aligned}$$

Since $P > 0$ by (6.3) this implies that

$$u(z) \ \le\ h(z) - \alpha|z - z_0|^2 \tag{6.6}$$

near z_0 with equality at z_0. This implies that u is not sub the function h on any small ball about z_0. Hence, u is not locally "sub" the degree-2 edge polynomial h. ∎

7. Further Discussion of Examples

Before turning to the examples we define the **(compact) invariance group of** $\mathcal{P}^+$ to be

$$\{g \in \mathrm{O}_n : g^*\mathcal{P}^+ \ = \ \mathcal{P}^+\}. \tag{7.1}$$

It is easy to see that for the minimal subequation $\mathcal{P}^+$ for a basic E,

$$g^*\mathcal{P}^+ = \mathcal{P}^+ \iff g^*S = S \iff g^*E = E \tag{7.2}$$

by using the conditions in Theorems 5.4 and 5.5, and this yields two equivalent definitions of this group.

Definition 7.1. (Self Duality). If the two convex cones $\mathcal{P}_0^+$ (the reduced constraint set) and $\mathcal{P}_+$ are polars of each other in the vector space S, then we say the subequation $\mathcal{P}^+$ is **polar self dual** (not to be confused with a subequation which equals its dual subequation in the sense of [**5**]).

Remark 7.2. Note that this can only happen for a minimal subequation $\mathcal{P}^+$. This is because if $\mathcal{P}_0^+ = \mathcal{P}_+$ (self duality), then $\mathcal{P}_0^+ = \mathcal{P}_+ \subset \mathcal{P}$, and hence $\mathcal{P}_0^+ = \pi(\mathcal{P}_0^+) \subset \pi(\mathcal{P})$. Note that $\mathcal{P} \subset \mathcal{P}^+$ so that $\pi(\mathcal{P}) \subset \mathcal{P}_0^+$ is always true. This proves $\mathcal{P}_0^+ = \pi(\mathcal{P})$, so by Theorems 5.5 and 5.4(1a), $\mathcal{P}^+$ is minimal.

Given a closed subset $\mathbb{G} \subset G(k, \mathbb{R}^n)$ consider the *subequation geometrically defined by* $\mathbb{G}$:

$$\mathcal{P}(\mathbb{G}) \ \equiv \ \left\{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \langle A, P_W\rangle = \mathrm{tr}(A\big|_W) \geq 0 \ \ \forall W \in \mathbb{G}\right\}.$$

We shall use the following notations introduced in Example 2.2:

$$\mathcal{P}^+ \equiv \mathcal{P}(\mathbb{G}), \ \ \mathcal{P}_+ = CCH(\mathbb{G}), \ \ S = \mathrm{span}\,(\mathbb{G}), \ \ E = S^\perp, \ \ \text{and} \ \ \mathcal{P}_0^+.$$

Note that the compact invariance group of the subequation $\mathcal{P}^+ = \mathcal{P}(\mathbb{G})$ can also be defined by

$$\{g \in \mathrm{O}(n) : g(\mathbb{G}) = \mathbb{G}\}. \tag{7.2$'$}$$

The O(n)-Invariance Group

For our first two examples of minimal subequations we focus on the O_n-**orthogonal decomposition**

$$\mathrm{Sym}^2(\mathbb{R}^n) \ = \ \mathbb{R}\cdot\mathrm{Id} \oplus \mathrm{Sym}^2_0(\mathbb{R}^n) \tag{7.3}$$

into irreducible components under O_n.

Example 7.1. (Real Monge-Ampère). The subequation is $\mathcal{P}^+ = \mathcal{P}$. Here the edge $E = \{0\}$ is as small as possible, and $S = \mathrm{Sym}^2(\mathbb{R}^n)$, $\mathcal{P}_+ = \mathcal{P}$, so the subequation is self-dual, and we have $\mathbb{G} = G(1, \mathbb{R}^n)$. Obviously E, S is a basic edge-span pair (Definition 3.12c). The conditions in Theorem 5.4 are obvious as well as the fact that $\mathcal{P} = \mathcal{P}^+ = \mathcal{P}_+ = \mathcal{P}_0^+$ is dimensionally complete. The invariance group is O_n, and the extreme rays are

$$\mathrm{Ext}(\mathcal{P}) \ = \ \{\mathrm{Ray}(P_e) : |e| = 1\}.$$

Each $A \in S$ can be put in canonical form $A = \sum_j \lambda_j P_{e_j}$ under the action of O_n, and $\det(A) = \prod_j \lambda_j$, provides a nonlinear operator for $\mathcal{P}^+ = \{\lambda_{\min} \geq 0\}$ (the standard real Monge-Ampère operator).

Example 7.2. (The Laplacian). Here $\mathcal{P}^+ = \Delta = \{A : \mathrm{tr}(A) \geq 0\}$ is a closed half space, and $\mathbb{G} = \{\mathrm{Id}\} = G(n, \mathbb{R}^n)$, $E = \mathrm{Sym}^2_0(\mathbb{R}^n)$, the traceless part of $\mathrm{Sym}^2(\mathbb{R}^n)$, $S = \mathbb{R}\cdot\mathrm{Id}$, and $\mathcal{P}_+ = \mathbb{R}_+\cdot\mathrm{Id}$ is a ray. The invariance group is O_n. The reduced constraint set is $\mathcal{P}_0^+ = \mathcal{P}_+$ so Δ is self dual. Now it is obvious that Δ is a minimal subequation.

The U(n)-Invariance Group

We now consider $\mathbb{C}^n$ and the following U(n)-**orthogonal decomposition** of real symmetric matrices into U_n-irreducible subspaces:

$$\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n) \;=\; \mathbb{R}\cdot\mathrm{Id}\oplus \mathrm{Herm}_0^{\mathbb{C}-\mathrm{sym}}(\mathbb{C}^n)\oplus \mathrm{Herm}^{\mathbb{C}-\mathrm{skew}}(\mathbb{C}^n) \tag{7.4}$$

multiples of the identity, traceless complex hermitian symmetric, and complex hermitian skew components. Given $A\in \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$, this decomposition can be written as

$$A \;=\; \frac{\mathrm{tr}(A)}{2n}\,\mathrm{Id} + A_0^{\mathbb{C}-\mathrm{sym}} + A^{\mathbb{C}-\mathrm{skew}} \tag{7.5}$$

where with respect to multiplication I by i:

$$A^{\mathbb{C}-\mathrm{sym}} \;=\; \tfrac12(A-IAI) \qquad \text{and} \qquad A^{\mathbb{C}-\mathrm{skew}} \;=\; \tfrac12(A+IAI).$$

Example 7.3. (Complex Plurisubharmonics). The subequation is $\mathcal{P}^+ = \mathcal{P}(\mathbb{G})$ where $\mathbb{G} = \mathbb{P}(\mathbb{C}^n) \subset G_{\mathbb{R}}(2,\mathbb{C}^n)$ is the Grassmannian of complex lines in $\mathbb{C}^n$. The edge is $E = \mathrm{Herm}^{\mathbb{C}-\mathrm{skew}}(\mathbb{C}^n)$ and the span is $S = \mathrm{Herm}^{\mathbb{C}-\mathrm{sym}}(\mathbb{C}^n)$. Also $\mathcal{P}_0^+ = \mathcal{P}_+$ is the convex cone on non-negative complex hermitian symmetric bilinear forms on $\mathbb{C}^n$, so this third example is self dual. Note that the projection of $2P_e$ onto S is $P_{\mathbb{C}e}$, (orthogonal projection onto the complex line through e) since $P_e - IP_eI = P_{\mathbb{C}e}$. The convex cone $\mathcal{P}_0^+ = \mathcal{P}_+$ has extreme rays generated by $\{P_{\mathbb{C}e} : |e| = 1\} = \mathbb{P}(\mathbb{C}^n) = \mathbb{G}$. The invariance group is U_n. Each $A\in S$ can be put into canonical form $A = \sum_{j=1}^n \lambda_j P_{\mathbb{C}e_j}$ under the action of this group, and $\mathcal{P}_0^+ = \{\lambda_{\min}\geq 0\}$. The complex Monge-Ampère operator $\det(A) = \lambda_1(A)\cdots\lambda_n(A)$ provides the nonlinear operator for $\mathcal{P}^+ = \mathcal{P}(\mathbb{P}(\mathbb{C}^n))$, in tight analogue with the real case $\mathcal{P}$.

Now we finally get to a new example, which is the subject of [**10**].

Example 7.4. (Lagrangian Plurisubharmonics). The subequation is $\mathcal{P}^+ = \mathcal{P}(\mathrm{LAG})$, where $\mathrm{LAG}\subset G_{\mathbb{R}}(n,\mathbb{C}^n)$ is the set of Lagrangian n-planes in $\mathbb{C}^n = \mathbb{R}^{2n}$. The edge E and span S are given by

$$E = \mathrm{Herm}_0^{\mathbb{C}-\mathrm{sym}}(\mathbb{C}^n) \qquad \text{and} \qquad S = \mathbb{R}\cdot\mathrm{Id}\oplus \mathrm{Herm}^{\mathbb{C}-\mathrm{skew}}(\mathbb{C}^n).$$

In [**10**] we prove that E, S is a basic edge-span pair, so that $\mathcal{P}^+ = E + \mathcal{P}$ and $\mathcal{P}_+ = S\cap\mathcal{P}$. The extreme rays in $\mathcal{P}_+$ are generated by the projections P_W with $W\in\mathrm{LAG}$ a Lagrangian n-plane. The extreme rays in $\mathcal{P}_0^+$ are generated by the images $\pi(P_e)$ of P_e where e is a unit vector. Note that

$$\pi(P_e) = \tfrac{1}{2n}\,\mathrm{Id} + \tfrac12(P_e + IP_eI) = \tfrac{1}{2n}\,\mathrm{Id} + \tfrac12(P_e - P_{Ie}),$$

and that $\frac12(P_e - P_{Ie})$ is the $\mathbb{C}$-skew component of P_e. This example is *not* self dual. However, since each $A\in S$ can be put in canonical form

$$A \;=\; \frac{\mathrm{tr}(A)}{2n} + \frac12\sum_{j=1}^n \lambda_j\left(P_{e_j} - P_{Ie_j}\right)$$

there is again a nonlinear operator for $\mathcal{P}^+ = \mathcal{P}(\mathrm{LAG})$ (see [**10**]). The invariance group is U_n.

The Sp(n)·Sp(1)-Invariance Group

Let $M_n(\mathbb{H})$ denote the space of $n\times n$ matrices with entries in $\mathbb{H}$, and let $A^* = \overline{A}^t$ if $A\in M_n(\mathbb{H})$. Consider the two subspaces

$$M_n^{\rm sym}(\mathbb{H}) \ = \ \{A\in M_n(\mathbb{H}) : A^* = A\}, \qquad \text{and}$$
$$M_n^{\rm skew}(\mathbb{H}) \ = \ \{A\in M_n(\mathbb{H}) : A^* = -A\}.$$

We let the scalars $\mathbb{H}$ act on the right. Then by letting $M_n(\mathbb{H})$ act on $x = (x_1, ..., x_n)^t \in \mathbb{H}^n$ on the left, one can identify $M_n(\mathbb{H})$ with ${\rm End}_{\mathbb{H}}(\mathbb{H}^n)$, the vector space of $\mathbb{H}$-linear maps of $\mathbb{H}^n$. Let

$${\rm Herm}^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n) \ = \ \{A\in {\rm End}_{\mathbb{H}}(\mathbb{H}^n) : A = A^*\}, \qquad \text{and}$$
$${\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n) \ = \ \{A\in {\rm End}_{\mathbb{H}}(\mathbb{H}^n) : A = -A^*\}.$$

so that $M_n^{\rm sym}(\mathbb{H}) = {\rm Herm}^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n)$ are identified (same for the skew parts).

Let $\epsilon(x,y) = \sum_{\ell=1}^n \overline{x}_\ell y_\ell$ denote the *standard quaternionic hermitian bilinear form on* $\mathbb{H}^n$. The **quaternionic unitary group** is

$${\rm Sp}_n \ = \ \{A\in M_n(\mathbb{H}) : \epsilon(Ax, Ay) = \epsilon(x,y)\}.$$

For each scalar $u\in\mathbb{H}$ let $R_u x \equiv xu$ denote right multiplication, and set $I\equiv R_i$, $J\equiv R_j$, $K\equiv R_k$. Then the group of unit scalars ${\rm Sp}_1 \equiv S^3 = \{R_u : u\in\mathbb{H}, |u| = 1\}$ acts on $\mathbb{H}^n$ on the right and the **enhanced quaternionic unitary group** is the group

$${\rm Sp}_n\cdot{\rm Sp}_1 \ = \ {\rm Sp}_n\times{\rm Sp}_1/\mathbb{Z}_2.$$

Since the standard euclidean inner product on $\mathbb{R}^{4n} = \mathbb{H}^n$ is $\langle x,y\rangle = {\rm Re}\,\epsilon(x,y)$,

$$M_n^{\rm sym}(\mathbb{H}) = {\rm Herm}^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n) \ \text{ is a real subspace of } {\rm Sym}^2(\mathbb{R}^{4n})$$

and

$$M_n^{\rm skew}(\mathbb{H}) = {\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n) \ \text{ is a real subspace of } {\rm Skew}^2(\mathbb{R}^{4n})$$

where ${\rm End}_{\mathbb{R}}(\mathbb{R}^{4n}) = {\rm Sym}^2(\mathbb{R}^{4n})\oplus{\rm Skew}^2(\mathbb{R}^{4n})$ is the usual decomposition. Note also that for each unit imaginary quaternion $u\in{\rm Im}\mathbb{H}$, we have $R_u\in{\rm Skew}^2(\mathbb{R}^{4n})$, and hence $R_uA = AR_u\in{\rm Sym}^2(\mathbb{R}^{4n})$ for all $A\in M_n^{\rm skew}(\mathbb{H}) = {\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n)$. This embeds

$${\rm Im}\mathbb{H}\otimes{\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n) \ = \ {\rm Im}\mathbb{H}\otimes M_n^{\rm skew}(\mathbb{H}) \ \subset \ {\rm Sym}^2(\mathbb{R}^{4n}). \tag{7.6}$$

The ${\rm Sp}_n\cdot{\rm Sp}_1$-**orthogonal decomposition**

$${\rm Sym}^2(\mathbb{R}^{4n}) = \mathbb{R}\cdot{\rm Id}\oplus{\rm Herm}_0^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n)\oplus\Big({\rm Im}\mathbb{H}\otimes{\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n)\Big) \tag{7.7}$$

into irreducible components plays a role in the next two examples, and a key role in classifying all the ${\rm Sp}_n\cdot{\rm Sp}_1$-invariant minimal subequations. Projection onto ${\rm Herm}^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n) = \mathbb{R}\cdot{\rm Id}\oplus{\rm Herm}_0^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n)$ and ${\rm Im}\mathbb{H}\otimes{\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n)$ are given by $A = A^{\mathbb{H}-{\rm sym}} + A^{\mathbb{H}-{\rm skew}}$ where

$$A^{\mathbb{H}-{\rm sym}} \ = \ \tfrac14(A - IAI - JAJ - KAK), \qquad \text{and} \tag{7.8a}$$
$$A^{\mathbb{H}-{\rm skew}} \ = \ \tfrac14(3A + IAI + JAJ + KAK). \tag{7.8b}$$

Example 7.5. (Quaternionic Plurisubharmonics). The subequation $\mathcal{P}^+\equiv\mathcal{P}(\mathbb{P}(\mathbb{H}^n))$ is geometric with $\mathbb{G} = \mathbb{P}(\mathbb{H}^n)\subset G_{\mathbb{R}}(4,\mathbb{H}^n)$. The edge and span are given by

$$E = {\rm Im}\mathbb{H}\otimes{\rm Herm}^{\mathbb{H}-{\rm skew}}(\mathbb{H}^n), \qquad \text{and} \qquad S = {\rm Herm}^{\mathbb{H}-{\rm sym}}(\mathbb{H}^n). \tag{7.9}$$

The set $\mathcal{P}_0^+ = \mathcal{P}_+$ is the convex cone of non-negative quaternionic hermitian symmetric bilinear forms on $\mathbb{H}^n$ (see [**1**] or [**4**] for more details). Under the identification of $\mathrm{Herm}^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n)$ with the set of quaternionic $n \times n$ matrices $M_n(\mathbb{H})$ satisfying $A^* \equiv \overline{A}^t = A$, we have

$$\mathcal{P}_0^+ \;=\; \{A \in M_n(\mathbb{H}) :\; A^* = A \;\text{ and }\; \overline{x}^t A x \geq 0 \;\forall\, x \in \mathbb{H}^n\}.$$

This is a minimal subequation and has compact invariance group $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$. Note that by (7.7a) the projection of P_e ($|e| = 1$) onto $\mathrm{Herm}^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n)$ is just $P_{\mathbb{H}e}$, orthogonal projection onto the quaternionic line $\mathbb{H}e$. Hence, this example is self dual, i.e., $\mathcal{P}_0^+ = \mathcal{P}_+$. Each $A \in M_n(\mathbb{H})$ with $A^* = A$ can be put in canonical form

$$A^{\mathbb{H}-\mathrm{sym}} \;=\; \sum_{j=1}^n \lambda_j P_{\mathbb{H}e_j}$$

under the action of $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$. The quaternionic Monge-Ampère operator

$$\det{}_{\mathbb{H}}(A) \;\equiv\; \prod_{j=1}^n \lambda_j \left(A^{\mathbb{H}-\mathrm{sym}}\right)$$

provides the nonlinear operator for $\mathcal{P}^+ = \mathcal{P}(\mathbb{P}(\mathbb{H}^n))$.

Example 7.6a. Reversing the roles of $\mathrm{Im}\mathbb{H} \otimes \mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)$ and $\mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n)$ in (7.9) above results in a second $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$-invariant minimal subequation $\mathcal{P}^+ \equiv E + \mathcal{P}$ with $\mathcal{P}_+ = S \cap \mathcal{P}$, where

$$E \;\equiv\; \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \quad\text{and}\quad S \;\equiv\; \mathbb{R}\cdot \mathrm{Id} \oplus (\mathrm{Im}\mathbb{H} \otimes \mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)). \tag{7.10}$$

Note that for each $|e| = 1$,

$$\pi(P_e) \;=\; \tfrac{1}{4n}\,\mathrm{Id} + \tfrac{1}{4}(3P_e - P_{Ie} - P_{Je} - P_{Ke}). \tag{7.11}$$

We leave as a question: Does $\pi(P_e)$ generate an exposed ray in $\mathcal{P}_0^+ = \pi(\mathcal{P})$?

This edge $E \equiv \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n)$ is reminiscent of the edge in Example 7.4 in the complex case. We now pursue this analogy. We say that a real n-plane W in $\mathbb{H}^n$ is **$\mathbb{H}$-Lagrangian** if

$$W \oplus IW \oplus JW \oplus KW \;=\; \mathbb{H}^n \quad \text{(orthogonal direct sum)}, \tag{7.12}$$

and let $\mathbb{H}\,\mathrm{Lag}$ denote the set of all such n-planes.

Example 7.6b. (Quaternionic Lagrangian Plurisubharmonics). These are defined as the subharmonics for the geometrically defined subequation $\mathcal{P}(\mathbb{H}\,\mathrm{Lag})$. Note that $\mathbb{H}\,\mathrm{Lag}$ and hence $\mathcal{P}(\mathbb{H}\,\mathrm{Lag})$ has compact invariance group $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$. Furthremore, given $A \in \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H}^n)$ one can show that

$$\mathrm{tr}\, A\big|_W \;=\; 0 \quad \forall\, W \in \mathbb{H}\,\mathrm{Lag} \qquad \iff \qquad A \in \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n)). \tag{7.13}$$

Consequently, $\mathcal{P}(\mathbb{H}\,\mathrm{Lag})$ has edge-span given by (7.10). At the moment we do not know whether or not $\mathcal{P}(\mathbb{H}\,\mathrm{Lag})$ is the minimal subequation with this edge-span. Of course one has

$$E + \mathcal{P} \;\subset\; \mathcal{P}(\mathbb{H}\,\mathrm{Lag}) \qquad\text{and}\qquad \mathcal{P}_+(\mathbb{H}\,\mathrm{Lag}) \;\subset\; S \cap \mathcal{P}, \tag{7.14}$$

where $\mathcal{P}_+(\mathbb{H}\,\mathrm{Lag})$ is the convex cone hull of $\{P_W : W \in \mathbb{H}\,\mathrm{Lag}\}$.

The $\mathrm{Sp}_n \cdot \mathrm{S}^1$–Invariance group

If U_n is replaced by the smaller subgroup SU_n, the decomposition (7.4) of $\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$ remains the same, and so SU_n is not a compact invariance group for a minimal subequation. However, the decomposition (7.7) does not remain the same if we replace $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$ by Sp_n. The new decomposition can be written as

$$\mathrm{Sp}_n : \quad \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H}^n) \ = \ \mathbb{R}\cdot \mathrm{Id} \oplus \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \bigoplus_{j=1}^{3} I_j \,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n) \qquad (7.15)$$

where I_j vary over I, J, K, or in fact over any orthonormal basis of $\mathrm{Im}\mathbb{H}$. Note that the representations $I_j \,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)$ are all equivalent.

The next example is a minimal subequation which is new.

Example 7.7. (I-Complex and J,K-Lagrangian Plurisubharmonics). This is a geometrically defined subequation given by the set

$$\mathbb{G} \ = \ \mathbb{G}(I; J, K) \ \subset \ G_{\mathbb{R}}(2n, \mathbb{H}^n)$$

of real $2n$-planes which a simultaneously I-complex and both J and K Lagrangian. (Note that any two of these conditions implies the third.) The associated subequation is $\mathcal{P}(\mathbb{G}(I; J, K))$.

Now $\mathcal{P}(\mathrm{JLAG})$ has edge

$$\mathrm{Herm}_0^{J\mathbb{C}-\mathrm{sym}}(\mathbb{C}^{2n}) = \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \oplus J\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)$$

and $\mathcal{P}(\mathrm{KLAG})$ has edge

$$\mathrm{Herm}_0^{K\mathbb{C}-\mathrm{sym}}(\mathbb{C}^{2n}) = \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \oplus K\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n).$$

Hence the sum

$$\mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \oplus J\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n) \oplus K\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n) \ \subset \ \mathrm{Edge}(\mathcal{P}(\mathbb{G})).$$

Each $W \in \mathbb{G}$ has a real basis of the form

$$e_1, Ie_1, ..., e_n, Ie_n \qquad \text{where } e_1, ..., e_n \text{ is an } \mathbb{H}\text{-basis for } \mathbb{H}^n.$$

Thus $P_W = P_V + P_{IV}$ where $V \equiv \mathrm{span}_{\mathbb{R}}\{e_1, ..., e_n\}$. Note that $W \in \mathbb{G} \Rightarrow W^\perp = JW = KW \in \mathbb{G}$. Hence, $\mathrm{Id} = P_W + P_{W^\perp} \in S \equiv \mathrm{span}\,(\mathcal{P}(\mathbb{G})) \equiv \mathrm{span}\,(\mathbb{G})$. Now we have

$$P_W - P_{W^\perp} = P_W - P_{IW} = P_W + IP_W I \ \in \ I\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n).$$

One can show (direct proof and invariance proof) that

$$S \ = \ \mathbb{R}\cdot \mathrm{Id} \oplus I\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n) \qquad (7.16)$$

and hence

$$E \ = \ \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \oplus J\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n) \oplus K\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n). \qquad (7.17)$$

Lemma 7.8. *Each $A \in I\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)$ commutes with I and anti-commutes with J and K. If e is an eigenvector of A with eigenvalue λ, then Ie, Je, Ke are eigenvectors with eigenvalues $\lambda, -\lambda, -\lambda$. Hence, A can be put in the canonical form (where $e_1, ..., e_n$ is an $\mathbb{H}$-basis for $\mathbb{H}^n$):*

$$A \ \equiv \ \sum_{j=1}^{n} \lambda_j \left(P_{e_j} + P_{Ie_j} - P_{Je_j} - P_{Ke_j}\right).$$

Corollary 7.9. *The element $B \equiv \frac{t}{4n}\,\mathrm{Id} + A \in S$ is ≥ 0 if and only if each $|\lambda_j| \leq \frac{t}{2n}$. Hence, taking $t = \mathrm{tr}(B) = 2n$, the non-negativity condition becomes*

$$|\lambda_j| \ \leq\ \frac{1}{2}, \quad j = 1, ..., n.$$

This describes a cube in $\mathbb{R}^n$. The 2^n extreme points are $\epsilon = (\pm\frac{1}{2}, ..., \pm\frac{1}{2})$, which yields

$$B(\epsilon) \ \equiv\ \tfrac{1}{2}\,\mathrm{Id} + \sum_{j=1}^{n} \pm\tfrac{1}{2}\left(P_{e_j} + P_{Ie_j} - P_{Je_j} - P_{Ke_j}\right) \ =\ P_{W(\epsilon)}$$

where

$$W(\epsilon) \ =\ \mathrm{span}\left\{(e_1, Ie_1 \ \text{if} \ \ \epsilon_1 = \tfrac{1}{2}) \ \ \text{or} \ \ (Je_1, Ke_1 \ \text{if} \ \ \epsilon_1 = -\tfrac{1}{2}), \ \ ... \ \ \text{etc.}\right\}.$$

This proves

Proposition 7.10.

$$S \cap \mathcal{P} \ =\ CCH\{P_W : W \in \mathbb{G}\} \ \equiv\ \mathcal{P}_+(\mathbb{G}).$$

Corollary 7.11. *The subequation $\mathcal{P}(\mathbb{G})$ is* **the** *minimal subequation with span S and edge E given by (7.9) and (7.10).*

Example 7.12. Consider the edge $E_I \equiv I\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)$ and the minimal subequation $\mathcal{P}^+ \equiv E_I + \mathcal{P}$. The compact invariance group is $\mathrm{Sp}_n \cdot S^1$, as in Example 7.7.

Lemma 7.13. *One has*

$$\begin{aligned} \mathcal{P}^+ \ \equiv\ E_I + \mathcal{P} \ &\subset\ \mathcal{P}(I\,\mathrm{Lag}) \cap \mathcal{P}(\mathbb{P}_J(\mathbb{C}^{2n})) \cap \mathcal{P}(\mathbb{P}_K(\mathbb{C}^{2n})) \\ &=\ \mathcal{P}\left((I\,\mathrm{Lag}) \cup \mathbb{P}_J(\mathbb{C}^{2n}) \cup \mathbb{P}_K(\mathbb{C}^{2n})\right) \end{aligned}$$

which has edge E_I.

Proof. Suppose for all $W \in I\,\mathrm{Lag} \cup \mathbb{P}_J(\mathbb{C}^{2n}) \cup \mathbb{P}_K(\mathbb{C}^{2n})$ that $\langle A, P_W\rangle \geq 0$. Taking $W \in I\,\mathrm{Lag}$ proves that $A \in \mathcal{P}(I\,\mathrm{Lag})$; taking $W \in \mathbb{P}_J(\mathbb{C}^{2n})$ proves that $A \in \mathcal{P}(\mathbb{P}_J(\mathbb{C}^{2n}))$; and taking $W \in \mathbb{P}_K(\mathbb{C}^{2n})$ proves that $A \in \mathcal{P}(\mathbb{P}_K(\mathbb{C}^{2n}))$. Conversely, if A belongs to the intersection of the three geometric subequations in the Lemma, then $\mathrm{tr}\,A\big|_W \geq 0$ for all $W \in I\,\mathrm{Lag} \cup \mathbb{P}_J(\mathbb{C}^{2n}) \cup \mathbb{P}_K(\mathbb{C}^{2n})$. This proves the last equality in the Lemma.

Since $E_I \subset E_{0,I}$, by Example 7.4,

$$\mathcal{P}^+ \ \equiv\ E_I + \mathcal{P} \subset E_{0,I} + \mathcal{P} \ =\ \mathcal{P}(I\,\mathrm{Lag}).$$

Since $E_I \subset E_{I,K}$, by Example 7.3,

$$\mathcal{P}^+ \ \equiv\ E_I + \mathcal{P} \subset E_{I,K} + \mathcal{P} \ =\ \mathcal{P}(\mathbb{P}_J(\mathbb{C}^{2n})).$$

Since $E_I \subset E_{I,J}$, by Example 7.3,

$$\mathcal{P}^+ \ \equiv\ E_I + \mathcal{P} \subset E_{I,J} + \mathcal{P} \ =\ \mathcal{P}(\mathbb{P}_K(\mathbb{C}^{2n})).$$

Finally since $E_I = E_{0,I} \cap E_{I,K} \cap E_{I,J}$, this proves that $\mathcal{P}\left((I\,\mathrm{Lag}) \cup \mathbb{P}_J(\mathbb{C}^{2n}) \cup \mathbb{P}_K(\mathbb{C}^{2n})\right)$ has edge E_I. ∎

It remains an open question whether or not $\mathcal{P}\left((I\,\mathrm{Lag}) \cup \mathbb{P}_J(\mathbb{C}^{2n}) \cup \mathbb{P}_K(\mathbb{C}^{2n})\right)$ is the minimal subequation $\mathcal{P}^+ \equiv E_I + \mathcal{P}$ with edge E_I.

8. Classifying the Invariant Minimal Subequations

Given a compact subgroup $G \subset \mathrm{O}_N$, one could ask which (if any) subequations have G as their exact invariance group. Now the compact invariance group for a minimal subequation $\mathcal{P}^+ = E + \mathcal{P}$ is the same as for its edge E (see (7.2)). Therefore we need only classify the possible invariant edges E. This is easily done as follows. First decompose $\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{R}^N)$ into irreducible pieces $\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{R}^N) = \mathbb{R} \cdot \mathrm{Id} \oplus E_0 \oplus E_1 \oplus \cdots \oplus E_k$, and note that $E_0 \oplus \cdots \oplus E_k = \mathrm{Sym}^2_0(\mathbb{R}^N)$, the traceless part. Hence any space $E = E_{i_1} \oplus \cdots \oplus E_{i_\ell}$, $0 \le i_1 < \cdot < \imath_\ell \le k$ can be chosen as a basic (invariant) edge. Note that $E = \{0\}$ is also a basic invariant edge, and $E + \mathcal{P} = \mathcal{P}$, which has compact invariance group O_N.

The $\mathbf{O}_n$-Case. Here we have

$$\mathrm{Sym}^2(\mathbb{R}^n) \ = \ \mathbb{R} \cdot \mathrm{Id} \oplus E_0 \qquad \text{with} \qquad E_0 \ \equiv \ \mathrm{Sym}^2_0(\mathbb{R}^n).$$

There are two examples: $E = \{0\}$ and $E = E_0$ given by Examples 7.1 and 7.2.

The $\mathbf{U}_n$-Case. Here it is more complicated:

$$\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n) \ = \ \mathbb{R} \cdot \mathrm{Id} \oplus E_0 \oplus E_1 \qquad \text{with}$$
$$E_0 \ \equiv \ \mathrm{Herm}_0^{\mathbb{C}-\mathrm{sym}}(\mathbb{C}^n) \qquad \text{and} \qquad E_1 \ \equiv \ \mathrm{Herm}^{\mathbb{C}-\mathrm{skew}}(\mathbb{C}^n),$$

which are Examples 7.3 and 7.4.

The $\mathbf{Sp}_n{\cdot}\mathbf{Sp}_1$-Case. Here we have

$$\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H}^n) \ = \ \mathbb{R} \cdot \mathrm{Id} \oplus E_0 \oplus E_1 \qquad \text{with}$$
$$E_0 \ \equiv \ \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n) \qquad \text{and} \qquad E_1 \ \equiv \ \mathrm{Im}\mathbb{H} \otimes \mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n).$$

Hence again there are two new examples $E = E_0$ and $E = E_1$ which are Examples 7.5 and 7.6a.

The $\mathbf{Sp}_n$ and $\mathbf{Sp}_n{\cdot}\mathbf{S}^1$-Cases. Under Sp_n we have

$$\mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H}^n) \ = \ \mathbb{R} \cdot \mathrm{Id} \oplus E_0 \oplus E_I \oplus E_J \oplus E_K \qquad \text{with}$$
$$E_0 \ \equiv \ \mathrm{Herm}_0^{\mathbb{H}-\mathrm{sym}}(\mathbb{H}^n), \quad E_I \ \equiv \ I\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n),$$
$$E_J \ \equiv \ J\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n), \quad E_K \ \equiv \ K\,\mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n),$$

(see (7.15)). Of the possible edges we can exclude most of them as coming from the previous cases. For example, $E_{0,I} \equiv E_0 \oplus E_I = \mathrm{Herm}_0^{\mathbb{C}-\mathrm{sym}}(\mathbb{C}^n)$ for the I-complex case (as well as $E_{0,J}$, $E_{0,K}$) come from Example 7.4. The case $E_{J,K} \equiv E_J \oplus E_K = \mathrm{Herm}^{\mathbb{C}-\mathrm{skew}}(\mathbb{C}^n)$ (for the complex structure I) can be excluded, since this is Example 7.3. Similarly we exclude $E_{I,K}$ and $E_{I,J}$. The case $E = E_0$ is just Example 7.6a, while the case $E \equiv E_{I,J,K} = E_I \oplus E_J \oplus E_K = \mathrm{Im}\mathbb{H} \otimes \mathrm{Herm}^{\mathbb{H}-\mathrm{skew}}(\mathbb{H}^n)$ is Example 7.5.

This leaves, up to permuting I, J, K, two examples: $E = E_I$, which is Example 7.12, and $E = E_{0,J,K} = E_0 \oplus E_J \oplus E_K$ as in (7.10), which is Example 7.7. These last two examples have compact invariance group $\mathrm{Sp}_n{\cdot}\mathrm{S}^1$. Note that this proves that there are no minimal subequations with compact invariance group Sp_n.

9. An Envelope Problem for Minimal Subequations.

Suppose that $\mathbf{F} \equiv \mathcal{P}^+$ is a minimal subequation. In this section we investigate the role played by the edge functions in solving the Dirichlet problem. The key fact about $\mathbf{F}$ that will be used below is the following from Theorem 5.4(2):

$$\mathrm{Int}\,\mathbf{F} \ = \ E + \mathrm{Int}\,\mathcal{P}. \tag{9.1}$$

We recall that existence and uniqueness for the (DP) on a bounded domain $\Omega \subset \mathbb{R}^n$ and arbitrary $\varphi \in C(\partial\Omega)$ was established in [**5**] if $\partial\Omega$ is smooth and strictly **F**- and $\widetilde{\mathbf{F}}$-convex (for any subequation $\mathbf{F} \subset \mathrm{Sym}^2(\mathbb{R}^n)$). Moreover, the solution H equals the Perron function

$$H(x) \equiv \sup_{u \in \mathcal{F}_{\mathbf{F}}(\varphi)} u(x) \qquad \text{for } x \in \overline{\Omega} \tag{9.2}$$

for the Perron family of **F**-subharmonics

$$\mathcal{F}_{\mathbf{F}}(\varphi) \equiv \left\{ u \in \mathbf{F}(\overline{\Omega}) : u\big|_{\partial\Omega} \leq \varphi \right\}. \tag{9.3}$$

By definition $u \in \mathbf{F}(\overline{\Omega})$ if u is $[-\infty, \infty)$-valued and upper semi-continuous on $\overline{\Omega}$ and $u\big|_{\Omega} \in \mathbf{F}(\Omega)$.

The proof of our main result here follows (as closely as possible) the existence proof for the Dirichlet Problem given in [**7**].

To begin we consider the following analogues of the above. Let

$$E(\overline{\Omega}) \equiv \left\{ u \in C(\overline{\Omega}) : u\big|_{\Omega} \in E(\Omega) \right\} \tag{9.4}$$

denote the space of edge functions on $\overline{\Omega}$, and consider the family of edge functions

$$\mathcal{F}_E(\varphi) \equiv \left\{ h \in E(\overline{\Omega}) : h\big|_{\partial\Omega} \leq \varphi \right\}. \tag{9.5}$$

A natural question to ask is:

Question 1. When is the envelope

$$U_E(x) \equiv \sup_{h \in \mathcal{F}_E(\varphi)} h(x) \ \text{ equal to the solution } H \text{ defined by (9.2)?}$$

There are two interesting extreme cases where the answer is positive.

Example 9.1. **($\mathbf{F} \equiv \mathcal{P}$).** Here $E(\Omega) \equiv \mathrm{Aff}(\mathbb{R}^n)$, the space of affine functions on $\mathbb{R}^n$. In this case

$$U_{\mathrm{Aff}} = H_{\mathcal{P}}$$

because, by the Hahn-Banach Theorem, for each point $x_0 \in \Omega$, there exists an affine function h with $h \leq H_{\mathcal{P}}$ on $\overline{\Omega}$ and $h(x_0) = H_{\mathcal{P}}((x_0)$.

Example 9.2. **($\mathbf{F} \equiv \Delta$).** Here $E(\Omega) \equiv \{h \in C(\overline{\Omega}) : h\big|_{\Omega}$ is Δ-harmonic$\}$. Therefore, $H \in \mathcal{F}_E(\varphi)$, proving that

$$U_{\Delta} = H_{\Delta}.$$

For other cases Question 1 remains open, so it is appropriate to consider larger families than $\mathcal{F}_E(\varphi)$. First, set

$$E^{\max}(\Omega) \equiv \{M : M = \max\{h_1, ..., h_N\} \ \text{ with } \ h_1, ..., h_N \in E(\Omega)\} \tag{9.6}$$

and consider the family

$$\mathcal{F}_{E^{\max}}(\varphi) \equiv \{M \in E^{\max}(\overline{\Omega}) : M\big|_{\partial\Omega} \leq \varphi\} \tag{9.7}$$

where by definition $M \in E^{\max}(\overline{\Omega})$ if $M \in \mathrm{USC}(\overline{\Omega})$ and $M\big|_{\Omega} \in E^{\max}(\Omega)$. Since the conditions $M \equiv \max\{h_1, ..., h_N\} \in E^{\max}(\overline{\Omega})$ and $M\big|_{\partial\Omega} \leq \varphi$ imply that each $h_k \in \mathcal{F}_E(\varphi)$, we have

$$U_{E^{\max}} = \sup_{M \in \mathcal{F}_{E^{\max}}(\varphi)} M = \sup_{h \in \mathcal{F}_E(\varphi)} h = U_E. \tag{9.8}$$

In particular, $U_{E^{\max}} = H \iff U_E = H$.

Now we consider a localized version

$$\mathcal{F}_{E^{\text{loc-max}}}(\varphi) \;=\; \{u \in E^{\text{loc-max}}(\overline{\Omega}) : u\big|_{\partial\Omega} \leq \varphi\} \tag{9.9}$$

where by definition $u \in E^{\text{loc-max}}(\overline{\Omega})$ if $u \in \text{USC}(\overline{\Omega})$ and for each point $x_0 \in \Omega$, there exists a neighborhood $B_r(x_0) \subset \Omega$ such that

$$u\big|_{B_r(x_0)} \;\in\; E^{\max}(B_r(x_0)). \tag{9.10}$$

Question 2. When is the envelope

$$U \;\equiv\; U_{E^{\text{loc-max}}} \;=\; \sup_{u \in \mathcal{F}_{E^{\text{loc-max}}(\varphi)}} u \quad \text{equal to the solution } H \text{ in (9.2)?}$$

We can answer this question.

THEOREM 9.3. *If* $\mathbf{F} = \mathcal{P}^+$ *is a minimal subequation and* $\partial\Omega$ *is smooth and strictly* $\mathbf{F}$*-convex, then*

$$U \;=\; H.$$

Proof. Since $\mathbf{F} \subset \widetilde{\mathbf{F}}$ (Thm. 5.7), the strict $\widetilde{\mathbf{F}}$-convexity of the boundary is automatic.

In what follows we shall shorten $\mathcal{F}_{E^{\text{loc-max}}}(\varphi)$ to $\mathcal{F}(\varphi)$.

Note that $\mathcal{F}(\varphi) \subset \mathcal{F}_{\mathbf{F}}(\varphi) \Rightarrow U \leq H \Rightarrow U^* \leq H \Rightarrow$

$$U^*\big|_{\partial\Omega} \;\leq\; \varphi, \qquad \text{and we also have} \tag{9.11a}$$

$$\varphi \;\leq\; U_*\big|_{\partial\Omega} \qquad \text{proved at the end.} \tag{9.11b}$$

Note 9.4. If $\partial\Omega$ is strictly convex, then Example 9.1 shows that $\varphi = U_{\mathcal{P}}\big|_{\partial\Omega}$ and $U_{\text{Aff}} = U_{\mathcal{P}}$. Since $\mathcal{P} \subset \mathbf{F}$ and $\text{Aff} \subset E$, we have $U_{\mathcal{P}} \leq U_E \leq U$. Hence $U_{\mathcal{P}} \leq U_*$, so that (9.11b) holds under strict $\mathcal{P}$-convexity of $\partial\Omega$.

These two properties imply the following.

$$\text{(Boundary Continuity)} \qquad U_*\big|_{\partial\Omega} \;=\; U\big|_{\partial\Omega} \;=\; U^*\big|_{\partial\Omega} \;=\; \varphi. \tag{9.11}$$

By the "families bounded above property" we have

$$U^* \in \mathbf{F}(\overline{\Omega}). \tag{9.12}$$

Note that H (or $\sup_{\partial\Omega}\varphi$ if you wish) provides an upper bound for $\mathcal{F}(\varphi)$.

Assume for the moment that:

$$-U_* \in \widetilde{\mathbf{F}}(\overline{\Omega}). \tag{9.13}$$

Then the proof is easily completed as follows. By (9.12) and (9.13), $U^* - U_* \in \widetilde{\mathcal{P}}(\overline{\Omega})$ is subaffine on Ω (see [**5**]). Moreover, it is ≥ 0 on $\overline{\Omega}$ and equal to zero on $\partial\Omega$. Hence, by the (MP) for $\widetilde{\mathcal{P}}$, $U^* - U_*$ vanishes on $\overline{\Omega}$. That is,

$$U_* \;=\; U \;=\; U^* \quad \text{on } \overline{\Omega}. \tag{9.14}$$

This proves that U is $\mathbf{F}$-harmonic on Ω and equal to φ on $\partial\Omega$. By uniqueness for the (DP) this proves that $U = H$ on $\overline{\Omega}$. Thus it remains to prove (9.11b) and the following.

Lemma 9.5. $\qquad -U_*\big|_{\Omega} \in \widetilde{\mathbf{F}}(\Omega).$

Proof. We follow that bump argument given in the proof of Lemma $\widetilde{F}$ in [**7**, p. 455] as closely as possible.

Suppose $-U_*\big|_\Omega \notin \widetilde{\mathbf{F}}(\Omega)$. Then there exists $x_0 \in \Omega$, $\epsilon > 0$ and ψ, a degree-2 polynomial, satisfying

$$\begin{aligned} &(a) \quad -U_* \ \leq\ \psi - \epsilon|x-x_0|^2 \quad \text{near} \quad x_0, \quad \text{and} \\ &(b) \quad -U_*(x_0) \ =\ \psi(x_0), \quad \text{and} \\ &(c) \quad D^2\psi \notin \widetilde{\mathbf{F}}. \end{aligned} \tag{9.15}$$

Rewrite (a) and (c) as

$$\begin{aligned} &(a)' \quad -\psi \ \leq\ U_* - \epsilon|x-x_0|^2 \quad \text{near} \quad x_0, \quad \text{and} \\ &(c)' \quad D^2(-\psi) \in \operatorname{Int}\mathbf{F}. \end{aligned}$$

By the key fact (9.1) above we have that

$$D^2(-\psi) \ =\ e + P \quad \text{with} \quad e \in E \quad \text{and} \quad P > 0. \tag{9.16}$$

Therefore

$$-\psi \ =\ h + \tfrac{1}{2}\langle P(x-x_0), x-x_0\rangle \tag{9.17}$$

with h a degree 2 polynomial satisfying

$$(i) \ \ D^2h \ =\ e \qquad \text{and} \qquad (ii) \ \ h(x_0) \ =\ U_*(x_0). \tag{9.18}$$

The first part is just the statement that

$$(i)' \ \ h \text{ is an edge function on } \mathbb{R}^n. \tag{9.18)(i)'}$$

Now by (9.17) the inequality (a)$'$ says

$$h + \tfrac{1}{2}\langle P(x-x_0), x-x_0\rangle \ \leq\ U_* - \epsilon|x-x_0|^2 \quad \text{on} \quad B_{r_2}(x_0). \tag{9.19}$$

Choose $0 < r_1 < r < r_2$. Then by (9.19)

$$h + \delta \ <\ U_* \quad \text{on} \quad B_{r_2}(x_0) - B_{r_1}(x_0) \tag{9.20}$$

where $\delta \equiv \inf_{|x-x_0|=r_1} \frac{1}{2}\langle P(x-x_0), x-x_0\rangle$ (or $\delta = \epsilon r_1^2$ also works). For each point $y \in \partial B_r(x_0)$ we have $h(y) + \delta < U(y)$ by (9.20). Hence, by the definition of the $U = U_{\mathcal{F}^{E\text{loc-max}}}$ given in Question 2, there exists $u_y \in \mathcal{F}(\varphi)$ with

$$h(y) + \delta \ <\ u_y(y), \tag{9.21}$$

and since h and u_y are continuous, this holds in a neighborhood of y. Therefore, by compactness, there exist $u_1, ..., u_N \in \mathcal{F}(\varphi)$ with

$$h + \delta \ <\ u \ \equiv\ \max\{u_1, ..., u_N\} \ \text{ in a neighborhood of } \partial B_r(x_0). \tag{9.22}$$

Since $\mathcal{F}(\varphi)$ is closed under taking the maximum of a finite number of elements, we have

$$h + \delta \ <\ u \text{ in a neighborhood of } \partial B_r(x_0) \text{ with } u \in \mathcal{F}(\varphi). \tag{9.23}$$

This implies that

$$u' \ \equiv\ \begin{cases} \quad u & \text{on } \ \overline{\Omega} - B_r(x_0) \\ \max\{u, h+\delta\} & \text{on } \ \overline{B_r(x_0)} \end{cases} \tag{9.24}$$

is an element of $\mathcal{F}(\varphi)$. (Note that $h + \delta$ and hence u' is not necessarily an element of $\mathcal{F}_E^{\max}(\varphi)$.) Since $u' \in \mathcal{F}(\varphi)$, we have $u' \leq U$ on $\overline{\Omega}$. In particular, $h + \delta \leq U$ on $B_r(x_0)$. Since h is continuous, this implies $h + \delta \leq U_*$, and hence

$$h(x_0) + \delta \ \leq\ U_*(x_0), \tag{9.25}$$

which contradicts (9.18 b) that $h(x_0) = U_*(x_0)$.

It only remains to do the following.

Proof of (9.11b). We fix $x_0 \in \partial\Omega$, and let ρ be a smooth, strictly **F**-convex defining function for $\partial\Omega$ defined in a neighborhood of x_0. Then by (9.1) there exist $\epsilon > 0$ and $r > 0$ such that

$$D_x^2\rho - \epsilon I \;\in\; \mathrm{Int}\,\mathbf{F} \;=\; E + \mathrm{Int}\,\mathcal{P} \qquad \forall\, x \in B_r(x_0).$$

In particular,

$$D_{x_0}^2\rho - \epsilon I \;=\; A + P \qquad \text{for } \; A \in E \; \text{ and } \; P > 0.$$

By adding a linear function to $\frac{1}{2}\langle Ax, x\rangle$, we get a quadratic ψ with $D_{x_0}^2\psi = A$ and $\psi(x_0) = 0$ so that

$$\rho(x) - \frac{\epsilon}{2}|x - x_0|^2 \;=\; \psi(x) + \tfrac{1}{2}\langle P(x - x_0), x - x_0\rangle + O(|x - x_0|^3).$$

Taking ϵ smaller, we can get a smaller $r > 0$ so that

$$\rho(x) - \frac{\epsilon}{2}|x - x_0|^2 \;>\; \psi(x) + \tfrac{1}{2}\langle P(x - x_0), x - x_0\rangle \qquad \text{for } \; x \in \overline{B_r(x_0)} - \{x_0\}.$$

Since $\rho \le 0$ on $\overline{\Omega}$ we have

$$-\frac{\epsilon}{2}|x - x_0|^2 - \tfrac{1}{2}\langle P(x - x_0), x - x_0\rangle \;\ge\; \psi(x) \qquad \text{for } \; x \in \overline{B_r(x_0)} \cap \overline{\Omega}. \tag{9.26}$$

We now fix $\delta > 0$ and shrink $r > 0$ so that

$$\varphi(x_0) - \delta \;<\; \varphi \qquad \text{for } \; x \in \overline{B_r(x_0)} \cap \partial\Omega. \tag{9.27}$$

From (9.26) above we have that there exists η with

$$0 \;>\; \eta \;\ge\; \psi(x) \qquad \text{for } \; x \in \left(\overline{B_r(x_0)} - B_{r/2}(x_0)\right) \cap \overline{\Omega}. \tag{9.28}$$

We now consider the edge function

$$\Psi(x) \;\equiv\; \varphi(x_0) - \delta + C\psi(x). \tag{9.29}$$

By (9.28) we see that for $C >> 0$ we will have

$$\Psi(x) \;<\; \inf \varphi \qquad \text{on} \quad \left(\overline{B_r(x_0)} - B_{r/2}(x_0)\right) \cap \Omega$$

Therefore

$$\underline{u} \;\equiv\; \begin{cases} \inf_{\partial\Omega} \varphi & \text{on } \; \overline{\Omega} - B_{r/2}(x_0) \\ \max\{\Psi, \inf_{\partial\Omega} \varphi\} & \text{on } \; \overline{B_r(x_0)} \cap \overline{\Omega} \end{cases}$$

is a well defined function on $\overline{\Omega}$, and it is locally the maximum of edge functions. Furthermore, by (9.27) and (9.29) we see that $\underline{u} \le \varphi$ on $\partial\Omega$. Hence, $\underline{u}$ is in our Perron family for the Dirichlet problem, and so we have $\underline{u} \le U$, which implies that $\underline{u} \le U_*$. In particular, $\underline{u}(x_0) = \varphi(x_0) - \delta \le U_*(x_0)$. Taking $\delta \to 0$ shows that $\varphi(x_0) \le U_*(x_0)$. This proves (9.11b) and therefore Theorem 9.3. ∎

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