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Orbifold regularity of weak Kähler-Einstein metrics

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Abstract. We show that weak Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties are smooth orbifold metrics away from analytic subsets of complex codimension at least three. Our proof uses orbifold resolution of singularities.

1. Introduction

It has been a fundamental problem to study how to compactify the moduli of Einstein metrics and what are degenerate metrics in the compactification. It has found many applications in Kähler geometry. For instance, in the resolution of the YTD conjecture on the existence of Kähler-Einstein metrics on Fano manifolds (see [**Tia4**] and also [**CDS**]), a crucial tool is a compactness result on Kähler-Einstein metrics. In its simplest form, this result says that the Gromov-Hausdorff limit of a sequence of smooth Kähler-Einstein manifolds $(X_i, \omega_{i,\mathrm{KE}})$ is a normal Fano variety $X := X_\infty$ with klt singularities and that there is a weak Kähler-Einstein metric $\omega_{\infty,\mathrm{KE}}$ on X_∞ which is smooth on X_∞^{reg}. The existence of a Gromov-Hausdorff limit follows from Gromov's compactness theorem. The problem is about the regularity of X_∞. It follows from Cheeger-Colding's theory and Cheeger-Colding-Tian's theory (see [**CCT**] and the reference therein) that X_∞ is smooth outside a closed subset S of Hausdorff codimension at least 4 and $\omega_{\infty,\mathrm{KE}}$ is a Kähler-Einstein metric. It was the second author ([**Tia1**], [**Tia2**], see also [**Li**]) who first pointed out the route to prove that X_∞ is an algebraic variety is to establish a so-called partial C^0-estimate. He demonstrated in [**Tia1**] how to achieve this when the complex dimension n is equal to 2 by showing that a sequence of Kähler-Einstein surfaces converges to a Fano orbifold with a smooth orbifold Kähler-Einstein metric. Note that when $n = 2$, klt singularities are nothing but isolated quotient singularities or orbifold singularities. Two key ingredients to prove the partial C^0-estimate in dimension 2 are orbifold compactness result of Kähler-Einstein 4-manifolds and Hörmander's L^2-estimates.

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Recently, Donaldson-Sun [**DS**], and the second author [**Tia3**] independently, generalized the partial C^0-estimate to higher dimensional Kähler-Einstein manifolds. Here they need to rely on compactness results on Kähler-Einstein metrics in higher dimensions developed by Cheeger-Colding and Cheeger-Colding-Tian. It follows from the partial C^0-estimate (see [**Tia2**], [**Li**] and [**DS**]) that X_∞ is a normal variety. Furthermore, the partial C^0-estimate implies that there is a uniform C^2-estimate of the potential of $\omega_{\infty,\mathrm{KE}}$ on the regular part X_∞^{reg} of X_∞. Then the Evans-Krylov theory or Calabi's 3rd derivative estimate allows one to show that $\omega_{\infty,\mathrm{KE}}$ is smooth on X_∞^{reg} (see [**Tia1**], [**DS**], [**Tia4**]). Alternatively, using Păun's Laplacian estimate in [**Pău**] and Evans-Krylov theory, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [**BBEGZ**] showed directly that any weak Kähler-Einstein metric $\omega^w_{\infty,\mathrm{KE}}$ (which is unique up to complex automorphism) on a klt Fano variety X_∞ is smooth on X_∞^{reg}. Hence, S is a subvariety of complex codimension at least 2 and $X_\infty^{\mathrm{reg}} = X_\infty \backslash S$.

It remains to study the structure of $(X_\infty, \omega_{\infty,\mathrm{KE}})$ around S. Compared to the complex dimension 2 case, the second author conjectured that $\omega_{\infty,\mathrm{KE}}$ is a smooth orbifold metric away from an analytic subvariety Z of complex codimension 3. In this short paper, we affirm this conjecture about the regularity of $\omega_{\infty,\mathrm{KE}}$ on the orbifold locus X_∞^{orb} of X_∞.

First, if $(X, -K_X)$ is a klt Fano variety, then we have the following result which says klt spaces have quotient singularities in codimension 2:

Theorem 1.1 ([**GKKP**, Theorem 9.3]). *Let X be a variety with klt singularities. Then there exists a closed subset $Z \subset X$ with $\mathrm{codim}_X Z \geq 3$ such that $X\backslash Z$ has quotient singularities. More precisely every point $x \in X \setminus Z$ has an analytic neighborhood that is biholomorphic to an analytic neighborhood of the origin in a variety of the form $\mathbb{C}^n/G$ where G is a finite subgroup of $GL(n, \mathbb{C})$ that does not contain any quasi-reflections.*

By the above result we just need to show the following regularity result. For the definition of weak Kähler-Einstein metric, see Definition 2.1.

Theorem 1.2. *Assume that ω^w_{KE} is a weak Kähler-Einstein metric on X_∞. Then ω^w_{KE} is a smooth orbifold metric on X_∞^{orb}.*

Our current proof uses the existence of an orbifold resolution, i.e., Theorem 3.3 which is proved by algebraic methods. Theorem 3.3 claims that there is a proper birational morphism $f^{\mathrm{par}} : X^{\mathrm{par}} \to X$ such that X^{par} only has quotient singularity and f^{par} is an isomorphic over X^{orb}. However, we believe that it is not necessary. There should be a purely differential geometric proof of Theorem 1.2 which does not rely on Theorem 3.3. In last section, we will discuss problems on analyzing further structures of singularities of higher codimension. We also believe that our analysis may be used to yield a complete understanding of the singularity for any 3-dimensional weak Kähler-Einstein metrics.

2. Regularity on the orbifold locus

From now on we will denote by X any $\mathbb{Q}$-Fano variety with klt singularities. Assume $\iota : X \to \mathbb{P}^N$ is an embedding given by the linear system $|-mK_X|$ for $m > 0 \in \mathbb{Z}$ sufficiently large and divisible. Let $h_0 = (\iota^* h_{FS})^{1/m}$ be the pull back of the Fubini-Study Hermitian metric h_{FS} on $\mathcal{O}_{\mathbb{P}^N}(1)$ normalized to be a Hermitian metric on $-K_X$. The Chern curvature form of h_0 is

$$\omega_0 = -\sqrt{-1}\partial\bar{\partial}\log h_0$$

which is a positive $(1,1)$-current on X. ω_0 is a smooth positive definite $(1,1)$-form on X^{reg}. However, on the singular locus X^{sing}, ω_0 in general is not canonically related to the local structure of X. Assume $p \in X^{\text{orb}}$ is a quotient singularity. By this, we mean that there exists a small neighborhood $\mathcal{U}_p$ which is isomorphic to a quotient of a smooth manifold by a finite group which acts freely outside a codimension 2 subset. In particular, there exists a branched covering map $\tilde{\mathcal{U}}_p \to \tilde{\mathcal{U}}_p/G \cong \mathcal{U}_p$. The lifting of metric ω_0 to the cover $\tilde{\mathcal{U}}_p$ in general is degenerate.

Now we define an adapted volume for on X by

$$\Omega = |v^*|_{h_0}^{2/m}(\sqrt{-1}^{mn^2} v \wedge \bar{v})^{1/m}.$$

Here v is any local generator of $\mathcal{O}(mK_X)$ and v^* is the dual generator of $\mathcal{O}(-mK_X)$. The Kähler-Einstein equation

$$Ric(\omega_\phi) = \omega_\phi. \tag{2.1}$$

can be transformed into a complex Monge-Ampère equation:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-\phi}\Omega. \tag{2.2}$$

Definition 2.1. A weak solution to the (2.2) is a bounded function $\phi \in L^\infty(X) \cap PSH(X,\omega)$ satisfying (2.2) in the sense of pluripotential theory.

Let's first recall the method to prove the regularity of ϕ on X^{reg} following [**BBEGZ**]. One first chooses a resolution $\pi : \tilde{X} \to X$ with simple normal crossing exceptional divisor $E = \pi^{-1}(X^{\text{sing}})$ such that π is an isomorphism over X^{reg}. Then we can pull back the equation (2.2) to $\tilde{X}$ and get:

$$(\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-\psi}\pi^*\Omega. \tag{2.3}$$

On the other hand we can write:

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i=1}^r a_i E_i - \sum_{j=1}^s b_j F_j,$$

such that $E = \cup_{i=1}^r E_i \bigcup \cup_{j=1}^s F_j$ and $a_i > 0$, $b_j > 0$. The klt property implies: $a_i > 0$, and $0 < b_j < 1$. Analytically, choosing a smooth Kähler metric η on $\tilde{X}$, there exists $f \in C^\infty(\tilde{X})$ such that:

$$\pi^*\Omega = e^f \frac{\prod_{i=1}^r |s_i|^{2a_i}}{\prod_{j=1}^s |\sigma_j|^{2b_j}}\eta^n.$$

where s_i and σ_j are defining sections of E_i for F_j respectively and $|s_i|^2$ and $|\sigma_j|^2$ are some fixed hermitian norms of them. So we have:

$$(\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-\psi+f+\sum_i a_i \log|s_i|^2 - \sum_j b_j \log|\sigma_j|^2}\eta^n = e^{\psi_+ - \psi_-}\eta^n, \tag{2.4}$$

Here we have denoted

$$\psi_+ = f + \sum_i a_i \log|s_i|^2, \quad \psi_- = \psi + \sum_j b_j \log|\sigma_j|^2.$$

It's easy to see that they satisfy the quasi-plurisubharmonic condition:

$$\sqrt{-1}\partial\bar{\partial}\psi_+ \geq -C\eta, \sqrt{-1}\partial\bar{\partial}\psi_- \geq -C\eta, \tag{2.5}$$

for some uniform constant $C>0$. To get Laplacian estimate of ψ away from E, we can first regularize (2.4) to

$$(\omega_\epsilon+\sqrt{-1}\partial\bar{\partial}\psi_\epsilon)^n=e^{\psi_{+,\epsilon}-\psi_{-,\epsilon}}\eta^n. \tag{2.6}$$

where $\omega_\epsilon=\pi^*\omega_0-\epsilon\theta_E$ is a Kähler metric on $\tilde{X}$, and $\psi_{\pm,\epsilon}\in C^\infty(\tilde{X})$ converges to $\psi_\pm$ in $L^\infty(\tilde{X}\backslash E)$ such that exponential $e^{\psi_{\pm,\epsilon}}$ converges to $e^{\psi_\pm}$ in $L^p(\tilde{X})$ for some $p>1$. This is possible thanks Demailly's regularization theorem ([**Dem**]). Then it follows from a result of Kołodziej (see Theorem 2.3) that ψ_ϵ converges to $\psi\in C^0(\tilde{X})$ which is a solution to the degenerate Monge-Ampère equation (2.3).

To get the higher regularity of ψ on $X^{\rm reg}_\infty$, Păun in the work [**Pău**] used the condition (2.5) and modified the Laplacian estimate of Aubin-Yau to prove the Laplacian estimate for the solutions ψ_ϵ away from E (see Theorem 2.4). More precisely, for any compact set $K\Subset\tilde{X}\backslash E$, there exists a constant $A=A(\|\psi\|_\infty,K)$, such that

$$\Delta_\eta\psi_\epsilon\le A(\|\psi\|_\infty,K)e^{-\psi_{-,\epsilon}}.$$

Because ω_ϵ is locally uniformly elliptic on $\tilde{X}\setminus E$, by the complex version of Evan-Krylov's theory (see [**Bło**]), we know that ψ_ϵ is locally uniformly $C^{2,\alpha}$ and hence by bootstrapping, $C^{k,\alpha}$ on $\tilde{X}\backslash E$. More precisely, for any compact set $K\Subset\tilde{X}\backslash E$, there exists a constant $C=C(K,\|\psi\|_\infty,k)$ such that:

$$\|\psi_\epsilon\|_{C^{k,\alpha}(K)}\le C. \tag{2.7}$$

As a consequence, ψ_ϵ converges to ψ in C^k norm locally uniformly away from E, and hence we know that ψ is smooth on $\tilde{X}\backslash E$.

One can also prove the regularity on $X^{\rm reg}$ with the help of Kähler-Ricci flow. Starting from the work in [**CTZ**], this idea has been used several times in the literature to prove the regularity of weak solutions to complex Monge-Ampère equations. Recall that the Kähler-Ricci flow is a solution to the following equation:

$$\begin{cases}\frac{\partial\omega_t}{\partial t}=-Ric(\omega_t)+\omega_t;\\ \omega(0)=\omega_{\phi_0}.\end{cases} \tag{2.8}$$

As in the elliptic case, this equation can be transformed into the following Monge-Ampère flow

$$\begin{cases}\frac{\partial\phi}{\partial t}=\log\frac{(\omega_0+\sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega}+\phi;\\ \phi(0,\cdot)=\phi_0.\end{cases} \tag{2.9}$$

To define a solution to this Monge-Ampère flow on the singular variety X, Song-Tian [**ST**] pulled up the flow equation in (2.9) to $\tilde{X}$ to get:

$$\begin{cases}\frac{\partial\tilde{\phi}}{\partial t}=\log\frac{(\pi^*\omega_0+\sqrt{-1}\partial\bar{\partial}\tilde{\phi})^n}{\pi^*\Omega}+\tilde{\phi};\\ \\ \tilde{\phi}(0,\cdot)=\pi^*\phi_0.\end{cases} \tag{2.10}$$

Theorem 2.2 ([**ST**]). *Let $\phi_0\in PSH_p(X,\omega_0)$ for some $p>1$. Then the Monge-Ampère flow (2.10) on $\tilde{X}\backslash E$ has a unique solution $\tilde{\phi}\in C^\infty((0,T_0)\times\tilde{X}\backslash E)\cap C^0([0,T_0)\times\tilde{X}\backslash E)$ such that for all $t\in[0,T_0)$, $\tilde{\phi}(t,\cdot)\in L^\infty(\tilde{X})\cap PSH(\tilde{X},\pi^*\omega_0)$.*

Since $\tilde{\phi}$ is constant along (connected) fibre of π, $\tilde{\phi}$ descends to a solution $\phi\in C^\infty((0,T_0)\times X^{\rm reg})\cap C^0([0,T_0)\times X^{\rm reg})$ of the Monge-Ampère flow .

Now suppose $\omega^w_{\rm KE}=\omega_0+\sqrt{-1}\partial\bar{\partial}\phi^w_{\rm KE}$ is a weak solution to the equation (2.2). If one can prove that the solution $\phi(t)$ to (2.9) with the initial condition $\phi(0)=\phi^w_{\rm KE}$ is stationary, then it follows from Theorem (2.2) that ω^w_{KE} is smooth on X^{reg}_∞. The idea to prove stationarity in [**CTZ**] is to show that the energy functional is decreasing along the flow solution $\phi(t)$ and to use the uniqueness of weak Kähler-Einstein metrics. These are indeed true in the current case by the work of [**BBEGZ**].

To prove Theorem 1.2, the main observation is that the above arguments can be used to prove the regularity of ω^w_{KE} on X^{orb} as long as one can find a partial resolution by orbifolds: $\pi^{\mathrm{par}}: X^{\mathrm{par}} \to X$. Indeed, by the next section, there exist orbifold (partial) resolutions. If $\pi^{\mathrm{par}}: X^{\mathrm{par}} \to X$ is an orbifold resolution, then we can write:

$$K_{X^{\mathrm{par}}} = (\pi^{\mathrm{par}})^* K_X + \sum_i^r a_i E_i - \sum_{j=1}^s b_j F_j,$$

where $E = \cup_{i=1}^r E_i \bigcup \cup_{j=1}^s F_j$ is now a simple normal crossing divisor within orbifold category (in the sense of Satake [**Sat1**, **Sat2**]). The centers of E_i and F_j on X are contained in the closed subvariety Z of codimension at least 3. The klt property of X again implies $a_i > 0$ and $0 < b_i < 1$. Indeed, a_i (resp. $-b_j$) are just discrepancies of E_i (resp. F_j) on X, which does not depend on the birational morphism π^{par}.

Then the similar arguments as in the proof of regularity of ω^w_{KE} on X^{reg}_∞ carry over to the orbifold setting to prove the orbifold regularity of ω^w_{KE} on X^{orb}. Indeed, the main arguments in proving the regularity on X^{reg}_∞ depend either on the maximum principle on compact Kähler manifolds or the local regularity theory. On the one hand, the maximum principle also works on compact orbifold Kähler manifolds. On the other hand, the local arguments remain true by working on local uniformizing charts. For convenience of the reader, we write down the orbifold version of Kołodziej's result:

THEOREM 2.3 (see [**Koł**, **EGZ**]). *Let M be a compact Kähler orbifold and $\omega_M > 0$ be a closed orbifold smooth $(1,1)$-form. Let $\{f_j\} \subset C^\infty(M)$ be a sequence of orbifold smooth functions on M, such that the following conditions are satisfied:*

(i) $\sup_j \|\exp(f_j)\|_{L^p(M)} < +\infty$ *for some* $p > 1$;

(ii) $\int_M e^{f_j}\omega_M^n = \int_M \omega_M^n$ *for* $j \geq 1$.

Then there exist a constant $C > 0$ such that for each solution φ_j of the equation

$$(\omega_M + \sqrt{-1}\partial\bar\partial\varphi_j)^n = e^{f_j}\omega_M^n \tag{2.11}$$

such that $\int_M \varphi_j \omega_M^n = 0$ we have $\sup_M |\varphi_j| \leq C$. Moreover, if $\exp(f_j) \to \exp(f_\infty)$ in L^p, then there exist a continuous function φ_∞ such that $\varphi_j \to \varphi_\infty$ in $C^0(M)$ and φ_∞ is a solution of the equation:

$$(\omega_M + \sqrt{-1}\partial\bar\partial\varphi_\infty)^n = e^{f_\infty}\omega_M^n. \tag{2.12}$$

Again one can prove the above result by following the same proof of Kołodziej, which depends on local pluripotential theory. For example the key lemma [**Koł**, Lemma 2.3.1] for plurisubharmonic functions on domains of $\mathbb{C}^n$ can be proved on domains of the form $U/G \subset \mathbb{C}^n/G$ by arguing on the local uniformization charts $U \to U/G$. Similar remarks apply to the following orbifold version of Păun's theorem:

THEOREM 2.4 (see [**Pău**, **BBEGZ**]). *Let M be an orbifold and $\omega_M \geq 0$ be a semi-positive orbifold smooth $(1,1)$-form. Let μ be a positive measure on an orbifold M of the form $\mu = e^{\psi^+ - \psi^-} dV$ with $\psi^\pm$ quasi-psh with respect to M and $e^{-\psi^-} \in L^p$ for some $p > 1$. Assume φ is a bounded ω_M-psh function such that*

$(\omega_M + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \mu$. Then we have $\Delta\varphi = O(e^{-\psi^-})$ locally in the ample locus where $\omega_M > 0$.

Note that it was already observed in [**ST**, Section 4.3] (via the same reasoning as above) that if X has only orbifold singularities, then the Kähler-Ricci flow smooths out initial metric to become an *orbifold* smooth metric immediately when $t > 0$.

3. Orbifold partial resolution

The results in this section were communicated to us by Chenyang Xu.

LEMMA 3.1 (Resolution of Deligne-Mumford stacks). Let $\mathcal{X}$ be an integral Deligne-Mumford stack which is of finite type over $\mathbb{C}$. Then there exists a birational proper representable morphism $g^{\mathrm{sm}} : \mathcal{X}^{\mathrm{sm}} \to \mathcal{X}$ from a smooth Deligne-Mumford stack $\mathcal{X}^{\mathrm{sm}}$. Furthermore, we can assume that g^{sm} is isomorphic over the smooth locus of $\mathcal{X}$, and the exceptional locus of g^{sm} is a normal crossing divisorial closed substacks of $\mathcal{X}^{\mathrm{sm}}$.

PROOF. This follows from the functoriality property of resolution of singularities (see [**Wło**], [**Kol**], [**BM**],[**Tem**]). Indeed, following the argument by Temkin in [**Tem**, Theorem 5.1], we first cover $\mathcal{X}$ with chart $U_\alpha \to U_\alpha/G_\alpha = V_\alpha$. Then we can resolve the singularity of U_α by the work of Hironaka. This means that there exists a birational morphism $\pi_\alpha : \tilde{U}_\alpha \to U_\alpha$ which is a composition of a sequence of blow-ups with regular centers such that $\tilde{U}_\alpha$ is regular and π_α is an isomorphism over the regular locs U_α^{reg} of U_α. It has been proved in [**BM**] that the resolution algorithm can be made functorial with respect to regular morphisms. This allows us to show that π_U is equivariant under the G_α action. Denote $\tilde{V}_\alpha = \tilde{U}_\alpha/G_\alpha$. Then the functoriality also implies $\tilde{V}_\alpha$ can be glued to become a smooth Deligne-Mumford stack $\mathcal{X}^{\mathrm{sm}}$. □

LEMMA 3.2 (Blow up the indeterminacy locus). Let X be a projective scheme. Let $\mathcal{X}$ be a normal Deligne-Mumford stack with a dense open set $i_U : U \hookrightarrow \mathcal{X}$, such that U admits a morphism $f_U : U \to X$. Then we can blow up an ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ to obtain a Deligne-Mumford stack $\tilde{\mathcal{X}}$ such that $\tilde{\mathcal{X}} \to \mathcal{X}$ is isomorphic over U and f_U extends to a morphism $f : \tilde{\mathcal{X}} \to X$.

PROOF. By the above lemma, we can assume $\mathcal{X}$ to be smooth. Because X is projective, we can replace X by $\mathbb{P}^N$. Let $\{H_i\}_{i=1}^{N+1}$ be the hyperplane section of $\mathbb{P}^N$. Let $D_i \subset U$ be the pull back of H_i by f_U and let $\bar{D}_i$ be the closure of D_i in $\mathcal{X}$. Let $\mathcal{I}$ be the ideal sheaf of $\cap_{i=1}^{N+1} D_i$. Then $\mathcal{I}$ is supported on $\mathcal{X} \setminus \mathcal{X}^\circ$. By using the same proof for the schemes as in [**Har**, II.7.17.3], $\mathrm{Bl}_{\mathcal{I}}\mathcal{X}$ satisfies the property in the statement. □

THEOREM 3.3. Let X be a quasi-projective normal variety. Let X^{orb} be the locus where X only has orbifold singularity. Then there exists $f^{\mathrm{par}} : X^{\mathrm{par}} \to X$ a proper birational morphism, such that X^{par} only has quotient singularity and f^{par} is an isomorphic over X^{orb}.

PROOF. After taking the closure of $X \subset \mathbb{P}^N$, we can assume X is projective.

By [**Vis**, 2.8], we know there is a smooth Deligne-Mumford stack $\mathcal{X}^0$ whose coarse moduli space is X^{orb}. It follows from [**Kre**, Theorem 4.4] that $\mathcal{X}^0 = [Z/G]$ for some quasi-projective scheme Z and linear algebraic group G. Actually, Z can be taken as the frame bundle of X^{orb} and $G = GL_n(\mathbb{C})$. Then by [**Kre**, Theorem

5.3], there is a proper Deligne-Mumford stack $\mathcal{X}$, such that $\mathcal{X}^0 \subset \mathcal{X}$ is a dense open set. We explain this by following the proof of [**Kre**]: for some N there is a linear action of G on $\mathbb{P}^N$ and an equivariant embedding $Z \to \mathbb{P}^N$ such that the embedding factors through the stable locus $(\mathbb{P}^N)^s$. By Kirwan's blow-up construction, there is a birational morphism $\mu : W \to \mathbb{P}^N$ which is a composition of blow-ups of nonsingular G-invariant subvarieties, such that μ is isomorphism over $(\mathbb{P}^N)^s$ and $W^s = W^{ss}$. Let $V \subset W$ be the closure of the strict transform of Z under μ. Then $\mathcal{X} = [V/G]$ satisfies the condition.

Consider the rational map $f : \mathcal{X} \dashrightarrow X$, by Lemma 3.2 we know that there is a blow up $\mathcal{Y} \to \mathcal{X}$ along the indeterminacy locus of f, such that there is a morphism $g : \mathcal{Y} \to X$. Moreover, by the construction, we know over X^{orb},

$$\mathcal{Y}^0 := g^{-1}(X^{\text{orb}}) \cong \mathcal{X}^0.$$

By Lemma 3.1, we know that there is a smooth Deligne-Mumford stack $h : \mathcal{Y}^{\text{sm}} \to \mathcal{Y}$, where h is a representable proper birational morphism which is isomorphic over the smooth locus of $\mathcal{Y}$. In particular, h is isomorphic over $\mathcal{Y}^0$.

As $\mathcal{X}$ has finite stabilizer and $\mathcal{Y}^{\text{sm}} \to \mathcal{Y} \to \mathcal{X}$ is proper, we know that $\mathcal{Y}^{\text{sm}}$ has also finite stabilizer. Thus it follows from [**KeM**] that $\mathcal{Y}^{\text{sm}}$ admits a coarse moduli space, which we denote by X^{par}. It has a morphism $f^{\text{par}} : X^{\text{par}} \to X$ by the universal property. We can then easily check that they satisfy all the properties. □

4. Further discussions

In this section, we discuss some possible extensions of our theorem and open questions. Assume that (X_i, ω_i) is a sequence of Kähler-Einstein metrics satisfying:
(1) $Ric(\omega_i) = \lambda\omega_i$, where $\lambda = -1$,0 or 1;
(2) There is a $v > 0$ such that $Vol(B_1(x_i, \omega_i)) \geq v$, where $x_i \in X_i$.

By the Gromov-Hausdorff compactness, by taking a subsequence if necessary, we may assume that the sequence of pointed spaces (X_i, ω_i, x_i) converges to a length space $(X_\infty, d_\infty, x_\infty)$ in the Gromov-Hausdorff topology. By Cheeger-Colding's theory and Cheeger-Colding-Tian's theory, X_∞ is smooth outside a closed subset S of codimension at least 4 and d_∞ is given by a Kähler-Einstein metric ω_∞ on $X_\infty \backslash S$. Moreover, (X_i, ω_i, x_i) converges to $(X_\infty, \omega_\infty, x_\infty)$ in the Cheeger-Gromov topology, and in particular, in the C^∞-topology outside S. We expect that X_∞ is a Kähler variety whose singular set S is a subvariety of complex codimension at least 2 and $(X_\infty, \omega_\infty)$ is a Kähler-Einstein orbifold outside a subvariety $Z \subset S$ of complex codimension at least 3. Theorem 1.2 confirms this in the case of Fano manifolds with Kähler-Einstein metrics. More generally, we have:

THEOREM 4.1. With the same assumption as above, if $(X_\infty, \omega_\infty)$ has finite diameter and $m\,[\omega_i] \in H^2(X_i, \mathbb{Z})$ is uniformly bounded for some fixed $m \in \mathbb{Z}$, then X_∞ is a normal projective variety and ω_∞ is a positive closed $(1,1)$-current which is a smooth Kähler-Einstein orbifold metric outside a subvariety Z of complex codimension at least 3.

PROOF. As mentioned in the introduction, the first statement follows from partial C^0-estimate (see [**Tia2**, **Tia3**], [**Li**] and [**DS**]). The second statement is obtained from the same proof as that of Theorem 1.2. □

The proof of this theorem as well as Theorem 1.2, relies on a global partial resolution X^{par} of X_∞ which resolves all the non-quotient singularities. It is desirable to have a proof which does not use such a partial resolution and works locally. More precisely, let U be an open subset in an affine subvariety $Y \subset \mathbb{C}^N$ of the form $\tilde{U}/\Gamma$, where $\tilde{U}$ is open subset in $\mathbb{C}^n$ and Γ is a finite group acting on $\mathbb{C}^n$ holomorphically. Assume that ω is a weak Kähler-Einstein metric on U, that is, it satisfies: (1) ω is a smooth Kähler-Einstein metric on the regular part of U; (2) Near each $x \in U$, it can be written as $\sqrt{-1}\,\partial\bar{\partial}\varphi_x$ in the sense of currents for some bounded function φ_x; (3) $\omega \geq \omega_0|_U$ for some smooth Kähler metric on $\mathbb{C}^N$. Then we expect that $\pi^*\omega$ extends to be a smooth metric on $\tilde{U}$, where $\pi : \tilde{U} \mapsto U$ is the natural projection. If Γ is trivial, it is true as explained in the introduction: The condition (3) implies that there is a uniform C^2-estimate of the potential of ω on any compact subset of U; next, the Evans-Krylov theory or Calabi's 3rd derivative estimate allows one to show that ω is smooth on U.

It should be possible to understand singularities of weak Kähler-Einstein metrics in higher codimensions in an inductive way. It is proved in [**Tia1**] that the only singularities for 2-dimensional weak Kähler-Einstein metrics are quotient singularities. The next case is dimension 3. One should be able to have a complete understanding of singularities for weak Kähler-Einstein metrics in dimension 3. Here is a heuristic reasoning: Let $(X_\infty, \omega_\infty)$ be a limit of 3-dimensional compact Kähler-Einstein metrics (X_i, ω_i) in the Cheeger-Gromov topology. Then any tangent cone C_x at $x \in X_\infty$ is a complex cone over some complex 2-dimensional Fano variety Z (see [**DS**]) and Z is a Kähler-Einstein orbifold. It is believed that the structure of X_∞ near x should be modeled on this tangent cone, so we may be able to analyze the structure of X_∞ near x. Moreover, if $x_i \in X_i$ converge to x, then we may also decode information on X_i near x_i.

Another possible extension of Theorem 1.2 is about conic Kähler-Einstein metrics. For simplicity, we consider only the following situation. The general cases are similar. Assume that M is a Fano manifold, $D \subset M$ is a pluri-anti-canonical divisor and ω_i are conic Kähler-Einstein metrics on M with angles $2\pi\beta$ along D, where $\beta \in (0,1)$. We further assume that $(M_\infty, \omega_\infty)$ is the limit of (M, ω_i) in the Cheeger-Gromov topology and D_∞ is the limit of D. In view of [**Tia4**] and [**CDS**], we know that M_∞ is a normal variety and D_∞ is a divisor in M_∞. Furthermore, if we write

$$D_\infty = \sum_{i=1}^{k} m_i\, D_{\infty,i},$$

where each $D_{\infty,i}$ is irreducible, then ω_∞ is a weak conic Kähler-Einstein metric with conic angles $2\pi\beta_i$ along each $D_{\infty,i}$. Here β_i is given by the equality

$$m_i\,(1-\beta) \,=\, (1-\beta_i).$$

If $k = 1$ and $m_1 = 1$, then it is not hard to show that ω_∞ is a conic Kähler-Einstein metric near any point of $D_\infty^{\mathrm{reg}} \cap M_\infty^{\mathrm{reg}}$, where D_∞^{reg} denotes the regular part of D_∞ and M_∞^{reg} denotes the regular part of M_∞. In particular, if M_∞ is smooth and D_∞ is a smooth divisor, ω_∞ is a genuine conic metric. In general, we expect a similar structure. More precisely, we conjecture that there is a subvariety Z of complex codimension at least 2 such that the restriction of ω_∞ to $M_\infty \backslash Z$ is a smooth conic Kähler-Einstein metric with angle $2\pi\beta_i$ along each $D_{\infty,i}$. It is possible that our arguments for proving Theorem 1.2 can be adapted to prove this conjecture by using the conic Kähler-Ricci flow.

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