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Analysis of the Laplacian on the moduli space of polarized Calabi-Yau manifolds

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Abstract. In this paper, we generalize the spectrum relation in the paper *On the spectrum of the Laplacian, Math. Ann., 359(1-2):211–238, 2014* (by Nelia Charalambous and Zhiqin Lu) to any Hermitian manifolds. We also prove that the closure of Laplace operator $\Box = \delta d$ on the moduli space of polarized Calabi-Yau manifolds is self-adjoint.

1. Introduction

Let (M, g) be a Hermitian manifold with a holomorphic vector bundle (E, h). Suppose $\Box$ is the Hodge Laplacian on smooth E-valued (p, q) forms. Though $\Box$ in general is only symmetric but not self-adjoint, one can consider self-adjoint extensions of the Hodge Laplacian. One well-known self-adjoint extension is the so-called Gaffney extension $\Box_G$ ([**5**]). In this note, we generalize the spectrum relations in [**2**] to the Gaffney extension on incomplete manifolds. One key ingredient for the spectrum relations is a generalized version of the Weyl's criterion.

Another well-know extension of $\Box$ is the Friedrichs extension $\Box_F$. $\Box_G$ and $\Box_F$ are in general different on incomplete manifolds. In the special case of the moduli space of polarized Calabi-Yau manifolds $\mathcal{M}$ with the Weil-Petersson metric ω_{WF}, we prove the Cauchy boundary of $\mathcal{M}$ has zero capacity, and therefore $\Box_G = \Box_F$ on functions. Furthermore, we also show that the Hodge Laplacian on functions with certain $Dom\Box$ is essentially self-adjoint, which is a generalization of the results in [**6**] and [**10**].

Using the spectrum results we obtain on different self-adjoint extensions of the Laplacians, we study the L^2-estimates on incomplete manifolds. The L^2-estimate played one of the most crucial roles in several complex variables and complex geometry. The method allows us to construct a lot of holomorphic functions and holomorphic sections in various function spaces.

One of the most important applications of the L^2-estimate is the proof of Kodaira's embedding theorem. Let L be a positive line bundle over a compact complex manifold X. Then there exists a positive integer k such that the line bundle $L^k = L \otimes \cdots \otimes L$ has a lot of (ample) holomorphic sections.

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In this paper, we study the case when X is *not* a complete complex manifold. As it is well known, on a incomplete manifold, the extension of the Laplacian as a self-adjoint operator is not unique. So we need to specify the extension. Secondly, the L^2 estimates heavily depends on the spectrum gap on the bundle-valued $(0,1)$ forms. Therefore, it is useful to generalize the results in [**2**] to the incomplete case.

The main result of this paper is in §5, where we re-prove the results of Masamune [**10, 11**]. We found a gap in his proof and we showed this gap by a counter-example in §8.

In §7 and §8, we apply our results to the moduli space of Calabi-Yau manifolds.

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2. Two Self-Adjoint Extensions of Hodge Laplacian

In this section, we assume (M,g) is a Hermitian manifold with a holomorphic Hermitian vector bundle (E,h). Consider Hodge Laplacian on E-valued (p,q) forms with compact support. As the Hodge Laplacian is symmetric but not self-adjoint, we consider the self-adjoint extensions of the Hodge Laplacian via the corresponding closed quadratic forms. By endowing the quadratic form with different domain of definition, we will get two important self-adjoint extensions, which are respectively Gaffney extension and Friedrichs extension. For more details about this section, we recommend references [**9, 15**].

We begin with the d-bar differential operator

$$\overline{\partial}_{p,q} : L^2(M, \Lambda^{p,q}(E)) \to L^2(M, \Lambda^{p,q+1}(E)),$$

with

$$Dom(\overline{\partial}_{p,q}) = \{\varphi \in L^2(M, \Lambda^{p,q}(E)) : \text{the distributional derivative } \overline{\partial}\varphi \in L^2(M, \Lambda^{p,q+1}(E))\}.$$

With the above domain of definition, the operator $\overline{\partial}_{p,q}$ is a densely defined closed operator. We denote the L^2 inner product on $L^2(M, \Lambda^{p,q}(E))$ as $(\cdot,\cdot)_{p,q}$. With respect to the L^2 inner product on $L^2(M, \Lambda^{p,q}(E))$ and $L^2(M, \Lambda^{p,q+1}(E))$, we have the adjoint operator of $\overline{\partial}_{p,q}$ as

$$\bar{\partial}^*_{p,q+1} : L^2(M, \Lambda^{p,q+1}(E)) \to L^2(M, \Lambda^{p,q}(E)),$$

with

$$Dom(\bar{\partial}^*_{p,q+1}) = \{\phi \in L^2(M, \Lambda^{p,q+1}(E)) : \exists\, \varphi \in L^2(M, \Lambda^{p,q}(E)) \text{ such that } (\overline{\partial}u, \phi)_{p,q+1} = (u, \varphi)_{p,q} \text{ for any } u \in Dom(\overline{\partial}_{p,q})\}.$$

And in the above notation, $\bar{\partial}^*\phi$ is defined to be φ.

In the following, we will suppress the indices p,q in the operators and inner product for simplicity when there is no confusion from context.

Now let us recall Hodge Laplacian and the associated quadratic form. We use the notation $\mathcal{D}(M, \Lambda^{p,q}(E))$ to denote the set of all smooth E-valued (p,q) forms with compact support.

DEFINITION 2.1. i) Let $\square : \mathcal{D}(M, \Lambda^{p,q}(E)) \to \mathcal{D}(M, \Lambda^{p,q}(E))$ be the Hodge Laplacian defined as

$$\square = \overline{\partial}\bar{\partial}^* + \bar{\partial}^*\overline{\partial}.$$

ii) Let $Q : \mathcal{D}(M, \Lambda^{p,q}(E)) \times \mathcal{D}(M, \Lambda^{p,q}(E)) \to \mathbb{C}$ be the quadratic form associated to $\square$ defined as

$$Q(\varphi, \phi) = (\overline{\partial}\varphi, \overline{\partial}\phi) + (\bar{\partial}^*\varphi, \bar{\partial}^*\phi) \text{ for any } \varphi, \phi \in \mathcal{D}(M, \Lambda^{p,q}(E)).$$

Since $\overline{\partial}, \bar{\partial}^*$ are closed operators, if we endow quadratic form Q with $Dom(Q) = Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*)$, then Q is closed. That means, for any sequence $\varphi_n \in Dom(Q)$, $\varphi_n \xrightarrow{L^2} \varphi$ and $Q(\varphi_m - \varphi_n, \varphi_m - \varphi_n) \to 0$ as $m, n \to \infty$, then $\varphi \in Dom(Q)$ and $Q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$.

We cite the following theorem from [**14**] in Chapter VIII.6.

THEOREM 2.1 ([**14**]). *If Q is a closed semibounded quadratic form, then Q is the quadratic form of a unique self-adjoint operator.*

By applying this theorem to our quadratic form Q with $Dom(Q) = Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) \subset L^2(M, \Lambda^{p,q}(E))$, we get a self-adjoint extension of $\square$, which is called Gaffney extension and denoted as $\square_G$. The domain of $\square_G$ is

$$\begin{aligned} Dom(\square_G) = \{\varphi \in Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) : \exists \eta \in L^2(M, \Lambda^{p,q}(E)) \text{ such that} \\ Q(\varphi, \phi) = (\eta, \phi) \text{ for any } \phi \in Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*)\}. \end{aligned} \tag{2.1}$$

And in the same notation as above, $\square_G\varphi$ is defined to be η.

The following Gaffney's Theorem from [**5**] (See also chapter 3 in [**9**]) tells us that Gaffney extension can be viewed as the composition of $\overline{\partial}$ and $\bar{\partial}^*$ as follows.

THEOREM 2.2 (Gaffney).

$$Dom(\square_G) = \{\varphi \in Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) : \overline{\partial}\varphi \in Dom(\bar{\partial}^*) \text{ and } \bar{\partial}^*\varphi \in Dom(\overline{\partial})\}. \tag{2.2}$$

And for any $\varphi \in Dom(\square_G)$, we have

$$\square_G\varphi = \overline{\partial}\bar{\partial}^*\varphi + \bar{\partial}^*\overline{\partial}\varphi.$$

Similarly, we will introduce Friedriechs extension by endowing Q with a different domain of definition. Let's first recall the following Sobolev spaces. We denote $Q_1(\cdot,\cdot) = Q(\cdot,\cdot) + (\cdot,\cdot)$. It is not hard to see Q_1 is an inner product on $\mathcal{D}(M, \Lambda^{p,q}(E))$.

DEFINITION 2.2 (Sobolev Spaces).

$$\begin{aligned} W_0^1(M, \Lambda^{p,q}(E) =& \text{Completion of } \mathcal{D}(M, \Lambda^{p,q}(E)) \\ & \text{with respect to } Q_1 \text{ inner product,} \end{aligned} \tag{2.3}$$

$$\begin{aligned} W^1(M, \Lambda^{p,q}(E)) =& \text{Completion of } \{\varphi \in \mathcal{C}^\infty(M, \Lambda^{p,q}(E)) : Q_1(\varphi, \varphi) < \infty\} \\ & \text{with respect to } Q_1 \text{ inner product.} \end{aligned} \tag{2.4}$$

REMARK 2.3. Note that φ is not necessarily in $Dom(\bar{\partial}^*_{p,q})$ when $\varphi \in \mathcal{C}^\infty(M, \Lambda^{p,q}(E))$. So in the definition of $W^1(M, \Lambda^{p,q}(E))$, to be precise, $Q_1(\varphi, \varphi) < \infty$ means $\varphi \in L^2(M, \Lambda^{p,q}(E))$ and the point-wise differentials $\overline{\partial}\varphi, \bar{\partial}^*\varphi$ belong to $L^2(M, \Lambda^{p,q+1}(E))$ and $L^2(M, \Lambda^{p,q-1}(E))$ respectively. And one can prove $\varphi \in$

$W^1(M, \Lambda^{p,q}(E))$ if and only if $\varphi \in L^2(M, \Lambda^{p,q}(E))$ and the distributional differentials $\overline{\partial}\varphi, \bar{\partial}^*\varphi$ belong to $L^2(M, \Lambda^{p,q+1}(E))$ and $L^2(M, \Lambda^{p,q-1}(E))$ respectively.

REMARK 2.4. Note that $W_0^1 \subset Dom(\overline{\partial}) \cap Dom(\bar{\partial}^*) \subset W^1$. But they are generally not equal to each other.

If we endow Q with $Dom(Q) = W_0^1$, then it becomes a closed quadratic form. By applying Theorem 2.1 again, we will get a different self-adjoint extension of Hodge Laplacian $\Box$, which is called Friedrichs extension and denoted as $\Box_F$. Note that $\Box_F$ is generally different from $\Box_G$ by Remark 2.4.

EXAMPLE 2.5. Take the Hermitian manifold $M = \Omega \subset \mathbb{C}^n$ be a bounded open set with smooth boundary. Let Hermitian vector bundle E be the trivial line bundle. Assume $u \in \mathcal{C}^\infty(\overline{\Omega}, \Lambda^{p,q})$. Let us investigate the boundary conditions induced from $\Box_G$ and $\Box_F$ in this case.

If $u \in Dom(\bar{\partial}^*)$, then

$$(\overline{\partial}\varphi, u) = (\varphi, \bar{\partial}^* u) \text{ for any } \varphi \in \mathcal{C}^\infty(\overline{\Omega}, \Lambda^{p,q-1}).$$

Note

$$(\overline{\partial}\varphi, u) = \int_\Omega \overline{\partial}\varphi \wedge *\overline{u} = \int_{\partial\Omega} \varphi \wedge *\overline{u} + (-1)^{p+q} \int_\Omega \varphi \wedge \overline{\partial} * \overline{u} = \int_{\partial\Omega} \varphi \wedge *\overline{u} + (\varphi, \bar{\partial}^* u).$$

Here $*$ is the Hodge star operator. The second equality follows from Stokes Theorem and the last one is based on the identity $\bar{\partial}^* = - * \partial *$. Therefore we have

$$\int_{\partial\Omega} \varphi \wedge *\overline{u} = 0 \text{ for any } \varphi \in \mathcal{C}^\infty(\overline{\Omega}, \Lambda^{p,q-1}).$$

It implies $*u|_{\partial\Omega} = 0$(the restriction of $*u$ to $\partial\Omega$). So by Theorem 2.2, $u \in Dom(\Box_G)$ implies the boundary condition $*u|_{\partial\Omega} = 0$ and $*\overline{\partial}u|_{\partial\Omega} = 0$.

For the Friedrichs extension, $u \in Dom(\Box_F)$ implies $u \in W_0^1$. Then there exists a sequence $u_j \in \mathcal{D}(M, \Lambda^{p,q})$ such that $u_n \to u$ in W_0^1. By Weitzenböck formula, we have

$$\Box u_j = -\sum_{i=1}^n \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^{\bar{i}}} u_j = -\sum_{i=1}^n \frac{\partial}{\partial z^{\bar{i}}} \frac{\partial}{\partial z^i} u_j. \tag{2.5}$$

Therefore by taking the inner product with u_j,

$$Q(u_j, u_j) = \sum_{i=1}^n (\frac{\partial}{\partial z^{\bar{i}}} u_j, \frac{\partial}{\partial z^{\bar{i}}} u_j) = \sum_{i=1}^n (\frac{\partial}{\partial z^i} u_j, \frac{\partial}{\partial z^i} u_j). \tag{2.6}$$

Then we have

$$\frac{\partial}{\partial z^{\bar{i}}} u_j \to \frac{\partial}{\partial z^{\bar{i}}} u \quad \text{and} \quad \frac{\partial}{\partial z^i} u_j \to \frac{\partial}{\partial z^i} u \quad \text{in } L^2 \text{ norms for each } i. \tag{2.7}$$

If we write $u = u_{I\overline{J}} dz^I \wedge dz^{\overline{J}}$, then each function $u_{I\overline{J}}$ is in the standard Sobolev space $H_0^1(\Omega)$, which implies $u_{I\overline{J}}|_{\partial\Omega} = 0$ for each multi-index I, J.

3. Spectrums of Gaffney Extension

The main goal of this section is to prove the following spectrum relations of Gaffney extension.

THEOREM 3.1. *Let (M, g) be a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h). Consider Gaffney extension of Hodge Laplacian, $\Box_{p,q}$: $L^2(M, \Lambda^{p,q}(E)) \to L^2(M, \Lambda^{p,q}(E))$. We have the following spectrum relations.*

$$\mathit{Spec}(\Box_{p,q}) \cup \{0\} = \mathit{Spec}(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup \mathit{Spec}(\bar{\partial}^*\overline{\partial}_{p,q-1}) \cup \{0\}. \tag{3.1}$$

$$\mathit{Spec}(\Box_{p,q}) \cup \{0\} = \mathit{Spec}(\overline{\partial}\bar{\partial}^*_{p,q}) \cup \mathit{Spec}(\bar{\partial}^*\overline{\partial}_{p,q}) \cup \{0\}. \tag{3.2}$$

REMARK 3.1. The above notation $\overline{\partial}\bar{\partial}^*_{p,q}$ means $\overline{\partial}_{p,q-1}\bar{\partial}^*_{p,q}$ and $\bar{\partial}^*\overline{\partial}_{p,q}$ means $\bar{\partial}^*_{p,q+1}\overline{\partial}_{p,q}$. Note that $\overline{\partial}\bar{\partial}^*_{p,q}$ and $\bar{\partial}^*\overline{\partial}_{p,q}$ are self-adjoint operators by Von Neumann's Theorem (see Chapter X in [**13**]) since both $\overline{\partial}_{p,q}$ and $\bar{\partial}^*_{p,q}$ are densely defined closed operators. In the following we will omit the sub-indices p, q when there is not confusion from context.

This is a generalization of results in [**2**], where similar spectrum relations were proved for complete Riemannian manifolds. One main tool we are going to use is the generalized Weyl criterion from [**2**]. The advantage of this generalized Weyl criterion is that we do not necessarily pick the test sequence from the domain of an unbounded operator. After proving it, we will mention a well known relation between Gaffney extension and L^2 estimates, which serves a preparation for later sections.

We will split the proof of Theorem 3.1 into to several Lemmas. First, we prove one containment relation of (3.1).

LEMMA 3.2. *Under the same assumption as Theorem* 3.1*, we have*

$$\mathit{Spec}(\Box_{p,q}) \subset \mathit{Spec}(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup \mathit{Spec}(\bar{\partial}^*\overline{\partial}_{p,q-1}) \cup \{0\}. \tag{3.3}$$

PROOF. In this proof, we will use $\Box$ to represent $\Box_{p,q}$ for simplicity. Take $\lambda_0 \in$ Spec($\Box$) and $\lambda_0 > 0$. By Weyl's criterion, there exists a sequence $u_j \in \mathit{Dom}(\Box)$ with $(u_j, u_j) = 1$ such that

$$(\Box - \lambda_0)u_j \to 0 \text{ as } j \to \infty.$$

Since $\Box$ is non-negative and self-adjoint, $(1+\Box)^{-1} : L^2(M, \Lambda^{p,q}(E)) \to \mathit{Dom}(\Box_{p,q}) \subset L^2(M, \Lambda^{p,q}(E))$ is a bounded operator. By identity (2.1), we have

$$Q((1+\Box)^{-2}u_j, (1+\Box)^{-2}u_j) = (\Box(1+\Box)^{-2}u_j, (1+\Box)^{-2}u_j). \tag{3.4}$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of $\Box$. Then

$$(\Box(1+\Box)^{-2}u_j, (1+\Box)^{-2}u_j) = \int_0^\infty \frac{\lambda}{(1+\lambda)^4} d(P_\lambda u_j, u_j). \tag{3.5}$$

Take $C(\lambda_0) = \min\limits_{\lambda \in [\frac{\lambda_0}{2}, \frac{3\lambda_0}{2}]} \frac{\lambda}{(1+\lambda)^4} > 0$. Then

(3.6)
$$\int_0^\infty \frac{\lambda}{(1+\lambda)^4} d(P_\lambda u_j, u_j) \geq C(\lambda_0) \int_{\frac{1}{2}\lambda_0}^{\frac{3}{2}\lambda_0} d(P_\lambda u_j, u_j) \geq C(\lambda_0) \|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)} u_j\|^2.$$

We denote $u_j^{(1)} = P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)} u_j$ and $u_j^{(2)} = u_j - u_j^{(1)}$. By using the Projection Valued Measure again, we have

$$((\Box - \lambda_0)u_j, (\Box - \lambda_0)u_j) = \int_0^\infty (\lambda - \lambda_0)^2 d(P_\lambda u_j, u_j) \geq \frac{\lambda_0^2}{4} \|u_j^{(2)}\|^2.$$

Since we know $(\Box - \lambda_0)u_j \to 0$ as j goes to infinity, we have

$$\|u_j^{(2)}\| \to 0 \text{ as } j \to \infty,$$

whence

$$\|u_j^{(1)}\| \to 1 \text{ as } j \to \infty.$$

Together with (3.4), (3.5) and (3.6), we have for sufficiently large j

$$\|\overline{\partial}(1+\Box)^{-2}u_j\|^2 + \|\overline{\partial}^*(1+\Box)^{-2}u_j\|^2 \geq \frac{C(\lambda_0)}{2} > 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} &\|(\overline{\partial}\bar{\partial}^* - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 + \|(\bar{\partial}^*\overline{\partial} - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 \\ =&\|\overline{\partial}(\Box - \lambda_0)(1+\Box)^{-2}u_j\|^2 + \|\bar{\partial}^*(\Box - \lambda_0)(1+\Box)^{-2}u_j\|^2 \\ =&\big(\Box(1+\Box)^{-2}(\Box - \lambda_0)u_j, (1+\Box)^{-2}(\Box - \lambda_0)u_j\big) \\ \leq&\|(\Box - \lambda_0)u_j\|^2. \end{aligned}$$

The first equality is because $\overline{\partial}\circ\overline{\partial} = 0$ on $Dom(\overline{\partial})$ and $\bar{\partial}^*\circ\bar{\partial}^* = 0$ on $Dom(\bar{\partial}^*)$. The second one follows from (2.1) and the commutativity of $\Box$ and $(1+\Box)^{-1}$. And the last inequality follows from $\|(1+\Box)^{-1}\|_{L^2\to L^2} \leq 1$ and $\|\Box(1+\Box)^{-1}\|_{L^2\to L^2} \leq 1$. Therefore

$$\|(\overline{\partial}\bar{\partial}^* - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 + \|(\bar{\partial}^*\overline{\partial} - \lambda_0)\overline{\partial}(1+\Box)^{-2}u_j\|^2 \to 0. \tag{3.8}$$

Combining (3.7) and (3.8), we have $\lambda_0 \in \mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup \mathrm{Spec}(\bar{\partial}^*\overline{\partial}_{p,q-1})$ by Weyl criterion. So the result follows. $\Box$

Now we prove the other containment of (3.1).

LEMMA 3.3. *Under the same assumption as Theorem* 3.1*, we have*

$$Spec(\overline{\partial}\bar{\partial}^*_{p,q+1}) \cup Spec(\bar{\partial}^*\overline{\partial}_{p,q-1}) \subset Spec(\Box_{p,q}) \cup \{0\}.$$

In order to prove this lemma, we will use one generalized Weyl criterion from [**2**].

THEOREM 3.2 (Charalambous-Lu). *Let H be a non-negative self-adjoint operator on Hilbert space $\mathcal{H}$. A positive real number λ_0 is contained in Spec(H) if there exists a sequence $u_j \in \mathcal{H}$ such that*

(1) *For any j, $\|u_j\| = 1$.*
(2) *$((H-\lambda_0)(1+H)^{-m}u_j, u_j) \to 0$ for $m = 1, 2$.*

Note that compared to the classical Weyl criterion, the above theorem does not require $u_j \in Dom(H)$. We give a proof of this theorem here for the completeness.

PROOF. Note that

$$(H-\lambda_0)^2(1+H)^{-2} = (H-\lambda_0)(1+H)^{-1} - (\lambda_0+1)(H-\lambda_0)(1+H)^{-2}.$$

The assumptions imply that

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) \to 0. \tag{3.9}$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of H. Then

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) = \int_0^\infty \frac{(\lambda-\lambda_0)^2}{(1+\lambda)^2} d(P_\lambda u_j, u_j). \tag{3.10}$$

Define $u_j^{(1)} = P_{(\lambda_0-\varepsilon_j,\lambda_0+\varepsilon_j)}u_j$ and $u_j^{(2)} = u_j - u_j^{(1)}$. The constants $\varepsilon_j \in (0, \frac{\lambda_0}{2})$ are to be selected later. Note the integrand $\frac{(\lambda-\lambda_0)^2}{(1+\lambda)^2}$ in (3.10) has the following lower bound for $\lambda \notin (\lambda_0 - \varepsilon_j, \lambda_0 + \varepsilon_j)$.

$$\frac{(\lambda-\lambda_0)^2}{(1+\lambda)^2} \geq \min\left(\frac{\varepsilon_j^2}{(1+\lambda_0-\varepsilon_j)^2}, \frac{\varepsilon_j^2}{(1+\lambda_0+\varepsilon_j)^2}\right) \geq \frac{\varepsilon_j^2}{(1+\frac{3}{2}\lambda_0)^2}. \tag{3.11}$$

Therefore

$$\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) \geq \frac{\varepsilon_j^2}{(1+\frac{3}{2}\lambda_0)^2}\|u_j^{(2)}\|^2. \tag{3.12}$$

Choose a sequence $\varepsilon_j \in (0, \frac{\lambda_0}{2})$ such that

i) $\varepsilon_j \to 0$.
ii) $\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big)/\varepsilon_j^2 \to 0$.

For example, we can take $\varepsilon_j = \big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big)^{\frac{1}{3}}$. Therefore (3.12) implies

$$\|u_j^{(2)}\| \to 0 \text{ as } j \to \infty,$$

whence

$$\|u_j^{(1)}\| \to 1 \text{ as } j \to \infty. \tag{3.13}$$

On the other hand, as

$$\int_0^\infty \lambda^2 d(P_\lambda u_j^{(1)}, u_j^{(1)}) \leq (\lambda_0+\varepsilon_j)^2\|u_j\|^2 < \infty,$$

the sequence $u_j^{(1)} \in Dom(H)$. So we can apply the classical Weyl Criterion to the sequence $u_j^{(1)}$. By Projection Valued Measure again,

$$\|(H-\lambda_0)u_j^{(1)}\|^2 = \int_0^\infty (\lambda-\lambda_0)^2 d(P_\lambda u_j^{(1)}, u_j^{(1)}) \leq \varepsilon_j^2 \to 0, \tag{3.14}$$

which implies $\lambda_0 \in \operatorname{Spec}(H)$. So the result follows. □

REMARK 3.4. Note that the condition (2) in the theorem can be weaken to $\big((H-\lambda_0)^2(1+H)^{-2}u_j, u_j\big) \to 0$ by the proof.

REMARK 3.5. The above theorem also holds for $\lambda_0 = 0$. And in fact we can also prove conditions (1) and (2) are not only sufficient but also necessary for $\lambda_0 \in \operatorname{Spec}(H)$. More details can be found in [**2**].

With the generalized Weyl criterion 3.2, we are ready to prove Lemma 3.3.

PROOF. Here we prove $\operatorname{Spec}(\overline{\partial}\bar{\partial}^*_{p,q+1}) \subset \operatorname{Spec}(\square_{p,q}) \cup \{0\}$. The other containment $\operatorname{Spec}(\bar{\partial}^*\overline{\partial}_{p,q-1}) \subset \operatorname{Spec}(\square_{p,q}) \cup \{0\}$ can be proved similarly.

Take $\lambda_0 \in \operatorname{Spec}(\overline{\partial}\bar{\partial}^*)$ and $\lambda_0 > 0$. By classical Weyl criterion, there exists a sequence $u_j \in Dom(\overline{\partial}\bar{\partial}^*)$ with $(u_j, u_j) = 1$ such that

$$((\overline{\partial}\bar{\partial}^* - \lambda_0)u_j, (\overline{\partial}\bar{\partial}^* - \lambda_0)u_j) \to 0. \tag{3.15}$$

We will verify that the sequence $\bar\partial^* u_j$ satisfies conditions in Theorem 3.2. For $m = 1, 2$,

$$\begin{aligned} &\big((\Box - \lambda_0)(1+\Box)^{-m}\bar\partial^* u_j, \bar\partial^* u_j\big) \\ =& \big((\Box - \lambda_0)(1+\Box)^{-m} u_j, \overline{\partial}\bar\partial^* u_j\big) \\ =& \big((\overline{\partial}\bar\partial^* - \lambda_0)(1+\Box)^{-m} u_j, \overline{\partial}\bar\partial^* u_j\big) \\ =& \big(\overline{\partial}\bar\partial^*(1+\Box)^{-m} u_j, (\overline{\partial}\bar\partial^* - \lambda_0) u_j\big)\,. \end{aligned}$$

The first equality is because $(1+\Box)^{-1}\bar\partial^* = \bar\partial^*(1+\Box)^{-1}$ on $Dom(\bar\partial^*)$, which follows from Theorem 2.2. The second one follows from $\overline{\partial}\circ\overline{\partial} = 0$ on $Dom(\overline{\partial})$. The third one comes from the self-adjointness of $\overline{\partial}\bar\partial^*$ and straightforward calculations. Since

$$\|\overline{\partial}\bar\partial^*(1+\Box)^{-m}u_j\| \le \|\Box(1+\Box)^{-m}u_j\| \le \|u_j\| = 1, \tag{3.16}$$

(3.15) implies

$$\big((\Box - \lambda_0)(1+\Box)^{-m}\bar\partial^* u_j, \bar\partial^* u_j\big) \to 0 \text{ for } m = 1, 2. \tag{3.17}$$

The other thing we need to verify is that $\|\bar\partial^* u_j\|$ has a positive lower bound uniformly for all j. This is from the following calculations:

$$(\bar\partial^* u_j, \bar\partial^* u_j) = ((\overline{\partial}\bar\partial^* - \lambda_0)u_j, u_j) + \lambda_0 \to \lambda_0 > 0. \tag{3.18}$$

Since $\|\bar\partial^* u_j\|$ has a uniform lower bound, we can apply Theorem 3.2 to the scaled sequence $\bar\partial^* u_j/\|\bar\partial^* u_j\|$ and the result follows immediately. □

Now we are going to finish the proof of Theorem 3.1 in next lemma.

Lemma 3.6. *Under the same assumption as Theorem* 3.1*, we have*

$$Spec(\Box_{p,q}) \subset Spec(\overline{\partial}\bar\partial^*_{p,q}) \cup Spec(\bar\partial^*\overline{\partial}_{p,q}) \cup \{0\}.$$

Proof. Take $\lambda_0 \in \mathrm{Spec}(\Box)$ and $\lambda_0 > 0$. Then by classical Weyl criterion, there exists a sequence $u_j \in Dom(\Box)$ with $\|u_j\| = 1$ such that

$$(\Box - \lambda_0)u_j \to 0. \tag{3.19}$$

We will use $\overline{\partial}\bar\partial^*(1+\Box)^{-2}u_j$ and $\bar\partial^*\overline{\partial}(1+\Box)^{-2}u_j$ as the test sequences. By the fact that $\overline{\partial}\circ\overline{\partial} = 0$ on $Dom(\overline{\partial})$ and $(1+\Box)^{-1}\Box = \Box(1+\Box)^{-1}$ on $Dom(\Box)$, we have

$$(\overline{\partial}\bar\partial^* - \lambda_0)\overline{\partial}\bar\partial^*(1+\Box)^{-2}u_j = \overline{\partial}\bar\partial^*(1+\Box)^{-2}(\Box - \lambda_0)u_j. \tag{3.20}$$

Since $\|\overline{\partial}\bar\partial^*(1+\Box)^{-2}\|_{L^2\to L^2} \le 1$, it implies

$$\|(\overline{\partial}\bar\partial^* - \lambda_0)\overline{\partial}\bar\partial^*(1+\Box)^{-2}u_j\| \le \|(\Box - \lambda_0)u_j\| \to 0. \tag{3.21}$$

Similarly, we also have

$$\|(\bar\partial^*\overline{\partial} - \lambda_0)\bar\partial^*\overline{\partial}(1+\Box)^{-2}u_j\| \le \|(\Box - \lambda_0)u_j\| \to 0. \tag{3.22}$$

Now we need to check either $\|\overline{\partial}\bar\partial^*(1+\Box)^{-2}u_j\|$ or $\|\bar\partial^*\overline{\partial}(1+\Box)^{-2}u_j\|$ has a positive lower bound. Note

$$\|\overline{\partial}\bar\partial^*(1+\Box)^{-2}u_j\|^2 + \|\bar\partial^*\overline{\partial}(1+\Box)^{-2}u_j\|^2 = \|\Box(1+\Box)^{-2}u_j\|^2. \tag{3.23}$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of $\Box$. Then

$$\|\Box(1+\Box)^{-2}u_j\|^2 = \int_0^\infty \frac{\lambda^2}{(1+\lambda)^4} d(P_\lambda u_j, u_j) \ge C(\lambda_0)\|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)}u_j\|^2. \tag{3.24}$$

Note $(\Box - \lambda_0)u_j \to 0$ implies

$$\|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)} u_j\| \to 1. \tag{3.25}$$

Therefore for sufficiently large j,

$$\|\overline{\partial}\bar{\partial}^*(1+\Box)^{-2}u_j\|^2 + \|\bar{\partial}^*\overline{\partial}(1+\Box)^{-2}u_j\|^2 \geq \frac{C(\lambda_0)}{2} > 0. \tag{3.26}$$

So $\lambda_0 \in \mathrm{Spec}(\overline{\partial}\bar{\partial}^*_{p,q}) \cup \mathrm{Spec}(\bar{\partial}^*\overline{\partial}_{p,q})$ by classical Weyl criterion and the result follows. $\Box$

One direct corollary from Theorem 3.1 is the following spectrum relations of Gaffney extensions.

Corollary 3.7. *Under the same assumption as Theorem* 3.1*, we have*

$$\mathit{Spec}(\Box_{p,q}) \subset \mathit{Spec}(\Box_{p,q+1}) \cup \mathit{Spec}(\Box_{p,q-1}) \cup \{0\}. \tag{3.27}$$

At the end of this section, let us recall the well known relation between the spectrum of Gaffney extension and L^2 estimates.

Theorem 3.3. *Let (M, g) be a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h). Assume the Gaffney extension of Hodge Laplacian $\Box_{p,q+1} : L^2(M, \Lambda^{p,q+1}(E)) \to L^2(M, \Lambda^{p,q+1}(E))$ satisfies $\mathit{Spec}(\Box_{p,q+1}) \subset [a, \infty)$ for some positive number a. Then for any $f \in \ker \overline{\partial}_{p,q+1} \subset L^2(M, \Lambda^{p,q+1}(E))$, there exists $u \in L^2(M, \Lambda^{p,q}(E))$ such that $\overline{\partial}u = f$ with the following estimate*

$$(u, u) \leq \frac{1}{a}(f, f). \tag{3.28}$$

Proof. In the proof, we will use $\Box$ to represent $\Box_{p,q+1}$ for simplicity. By the condition $\mathrm{Spec}\Box \subset [a, \infty)$, we have $\Box^{-1} : L^2(M, \Lambda^{p,q+1}(E)) \to \mathit{Dom}(\Box) \subset L^2(M, \Lambda^{p,q+1}(E))$ is a bounded operator with

$$\|\Box^{-1}\|_{L^2 \to L^2} \leq \frac{1}{a}. \tag{3.29}$$

Take $u = \bar{\partial}^*\Box^{-1}f$ and we will verify u satisfies all the conclusions. First, since the Gaffney extension satisfies $\Box = \overline{\partial}\bar{\partial}^* + \bar{\partial}^*\overline{\partial}$ by Theorem 2.2, we have

$$\overline{\partial}u = \overline{\partial}\bar{\partial}^*\Box^{-1}f = f - \bar{\partial}^*\overline{\partial}\Box^{-1}f. \tag{3.30}$$

Therefore $f \in \ker \overline{\partial}$ implies $\bar{\partial}^*\overline{\partial}\Box^{-1}f \in \ker \overline{\partial}$. By taking the following inner product

$$0 = \left(\overline{\partial}\bar{\partial}^*\overline{\partial}\Box^{-1}f, \overline{\partial}\Box^{-1}f\right) = \left(\bar{\partial}^*\overline{\partial}\Box^{-1}f, \bar{\partial}^*\overline{\partial}\Box^{-1}f\right), \tag{3.31}$$

we have

$$\bar{\partial}^*\overline{\partial}\Box^{-1}f = 0. \tag{3.32}$$

Again by taking the following inner product with $\Box^{-1}f$

$$0 = \left(\bar{\partial}^*\overline{\partial}\Box^{-1}f, \Box^{-1}f\right) = \left(\overline{\partial}\Box^{-1}f, \overline{\partial}\Box^{-1}f\right), \tag{3.33}$$

we have

$$\overline{\partial}\Box^{-1}f = 0. \tag{3.34}$$

Together with (3.30), we have

$$\overline{\partial}u = f.$$

Second, we will verify the estimate (3.28). By (3.34) and straightforward calculations, we have

$$(u,u) = (\bar{\partial}^*\Box^{-1}f, \bar{\partial}^*\Box^{-1}f) = (\overline{\partial}\bar{\partial}^*\Box^{-1}f, \Box^{-1}f) = (f, \Box^{-1}f).$$

Therefore (3.29) implies the result. $\Box$

REMARK 3.8. Note we cannot directly use $\overline{\partial}_{p,q+2}\Box^{-1}_{p,q+1}f = \Box^{-1}_{p,q+2}\overline{\partial}_{p.q+1}f = 0$ in the proof as we do not know the existence of $\Box^{-1}_{p,q+2}$.

4. Spectrums of Friedrichs Extension

In this section, we assume (M,ω) is a Kähler manifold with a holomorphic Hermitian line bundle (L,h) . Since $Dom(\Box_F) \subset W^1_0$, we will not get any boundary term when doing integration by parts for sections in $Dom(\Box_F)$. By using the Weitzenböck formula, we will prove the following spectrum lower bound for Friedrichs extension under certain curvature conditions.

THEOREM 4.1. *Let (M,ω) be a Kähler manifold with a holomorphic Hermitian line bundle (L,h). Consider Friedrichs extension of Hodge Laplacian, $\Box_{0,q}$: $L^2(M,\Lambda^{0,q}(L)) \to L^2(M,\Lambda^{0,q}(L))$. If $\mathrm{Ric}(TM) + \mathrm{Ric}(L) \geq a\omega$ for some positive number a, then*

$$Spec\Box_{0,q} \subset [aq,\infty). \tag{4.1}$$

REMARK 4.1. In this section, $\Box_{0,q}$ always represents the Friedrichs extension and we will omit the subindex $\{0,q\}$ when there is no ambiguity.

PROOF. Take $\varphi \in Dom(\Box_{0,q})$. As $Dom(\Box_{0,q}) \subset W^1_0(M,\Lambda^{0,q}(L))$, there exists a sequence $\varphi_n \in \mathcal{D}(M,\Lambda^{0,q}(L))$ such that $\varphi_n \to \varphi$ in W^1_0. By the Weitzenböck formula $\Box_{0,q} = -\overline{\nabla}\nabla + q\,\mathrm{Ric}(TM) + q\,\mathrm{Ric}(L)$, we have

$$Q(\varphi_n,\varphi_n) = (\nabla\varphi_n,\nabla\varphi_n) + (q(\mathrm{Ric}(TM)+\mathrm{Ric}(L))\varphi_n,\varphi_n) \geq aq(\varphi_n,\varphi_n). \tag{4.2}$$

Letting $n \to \infty$, we have

$$Q(\varphi,\varphi) \geq aq(\varphi,\varphi). \tag{4.3}$$

As $Q(\varphi,\varphi) = (\Box_{0,q}\varphi,\varphi)$, the result follows. $\Box$

REMARK 4.2. Let $n = \dim M$. As the Weitzenböck formula for L-valued (n,q) form is $\Box_{n,q} = -\overline{\nabla}\nabla + q\,\mathrm{Ric}(L)$. If $\mathrm{Ric}(L) \geq a\omega$ for some positive constant a, then the Friedrichs extension $\Box_{n,q}$ satisfies $\mathrm{Spec}\Box_{n,q} \subset [aq,\infty)$.

5. Manifolds with Almost Polar Boundary

Let (M,g) be a Riemannian manifold. Similar as the Definition 2.2, we can define the Sobolev space for functions by taking the quadratic form $Q_1(\cdot,\cdot) = (\cdot,\cdot) + (d\cdot,d\cdot)$.

DEFINITION 5.1.

$$W^1_0(M) = \text{Completion of } \mathcal{D}(M) \text{ with respect to } Q_1 \text{ inner product}, \tag{5.1}$$

$$W^1(M) = \text{Completion of } \{\varphi \in \mathcal{C}^\infty(M) : Q_1(\varphi,\varphi) < \infty\} \text{ with respect to } Q_1 \text{ inner product}. \tag{5.2}$$

Generally we know $W^1(M) = W_0^1(M)$ for complete Riemannian manifolds. In [**10, 11**], Masamune proved $W^1(M) = W_0^1(M)$ for Riemannian manifolds with almost polar boundary. We will repeat the proof here for the sake of completeness and because there is a gap in Masamune's proof.

We first introduce the definition and notations. Let d be the distance function induced by the length of piecewise curves on M. Then (M, d) is a metric space. We use $(\overline{M}_c, d)$ to denote the Cauchy completion of (M, d). We define the Cauchy boundary $\partial_c M = \overline{M}_c - M$.

Definition 5.2. We define the capacity of an open set $O \subset \overline{M}_c$ by

$$\mathrm{cap}(O) = \inf\{Q_1(u,u) : u \in W^1(M), 0 \le u \le 1 \text{ and } u|_{O\cap M} = 1\}. \tag{5.3}$$

We also define the capacity of an arbitrary set $\Sigma \subset \overline{M}_c$ by

$$\mathrm{cap}(\Sigma) = \inf\{\mathrm{cap}(O), \Sigma \subset O, O \subset \overline{M}_c \text{ is open}\}. \tag{5.4}$$

A set Σ is said to be almost polar if $\mathrm{cap}(\Sigma) = 0$.

Remark 5.3. For any open set $O \subset \overline{M}_c$, $e \in W^1(M)$ is called the equilibrium potential of O if it satisfies

1. $Q_1(e, e) = \mathrm{cap}(O)$.
2. $e|_O = 1$.
3. $0 \le e \le 1$.

It is know that the equilibrium potential exists for any open set $O \subset \overline{M}_c$. See [**3**] for more details.

Here is the main theorem we are going to prove.

Theorem 5.1. *Let (M, g) be a Riemannian manifold. If cap$(\partial_c M) = 0$, then*

$$W^1(M) = W_0^1(M). \tag{5.5}$$

Before going to the proof, let's explain the main idea. First we show that $L^\infty(M)\cap W^1(M) \subset W^1(M)$ is dense. Then it is sufficient to consider $f \in L^\infty(M)\cap W^1(M)$. Choosing a sequence of open sets $\{V_n\}$ decreasing to $\partial_c M$, by using the equilibrium potential of V_n, say e_n, we can approximate f by $(1 - e_n)f$ whose support is contained in $M - V_n$. In the last, we want to modify the function $(1 - e_n)f$ to be compactly supported. As $(\overline{M}_c, d)$ is only a complete metric space, the closed metric ball excluding an open set containing $\partial_c M$ might not be a compact set even if $\mathrm{cap}(\partial_c M) = 0$(See Section 8 for more details). So we will refer to the intrinsic distance and verify that the intrinsic distance induces the same topology as d on $M - V_n$. As the closed metric ball with respect to the intrinsic distance is compact by Hopf-Rinow-Cohn-Vossen Theorem (see Theorem 2.5.28 in [**1**]). And we will use some cut-off function to finish the modification on support.

We begin the proof with the following lemma described above.

Lemma 5.4. *For any Riemannian manifold (M, g), $L^\infty(M) \cap W^1(M)$ is dense in $W^1(M)$.*

Proof. Take $f \in W^1(M)$. Define a cut-off function $\rho \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\rho(x) = \begin{cases} 1 & x \le 1 \\ 0 & x \ge 2 \end{cases},$$

and

$$0 \le \rho \le 1, \quad -C \le \rho' \le 0.$$

We define $\rho_m(x) = \rho(\frac{x}{m})$ and $f_m = \rho_m(|f|)f$. Note $f_m \in L^\infty(M) \cap W^1(M)$ and we will prove $f_m \to f$ in $W^1(M)$. By dominated convergence theorem, we directly get $f_m \to f$ in $L^2(M)$.

As to df_m, we have

$$df_m - df = \big(\rho(\frac{|f|}{m}) - 1\big)df + \frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f|. \tag{5.6}$$

The first term on the right hand side converges to 0 in $L^2(M, \Lambda^1)$ as $|\rho(\frac{|f|}{m}) - 1| \le \chi_{\{|f| \ge m\}}$. For the second term, since

$$\big|\frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f|\big| \le 2C\chi_{\{m \le |f| \le 2m\}}|df|, \tag{5.7}$$

it follows that $\frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f| \to 0$ in $L^2(M, \Lambda^1)$. So we have $f_m \to f$ in $W^1(M)$ and the result follows. □

In next two lemmas, we will construct open sets containing $\partial_c M$ with smooth boundary.

Lemma 5.5. *$\partial_c M \subset \overline{M}_c$ is a closed subset.*

Proof. Since M is the complement of $\partial_c M$ in $\overline{M}_c$, it is equivalent to check that $M \subset \overline{M}_c$ is an open subset. For any $x \in M$, let i_x be the injectivity radius of (M, g) at x. Then for any $r \in (0, i_x)$, by considering the exponential map at x, we know $\overline{B_M(x,r)} = \{y \in M, d(x,y) \le r\}$ is compact, whence complete. Therefore $B_M(x,r) = B_{\overline{M}_c}(x,r) = \{y \in \overline{M}_c, d(x,y) < r\}$ since we will not add any new point to $B_M(x,r)$ during the Cauchy completion of M. So $B_{\overline{M}_c}(x,r) \subset M$ and the result follows. □

Lemma 5.6. *For any open set $U \subset \overline{M}_c$ containing $\partial_c M$, there exists an open set $V \subset \overline{M}_c$ such that $\partial_c M \subset V \subset \overline{V} \subset U$ and $\partial(\overline{M}_c \setminus V) \subset M$ is a smooth submanifold of codimension 1.*

Proof. Let U^C be the complement of U in $\overline{M}_c$. Since $\partial_c M$ and U^C are both closed in $(\overline{M}_c, d)$. By Urysohn's Lemma, there exists a function $f \in \mathcal{C}(\overline{M}_c)$ such that $0 \le f \le 1$, $f^{-1}(\{0\}) = \partial_c M$ and $f^{-1}(\{1\}) = U^C$. Take $S = f^{-1}([0, \frac{1}{2}))$. Then S is an open subset of $\overline{M}_c$ such that $\partial_c M \subset S \subset \overline{S} \subset U$.

Note that $\overline{S} \setminus \partial_c M = \overline{S} \cap M$ and U^C are both closed in M. By the Smooth Urysohn's Lemma in [**12**], there exists a function $g \in \mathcal{C}^\infty(M)$ such that $0 \le g \le 1$, $g^{-1}(\{0\}) = \overline{S} \setminus \partial_c M$ and $g^{-1}(\{1\}) = U^C$. By Sard's Theorem, without loss of generality, we can assume $\frac{1}{2}$ is a regular value of g. Take $V = g^{-1}([0, \frac{1}{2})) \cup \partial_c M \subset \overline{M}_c$. Then it's easy to see $V = g^{-1}([0, \frac{1}{2})) \cup S$. Therefore V is open in $\overline{M}_c$ such that $\partial_c M \subset V \subset \overline{V} = \overline{g^{-1}([0, \frac{1}{2}))} \cup \overline{S} \subset U$. The remaining part of the lemma follows from $\partial(\overline{M}_c \setminus V) = g^{-1}(\{\frac{1}{2}\})$ and $\frac{1}{2}$ is a regular value of g. □

Let V be an open subset satisfying the conclusion in the above lemma. Denote $V^C = \overline{M}_c \setminus V$ as the complement of V in $\overline{M}_c$. Then $V^C = \cup_{\lambda \in \Lambda} A_\lambda$, where each A_λ is a connected component of V^C and Λ is the index set. Since V^C is locally path connected, each A_λ is both open and closed in V^C. Define the intrinsic distance function d_{A_λ} on A_λ as

Definition 5.7. Define the intrinsic distance on A_λ as $d_{A_\lambda}: A_\lambda \times A_\lambda \to [0,\infty)$,

$$d_{A_\lambda}(x,y) = \inf_{l \in L_{A_\lambda}} \|l\| \tag{5.8}$$

where $L_{A_\lambda} = \{$all piecewise smooth curves contained in A_λ from x to $y\}$ and $\|l\|$ denotes the length of curve l.

Remark 5.8. $d(x,y) \le d_{A_\lambda}(x,y)$ for any $x, y \in A_\lambda$ as d is the infimum over a larger set.

In general, d and d_{A_λ} are not globally equivalent to each other on A_λ. The next lemma shows that they are locally equivalent on A_λ.

Lemma 5.9. *For any $x \in A_\lambda$, there exists $r = r(x) > 0$ such that*

$$d_{A_\lambda}(x,y) \le 4d(x,y) \quad \textit{for any } y \in B_{A_\lambda}(x,r). \tag{5.9}$$

where $B_{A_\lambda}(x,r) = \{y \in A_\lambda, d(x,y) < r\}$.

Proof. For any $x \in A_\lambda \subset V^C \subset M$, either x is in the interior of V^C or $x \in \partial V^C$. In the first case, take $r < i_x$ (i_x denotes the injectivity radius at x) small enough such that $B_M(x,r) \subset A_\lambda$. Then for any $y \in B_M(x,r)$, there existed a minimizing geodesic $l \subset B_M(x,r)$ such that $\|l\| = d(x,y)$. Therefore $d_{A_\lambda}(x,y) = d(x,y)$ for any $y \in B_{A_\lambda}(x,r) = B_M(x,r)$.

In the second case, i.e. $x \in \partial V^C$, take $r < i_x$. We can identify $B_{\mathbb{R}^m}(o,r)$ (w.r.t the Euclidean metric g_x) with $B_M(x,r)$ by the exponential map Exp_x at x. By shrinking r, we can assume the Riemannian metric on $B_M(x,r)$ is equivalent to the metric at x, say $\frac{1}{2}g_x \le g \le 2g_x$. Let $\{e_i\}_{i=1}^m$ be the standard orthonormal basis of $\mathbb{R}^m$. Up to an orthonormal linear transformation, we can assume $\{e_i\}_{i=1}^{m-1} \subset T_x(\partial V^C)$ and e_m is the normal direction of ∂V^C at x. By Lemma 5.6, possibly shrinking r again, we can assume $\partial V^C = \{(x_1, x_2, \cdots, x_m) \in B(o,r), x_m = h(x_1, x_2 \cdots, x_{m-1})\}$ where $h \in \mathcal{C}^\infty(\mathbb{R}^{m-1})$ and $h(0,\cdots,0) = 0$. Since $\{e_i\}_{i=1}^{m-1}$ are tangent vectors of ∂V^C at x, $\nabla h(0,\cdots,0) = 0$. By shrinking r again, we can assume $|\nabla h| \le 1$ in $B_{\mathbb{R}^{m-1}}(o,r)$.

For any point $y \in B_{\mathbb{R}^m}(o,r)$, consider the curve $l_1 = (ty_1, ty_2, \cdots, ty_{m-1}, h(ty_1, \cdots, ty_{m-1}))$ for $t \in [0,1]$ and $l_2 = (y_1, y_2, \cdots, y_{m-1}, ty_m + (1-t)h(y_1, y_2, \cdots, y_{m-1}))$ for $t \in [0,1]$. Then the concatenation $l_1 \cup l_2 \subset V^C$ is from x to y. The Euclidean length of l_1, l_2 are respectively

$$\begin{aligned}
\|l_1\|_{\mathbb{R}^m} =& \int_0^1 \sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2 + |\nabla h(ty_1, ty_2, \cdots, ty_{m-1}) \cdot (y_1, y_2, \cdots, y_{m-1})|^2} dt \\
\le& 2\sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2}, \\
\|l_2\|_{\mathbb{R}^m} =& |y_m - h(y_1, y_2, \cdots, y_{m-1})| \\
\le& |y_m| + \sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
d_{A_\lambda}(x,y) \le& \|l_1\| + \|l_2\| \\
\le& 2\|l_1\|_{\mathbb{R}^m} + 2\|l_2\|_{\mathbb{R}^m} \\
\le& 4\sqrt{y_1^2 + y_2^2 + \cdots + y_{m-1}^2 + y_m^2} \\
=& 4d(x,y).
\end{aligned}$$

The second inequality is because $\frac{1}{2}g_x \le g \le 2g_x$. So the result follows. $\square$

Base on Remark 5.8 and Lemma 5.9, we have the following properties on $(A_\lambda, d_{A_\lambda})$.

PROPOSITION 5.10. *$(A_\lambda, d_{A_\lambda})$ satisfies the following property.*

(a). *$(A_\lambda, d_{A_\lambda})$ and (A_λ, d) have the same topology.*
(b). *$(A_\lambda, d_{A_\lambda})$ is locally compact.*
(c). *$(A_\lambda, d_{A_\lambda})$ is complete.*

PROOF. Part (a) directly follows from Remark 5.8 and Lemma 5.9.

Now we prove part (b). Since V^C is a closed subset of (M, d) and (M, d) is locally compact, (V^C, d) is locally compact. And we know A_λ is a closed subset of (V^C, d), therefore (A_λ, d) is locally compact. The result follows by part (a).

Last we prove part (c). Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in $(A_\lambda, d_{A_\lambda})$. By Remark 5.8, $\{x_n\}_{n=1}^\infty$ is also a Cauchy sequence in (A_λ, d). Since A_λ is closed in (V^C, d) and V^C is closed in the complete space $(\overline{M}_c, d)$, (A_λ, d) is complete. Then there exists some $x \in A_\lambda$ such that $\lim d(x, x_n) = 0$. By Lemma 5.9, $\lim d_{A_\lambda}(x_n, x) = 0$ and therefore the result follows. $\square$

For any $x_0 \in A_\lambda$, define the function $r_{x_0} : A_\lambda \to [0, \infty)$ as $r_{x_0}(x) = d_{A_\lambda}(x_0, x)$. Then r_{x_0} has the following property.

PROPOSITION 5.11. *For the function r_{x_0} defined as above, we have*

$$|\nabla r|_g \le 4. \tag{5.10}$$

PROOF. Since $|r(x) - r(y)| \le d_{A_\lambda}(x, y)$, the result follows from Lemma 5.9. $\square$

The closed metric ball induced by d_{A_λ} is compact though it is not the case for the closed metric ball induced by d. The following lemma is essentially Hopf-Rinow-Cohn-Vossen Theorem. See Theorem 2.5.28 in [**1**] for more details.

LEMMA 5.12. *For any $x \in A_\lambda, r > 0$, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)}$ is compact. Here $B_{(A_\lambda, d_{A_\lambda})}(x, r)$ denotes the set $\{y \in A_\lambda, d_{A_\lambda}(x, y) < r\}$.*

REMARK 5.13. By part (a) in Proposition 5.10, the closures of $B_{(A_\lambda, d_{A_\lambda})}(x, r)$ in (A_λ, d) and in (A_λ, d_λ) are the same. The compactness in (A_λ, d) and that in (A_λ, d_λ) are also the same. So there is no ambiguity in the above lemma.

PROOF. By part (b) in Proposition 5.10, the set

$$\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)} \text{ is compact}\}$$

is nonempty. So we can define $r_0 = \sup\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)} \text{ is compact}\}$. Now it suffices to prove $r_0 = \infty$. Assume not. Then $r_0 \in (0, \infty)$.

First, we prove that $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is compact. Take an arbitrary $\varepsilon > 0$. For any $y \in \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$, since $d_{A_\lambda}(x, y) \le r_0$, there exists a piecewise smooth curve $l \subset A_\lambda$ from x to y such that $\|l\| < r_0 + \varepsilon$. Reparametrize the curve l by arc length. Then the restriction $l|_{[r_0 - \varepsilon, \|l\|]}$ is a piecewise smooth curve from a point in $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}$ to y. Since

$$\|l|_{[r_0 - \varepsilon, \|l\|]}\| < 2\varepsilon, y \in B_{(A_\lambda, d_{A_\lambda})}(\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}, 2\varepsilon).$$

Therefore

$$\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0)} \subset B_{(A_\lambda,d_{A_\lambda})}(\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0-\varepsilon)},2\varepsilon).$$

Since $\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0-\varepsilon)}$ is compact by the definition of r_0, $\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0)}$ is totally bounded in $(A_\lambda, d_{A_\lambda})$. Therefore $\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0)}$ is compact by part (c) in Proposition 5.10.

Second, we prove that $\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0+\delta)}$ is compact for some $\delta > 0$, which contradicts the definition of r_0 and therefore we get the result. Since $\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0)}$ is also compact, together with part (b) in Proposition 5.10, $\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0)}$ has a finite cover $\{B_{(A_\lambda,d_{A_\lambda})}(y_i,\delta_i)\}_{i=1}^N$, such that $y_i \in \overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0)}$, $\delta_i > 0$ and $\overline{B_{(A_\lambda,d_{A_\lambda})}(y_i,2\delta_i)}$ is compact for each i. Take $\delta = \min_{1\le i\le N}\delta_i$. Then

$$\overline{B_{(A_\lambda,d_{A_\lambda})}(x,r_0+\delta)} \subset \cup_{i=1}^N \overline{B_{(A_\lambda,d_{A_\lambda})}(y_i,2\delta_i)}$$

is compact. □

Now we are ready to prove the Theorem 5.1.

Proof. Since $\mathrm{cap}(\partial_c M) = 0$, there exists a sequence of open sets $\{U_n\}_{n=1}^\infty$ such that $\partial_c M \subset U_n$ and $\lim \mathrm{cap}(U_n) = 0$. For U_1, by Lemma 5.6, there exists an open set V_1 such that $\partial_c M \subset V_1 \subset \overline{V_1} \subset U_1$ and $\partial(V_1^C)$ is a smooth submanifold. Then for $V_1 \cap U_2$, by Lemma 5.6, there exists an open set V_2 such that $\partial_c M \subset V_2 \subset \overline{V_2} \subset V_1 \cap U_2$ and $\partial(V_2^C)$ is a smooth submanifold. Inductively, we construct V_{i+1} by applying Lemma 5.6 to $V_i \cap U_{i+1}$. So we get a sequence of decreasing open sets $\{V_n\}_{n=1}^\infty$ such that $\partial_c M \subset V_n \subset \overline{V_n} \subset V_{n-1} \cap U_n$ and $\partial(V_n^C)$ is a smooth submanifold. Since in particular $V_n \subset U_n$, we have $\lim \mathrm{cap}(V_n) = 0$.

Take $f \in W^1(M) \cap L^\infty$. It suffices to prove $f \in W_0^1(M)$.

First, we approximate f by functions with support in some V_n^C. Let e_n be the equilibrium potential (see Remark 5.3) of V_n, i.e. e_n satisfies

- $e_n \in W$ and $Q_1(e_n, e_n) = \mathrm{cap}(V_n)$.
- $e_n|_{V_n} = 1$.
- $0 \le e_n \le 1$.

Since $\|e_n\|_W = \mathrm{cap}(V_n) \to 0$, we can assume $e_n \to 0$ a.e. by passing to a subsequence. Let $f_n = (1-e_{n-1})f$. Then $f_n \to f$ in $W^1(M)$ and $\mathrm{supp}(f_n) \subset V_{n-1}^C \subset \overline{V_n}^C \subset \mathrm{interior}(V_n^C)$.

Secondly, we approximate each f_n with $\mathrm{supp}(f_n) \subset \mathrm{interior}(V_n^C)$ by functions with compact support. From now on, we fix f_n and V_n^C. For economy we suppress the index n. Write V^C into the disjoint union of connected component, $V^C = \cup_{\lambda\in\Lambda} A_\lambda$. Since $f \in W^1(M)$ and $\{A_\lambda\}_{\lambda\in\Lambda}$ is pairly disjoint, f vanishes on all but countably many A_λ, say $\{A_{\lambda_j}\}_{j=1}^\infty$. Denote $g_j = f\chi_{A_{\lambda_j}}$ where $\chi_{A_{\lambda_j}}$ is the characteristic function of A_{λ_j}. Note $g_j \in W^1(M)$ and $\nabla g_j = (\nabla f)\chi_{A_{\lambda_j}}$ by the fact that $\partial A_{\lambda_j} \subset \partial(V^C)$ and f vanishes close to $\partial(V^C)$ as $\mathrm{supp}\, f \subset \mathrm{interior}(V^C)$. Then $f = \sum_{j=1}^\infty g_j$ and $\|f\|_W^2 = \sum_{j=1}^\infty \|g_j\|_W^2$. Therefore for any $\varepsilon > 0$, there exists $N > 0$ such that $\|f - \sum_{j=1}^N g_j\|_W < \varepsilon$.

Now it suffices to approximate each g_j be compact supported function. Take $x_j \in A_{\lambda_j}$ and define $r_j : A_{\lambda_j} \to [0,\infty)$ as $r_j(x) = d_{A_{\lambda_j}}(x_j, x)$. Then $|\nabla r_j|_g \le 4$ by Proposition 5.11. Let $\varphi \in \mathcal{C}^\infty(R)$ satisfy the following conditions:

- φ is a decreasing function and $0 \le \varphi \le 1$.

- $\varphi|_{(-\infty,0]} = 1$ and $\varphi_{[1,\infty)} = 0$.
- $|\varphi'| \leq C$ and C is a fixed constant.

Define $\varphi_k(x) = \varphi(\frac{x}{k})$. Then $\varphi_k \circ r_j \to 1$ a.e. on A_{λ_j} as $k \to \infty$ and $|\nabla(\varphi_k \circ r_j)|_g \leq \frac{4C}{k}$. Therefore we have $(\varphi_k \circ r_j)g_j \to g_j$ in $W^1(M)$. And $\mathrm{supp}((\varphi_k \circ r_j)g_j) \subset \mathrm{supp}(\varphi_k \circ r_j) \subset \overline{B_{(A_{\lambda_j}, d_{A_{\lambda_j}})}(x_j, 2k)}$, which is compact by Lemma 5.12. So the result follows. $\square$

6. Moduli Space of Polarized Calabi-Yau Manifolds

Let (M, L) be a Calabi-Yau manifold polarized by a positive line bundle L. That is, M is a compact Kähler manifold with a Ricci flat Kähler metric ω and the metric ω is contained in the first Chern class of L. Let $\mathcal{M}$ be the moduli space of Calabi-Yau manifolds polarized by a fixed positive line bundle L. In [**16**], Viehweg proved the moduli space $\mathcal{M}$ is a quasi-projective variety. Take $\overline{\mathcal{M}}$ as the compactification of $\mathcal{M}$. With the classical result of Hironaka, by resolution of singularities, we can choose $\overline{\mathcal{M}}$ in such a way that the divisor $Y = \overline{\mathcal{M}} \setminus \mathcal{M}$ is a divisor of normal crossings. After passing to a finite cover, we may assume $\mathcal{M}$ and $\overline{\mathcal{M}}$ are smooth manifolds (see Lemma 4.1 in [**8**]). From now on, we will work on this quasi-projective Kähler manifold $(\mathcal{M}, \omega_{WP})$ with the compactification $\overline{\mathcal{M}}$ as a compact Kähler manifold.

Here is the main theorem we are going to prove in this section.

THEOREM 6.1. *The moduli space of polarized Calabi-Yau manifolds* $(\mathcal{M}, \omega_{WP})$ *has almost polar Cauchy boundary, i.e. cap*$(\partial_c\mathcal{M}) = 0$.

REMARK 6.1. In general, the Cauchy completion $\overline{\mathcal{M}}_c$ is not necessarily identical to the compactification $\overline{\mathcal{M}}$.

It is well-known that there is a complete Kähler metric on $\mathcal{M}$ such that it is asymptotical to the Poincaré metric near infinity. We call it Poincaré metric and denote it by ω_P(See Lemme 3.1 in [**8**]). The key ingredient to prove Theorem 6.1 is the following lemma in [**7**].

LEMMA 6.2. *For any* $\varepsilon > 0$ *small enough, there is a smooth real valued function* $\rho_\varepsilon \in \mathcal{D}(\mathcal{M})$ *such that*

(a). $0 \leq \rho_\epsilon \leq 1$;

(b). *There is a constant* C*, independent of* ϵ*, such that* $-C\omega_P \leq \sqrt{-1}\partial\overline{\partial}\rho_\epsilon \leq C\omega_P$;

(c). *In a neighborhood of* Y, $\rho_\epsilon = 0$ *and* $\rho_\epsilon(x) = 1$ *if the Euclidean distance of* $x \in M$ *to* Y *is greater than* 2ε.

PROOF. As $Y \subset \overline{\mathcal{M}}$ is a divisor of normal crossings, by [**8**] (see Lemma 4.1), we can find a finite cover $\{U_\alpha\}_{\alpha=1}^t$ of $\overline{\mathcal{M}}$ such that $Y \subset \cup_{\alpha=1}^s U_\alpha$ and $U_{s+1} \cup \cdots \cup U_t) \cap Y = \emptyset$. Furthermore, we can assume that $U_\alpha - Y = (\Delta^*)^{a_\alpha} \times (\Delta)^{b_\alpha}$ with the coordinates $(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha, w_1^\alpha, \cdots, w_{b_\alpha}^\alpha)$ for any $1 \leq \alpha \leq s$, where Δ^* and Δ are respectively the punctured unit disk and the unit disk in $\mathbb{C}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that $0 \leq \eta \leq 1$ and

$$\eta = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \end{cases}.$$

Let

$$\eta_\varepsilon(z) = \begin{cases} 1 & |z| \le e^{-\frac{1}{\varepsilon}} \\ \eta\left(\frac{(\log\frac{1}{|z|})^{-1}-\varepsilon}{\varepsilon}\right) & e^{-\frac{1}{\varepsilon}} \le |z| \le e^{-\frac{1}{2\varepsilon}} \\ 0 & |z| \ge e^{-\frac{1}{2\varepsilon}} \end{cases}.$$

And let

$$\eta_\varepsilon^\alpha(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha) = \prod_{j=1}^{a_\alpha}(1-\eta_\varepsilon(s_j^\alpha)).$$

Then define the function

$$\rho_\varepsilon = \sum_{\alpha=1}^{s}\psi_\alpha\eta_\varepsilon^\alpha + \sum_{\alpha=s+1}^{t}\psi_\alpha,$$

where $\{\psi_\alpha\}$ is a partition of unity subordinated to $\{U_\alpha\}$.

Then $0 \le \rho_\varepsilon \le 1$. By a straightforward calculation, we have

$$\begin{aligned} \overline{\partial}\eta_\varepsilon &= \frac{1}{2\varepsilon}\eta'\frac{d\bar z}{\bar z(\log\frac{1}{|z|})^2}, \\ \partial\overline{\partial}\eta_\varepsilon &= \frac{1}{4\varepsilon^2}\eta''\frac{dz\wedge d\bar z}{|z|^2(\log\frac{1}{|z|})^4} + \frac{1}{2\varepsilon}\eta'\frac{dz\wedge d\bar z}{|z|^2(\log\frac{1}{|z|})^3}. \end{aligned}$$

Note that $\eta' = 0$ unless $\varepsilon \le (\log\frac{1}{|z|})^{-1} \le 2\varepsilon$. Therefore

$$|\overline{\partial}\eta_\varepsilon| \le C\left|\frac{d\bar z}{|z|\log\frac{1}{|z|}}\right|, \quad |\partial\overline{\partial}\eta_\varepsilon| \le C\left|\frac{dz\wedge d\bar z}{|z|^2(\log\frac{1}{|z|})^2}\right|,$$

where C is a constant independent of ϵ. Therefore we obtain part (b) as ψ_α are fixed smooth functions on $\overline{\mathcal{M}}$.

Let $x \in \mathcal{M}$. When x is sufficiently close to Y, $\psi_\alpha = 0$ for any $\alpha \ge s+1$ and $\eta_\varepsilon^\alpha = 0$ for any $\alpha \le s$. Therefore $\rho_\varepsilon = 0$ in a neighborhood of Y. If the distance of x to Y is at least 2ε, then there is a constant $C > 0$ such that $|s_j^\alpha| \ge C\varepsilon$ for any $1 \le j \le a_\alpha$ and $1 \le \alpha \le s$. Since $\varepsilon e^{\frac{1}{2\varepsilon}} \to \infty$ as $\varepsilon \to 0$, when ε is small enough we have $\rho_\varepsilon(x) = \sum\psi_\alpha = 1$. □

Now we are ready to prove Theorem 6.1.

Proof. Take the function ρ_ε constructed in Lemma 6.2. As $\rho_\varepsilon \in \mathcal{D}(\mathcal{M})$ and $0 \le \rho_\varepsilon \le 1$, we have

$$\text{(6.1)} \qquad \operatorname{cap}(\partial_c\mathcal{M}) \le \int_{\mathcal{M}}|1-\rho_\varepsilon|^2\frac{\omega_{WP}^n}{n!} + \int_{\mathcal{M}}|d(1-\rho_\varepsilon)|^2\frac{\omega_{WP}^n}{n!}, \text{ for any } \varepsilon > 0.$$

Since $\rho_\varepsilon \to 1$ pointwise on $\mathcal{M}$ and the volume of Weil-Petersson metric is finite by Theorem 1.1 in [**8**],

$$\text{(6.2)} \qquad \lim_{\varepsilon\to 0}\int_{\mathcal{M}}|1-\rho_\varepsilon|^2\frac{\omega_{WP}^n}{n!} = 0.$$

It suffices to prove that $\int_{\mathcal{M}} |d\rho|^2 \to 0$. Note

$$\int_{\mathcal{M}} |d\rho_\varepsilon|^2 \omega_{WP}^n = 2\int_{\mathcal{M}} |\overline{\partial}\rho_\varepsilon|^2 \omega_{WP}^n = 2n\int_{\mathcal{M}} \sqrt{-1}\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon \wedge \omega_{WP}^{n-1}$$
$$= -2n\int_{\mathcal{M}} \sqrt{-1}\rho_\varepsilon \partial\overline{\partial}\rho_\varepsilon \wedge \omega_{WP}^{n-1}.$$

Since $-C\omega_P \le \sqrt{-1}\partial\overline{\partial}\rho_\varepsilon \le C\omega_P$ and $\omega_{WP} \le C\omega_P$(see Proposition 3.1 in [**8**]), we have

$$\int_{\mathcal{M}} |\overline{\partial}\rho_\varepsilon|^2 \omega_{WP}^n \le C\int_{\mathrm{supp}(\overline{\partial}\rho_\varepsilon)} \omega_P^n.$$

Use the same cover of $\{U_\alpha\}_{\alpha=1}^t$ of $\overline{\mathcal{M}}$ as in Lemma 6.2. Then $Y \subset \cup_{\alpha=1}^s U_\alpha$, $U_{s+1} \cup \cdots \cup U_t) \cap Y = \emptyset$ and $U_\alpha - Y = (\Delta^*)^{a_\alpha} \times (\Delta)^{b_\alpha}$ with the coordinates $(s_1^\alpha, \cdots, s_{a_\alpha}^\alpha, w_1^\alpha, \cdots, w_{b_\alpha}^\alpha)$ for any $1 \le \alpha \le s$. When ε is small enough, we can assume that $\mathrm{supp}(\overline{\partial}\rho_\varepsilon) \cap U_\alpha \subset \{|s_j^\alpha| \le \frac{1}{2}, |w_j^\alpha| \le \frac{1}{2}\}$ for any $1 \le \alpha \le s$. Since in $U_\alpha - Y$ for any $1 \le \alpha \le s$, the Poincaré metric ω_P is asymptotic to

$$\frac{\sqrt{-1}}{2}\left(\sum_{j=1}^{a_\alpha} \frac{ds_j^\alpha \wedge d\bar{s}_j^\alpha}{|s_j^\alpha|^2(\log\frac{1}{|s_j^\alpha|})^2} + \sum_{j=1}^{b_\alpha} dw_j^\alpha \wedge d\bar{w}_j^\alpha\right), \tag{6.3}$$

we have

(6.4)

$$\int_{\mathrm{supp}(\overline{\partial}\rho_\varepsilon)} \omega_P^n \le C\sum_{\alpha=1}^{s}\prod_{j=1}^{a_\alpha}\int_{e^{-\frac{1}{2\varepsilon}}}^{e^{-\frac{1}{\varepsilon}}} \frac{1}{|s_j^\alpha|(\log\frac{1}{|s_j^\alpha|})^2} d|s_j^\alpha| \prod_{j=1}^{b_\alpha}\int_0^{\frac{1}{2}} |w_j^\alpha| d|w_j^\alpha| \le C\varepsilon.$$

So we have

$$\lim_{\varepsilon\to 0}\int_{\mathcal{M}} |d\rho_\varepsilon|^2 \omega_{WP}^n = 0 \tag{6.5}$$

and the result follows. □

7. Self-Adjointness of the Laplacian on Moduli Space

In this section, we will consider the self-adjointness of Laplacian on $(\mathcal{M}, \omega_{WP})$. Let us consider the differential operators d and δ defined on C^1 functions and C^1 forms on $\mathcal{M}$ respectively. We define the domain $Dom(d)$ of d to be the set of C^1 functions f defined on $\mathcal{M}$ such that both f and df are in L^2. Similarly, we define the domain $Dom(\delta)$ of δ to be the set of C^1 1-forms w such that both w and δw are in L^2. We then define the Laplacian Δ with respect to ω_{WP} by Δ with $Dom(\Delta)$ given by the set of C^2 functions f such that $f \in Dom(d)$ and $df \in Dom(\delta)$. In this section, we will prove the closure $\overline{\Delta}$ of Δ is self-adjoint.

THEOREM 7.1. *On $(\mathcal{M}, \omega_{WP})$, the closure $\overline{\Delta}$ of Laplacian on functions is self-adjoint.*

It is proved in [**6**] that $\overline{\Delta}$ is self-adjoint on $M \setminus \Sigma_M$ when M is an algebraic variety with the induced Fubini-Study metric and Σ_M is the singular set at least of real codimension 2. Here our result is different as we are considering the Weil-Petersson metric.

PROOF. By the theorem of Gaffney in [**4**], in order to show $\overline{\Delta}$ is self-adjoint, it is sufficient to prove

$$(df, w) = (f, \delta w) \tag{7.1}$$

for any $f \in Dom(d)$ and $w \in Dom(\delta)$. By Theorem 6.1 and 5.1, we have $W^1(\mathcal{M}) = W_0^1(\mathcal{M})$. Since $Dom(d) \subset W^1(\mathcal{M})$, there exists a sequence $f_n \in \mathcal{D}(\mathcal{M})$ such that $f_n \to f$ in $W^1(\mathcal{M})$. As each f_n has compact support, by integration by parts, we have

$$(df_n, w) = (f_n, \delta w). \tag{7.2}$$

The result follows by taking $n \to \infty$. □

8. An Example

Let (M, g) be a Riemannian manifold. A closed metric ball in $(\overline{M}_c, d)$ excluding an open set containing $\partial_c M$ might not be compact even if $\text{cap}(\partial_c M) = 0$. In this section, we will give a concrete example.

Consider the Riemannian manifold (M, g) as follows. $M = \mathbb{R}^3$ and in terms of the cylindrical coordinates (r, θ, z),

$$g = e^{2z}(dr^2 + f^2(r)d\theta^2 + dz^2). \tag{8.1}$$

Here the function $f \in \mathcal{C}^\infty([0, \infty))$ satisfies the following properties:

- $f(r) = r$ for $r \in [0, \frac{1}{2}]$.
- f is increasing on $[0, 1]$ and $f(1) = 1$.
- f is decreasing on $[1, \infty)$.
- $f(r) = e^{-r}$ for $f \in [2, \infty)$.

For any piecewise smooth curve $l : [a, b] \to M$, we denote the length of l by $\|l\|$, i.e.

$$\|l\| = \int_a^b e^{z(t)} \sqrt{\dot{r}^2(t) + f^2(r(t))\dot{\theta}^2(t) + \dot{z}^2(t)}dt \tag{8.2}$$

And define the distance function d as

$$d(p, q) = \inf_{l \in L} \|l\|,$$

where $L = \{\text{all piecewise smooth curves from } p \text{ to } q\}$. Then we know (M, d) is a metric space.

LEMMA 8.1. *For any $P_1, P_2 \in M$, denote the coordinate of P_i as (r_i, θ_i, z_i) for $i = 1, 2$. Then*

$$d(P_1, P_2) \le e^{z_1} + e^{z_2}. \tag{8.3}$$

PROOF. For any $t_0 < \min(z_1, z_2)$. Define the following three smooth curves.

- $l_1 : (r_1, \theta_1, t)$ for $t \in [t_0, z_1]$ oriented form z_1 to t_0.
- $l_2 : (r_1 + (r_2 - r_1)t, \theta_1 + (\theta_2 - \theta_1)t, t_0)$ for $t \in [0, 1]$.
- $l_3 : (r_2, \theta_2, t)$ for $t \in [t_0, z_2]$.

Then $l_1 \cup l_2 \cup l_3$ is a piecewise smooth curve connecting P_1 and P_2. We can calculate the length of these curves straightforwardly.

$$\begin{aligned}
\|l_1\| &= \int_{t_0}^{z_1} e^t dt = e^{z_1} - e^{t_0}, \\
\|l_3\| &= \int_{t_0}^{z_2} e^t dt = e^{z_2} - e^{t_0}, \\
\|l_2\| &= \int_0^1 e^{t_0} \sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2 f^2(r_1 + (r_2 - r_1)t)} dt \\
&\leq e^{t_0} \sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2}.
\end{aligned}$$

Therefore

$$d(P_1, P_2) \leq e^{z_1} + e^{z_2} - 2e^{t_0} + e^{t_0} \sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2}.$$

Taking $t_0 \to -\infty$, the result follows. □

Define $H_I = \mathbb{R}^2 \times I = \{(r, \theta, z) : z \in I\}$ for any $I \subset \mathbb{R}$. And we will use diam S to denote the diameter of set $S \subset M$.

Corollary 8.2. *diam* $H_{(-\infty,0]} \leq 2$.

Proof. For any $P_1, P_2 \in H_{(-\infty,0]}$, we have $d(P_1, P_2) \leq e^{z_1} + e^{z_2} \leq 2$. □

Lemma 8.3. *For any* $P_1, P_2 \in M$,

$$d(P_1, P_2) \geq |e^{z_1} - e^{z_2}|. \tag{8.4}$$

Proof. For any piecewise smooth curve $l : [0, 1] \to M$ from P_1 to P_2, we have

$$\begin{aligned}
\|l\| &= \int_0^1 e^{z(t)} \sqrt{\dot{r}^2(t) + f^2(r(t))\dot{\theta}^2(t) + \dot{z}^2(t)} dt \\
&\geq \int_0^1 e^z |\dot{z}(t)| dt \\
&\geq |e^{z_1} - e^{z_2}|.
\end{aligned}$$

□

Note that the metric space (M, d) is not complete. $\{(0, 0, -n)\}_{n=1}^{\infty}$ is a Cauchy sequence since $d((0, 0, -m), (0, 0, -n)) \leq e^{-m} + e^{-n}$. But it is not convergent in M.

Theorem 8.1. *Let* $\overline{M}$ *be the completion of* M *with respect to metric* d. *Then* $\overline{M} = M \cup \{\infty\}$ *where* $\{\infty\}$ *is defined as the Cauchy sequence* $\{(0, 0, -n)\}_{n=1}^{\infty}$.

We want show that for any Cauchy sequence $\{P_n\}_{n=1}^{\infty}$, either it is convergent in M or it is equivalent to the Cauchy sequence $\{(0, 0, -n)\}_{n=1}^{\infty}$. We split the proof into following lemmas.

Lemma 8.4. *Let* $\{P_n\}_{n=1}^{\infty}$ *be a Cauchy sequence in* M *and denote* $P_n = (r_n, \theta_n, z_n)$. *Then* $\{z_n\}_{n=1}^{\infty}$ *is either convergent in* $\mathbb{R}$ *or* $\lim_{n\to\infty} z_n = -\infty$.

Proof. By inequality (8.4), we have $d(P_m, P_n) \geq |e^{z_m} - e^{z_n}|$. Therefore $\{e^{z_n}\}$ is a Cauchy sequence in $\mathbb{R}$. So the result follows. □

Lemma 8.5. *Let* $\{P_n\}_{n=1}^{\infty}$ *be a Cauchy sequence in* M *and denote* $P_n = (r_n, \theta_n, z_n)$. *If* $\{z_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $\mathbb{R}$, *then* $\{r_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $\mathbb{R}$.

PROOF. Let $z_0 = \lim z_n$. By dropping finitely many beginning terms, we can assume $z_n \in [z_0 - 1, z_0 + 1]$. Let $\delta = \delta(z_0) = e^{z_0-1} - e^{z_0-2}$. Since $\{P_n\}$ is Cauchy, by dropping more beginning terms, we can assume further that $d(P_m, P_n) < \frac{\delta}{3}$ for any $m, n \in \mathbb{Z}^+$. By the definition of metric d, there exists a piecewise smooth curve $l_{mn} : [0, 1] \to M$ from P_m to P_n such that $\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n)$. We claim

$$\min_{t\in[0,1]} z(t) > z_0 - 2. \tag{8.5}$$

Assume not. Take $t = t_0 \in [0, 1]$ be the first time such that $z(t) = z_0 - 2$, which implies that $z(t) \geq z_0 - 2$ for $t \in [0, t_0]$. Then

$$\|l_{mn}\| \geq \int_0^{t_0} e^{z(t)}|\dot{z}(t)|dt \geq e^{z_m} - e^{z(t_0)} \geq e^{z_0-1} - e^{z_0-2} = \delta.$$

However, according to our assumption on l_{mn}, we have

$$\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n) < \frac{\delta}{2}, \tag{8.6}$$

which is a contradiction and therefore the claim follows. Thus we have

$$\frac{3}{2}d(P_m, P_n) \geq l_{mn} \geq \int_0^1 e^{z(t)}|\dot{r}(t)|dt \geq e^{z_0-2}|r_m - r_n|.$$

Therefore $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. □

LEMMA 8.6. *Let $\{P_n\}_{n=1}^\infty$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. If $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ and $\lim r_n > 0$, then $\{\theta_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$.*

PROOF. Let $z_0 = \lim z_n$ and $r_0 = \lim r_n$. By dropping finitely many beginning terms, we can assume that $z_n \in [z_0 - 1, z_0 + 1]$ and $r_n \in [\frac{1}{2}r_0, \frac{3}{2}r_0]$ for any $n \in \mathbb{Z}$. Define $\delta(z_0) = e^{z_0-1} - e^{z_0-2}$ and $\delta(r_0, z_0) = \frac{1}{4}r_0e^{z_0-2}$. And take $\delta = \min\{\delta(z_0), \delta(r_0, z_0)\}$. By dropping more beginning terms, we can assume further $d(P_m, P_n) < \frac{\delta}{3}$ for any $m, n \in \mathbb{Z}$. Again we take a piecewise smooth curve $l_{mn} : [0, 1] \to M$ from P_m to P_n such that $\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n)$. By the proof in Lemma 8.5, we have $\min z(t) \geq z_0 - 2$. Here we claim

$$r(t) \in [\frac{1}{4}r_0, \frac{7}{4}r_0] \text{ for any } t \in [0, 1]. \tag{8.7}$$

Assume not. Then let $t = t_0$ be the first time such that $r(t_0) = \frac{1}{4}r_0$ or $\frac{7}{4}r_0$. Then

$$\|l_{mn}\| \geq \int_0^{t_0} e^{z(t)}|\dot{r}(t)|dt \geq e^{z_0-2}|r(t_0) - r_m| \geq \frac{1}{4}r_0e^{z_0-2} = \delta(r_0, z_0).$$

But we also have

$$\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n) < \frac{\delta}{2},$$

which is a contradiction. So the claim follows. Therefore

$$\begin{aligned}\frac{3}{2}d(P_m, P_n) \geq \|l_{mn}\| &\geq \int_0^1 e^{z(t)}f(r(t))|\dot{\theta}(t)|dt \\ &\geq e^{z_0-2}\min\{f(\frac{1}{4}r_0), f(\frac{7}{4}r_0)\}|\theta_m - \theta_n|.\end{aligned}$$

It follows that $\{\theta_n\}$ is a Cauchy sequence. □

Lemma 8.7. *Let $P_n = (r_n, \theta_n, z_n)$ be a sequence in M. If $r_n \to r_0, \theta_n \to r_0, z_n \to z_0$ in $\mathbb{R}$, then P_n converges to $P_0 = (r_0, \theta_0, z_0)$ with respect to metric d.*

Proof. Since $z_n \to z_0$ in $\mathbb{R}$. By dropping finitely many beginning terms, we can assume $z_n \in [z_0 - 1, z_0 + 1]$. Define a smooth curve from P_0 to P_n as $l(t) = (r_0 + (r_n - r_0)t, \theta_0 + (\theta_n - \theta_0)t, z_0 + (z_n - z_0)t)$. Then

$$
\begin{aligned}
d(P_0, P_n) \leq \|l\| =& int_0^1 e^{z(t)}\sqrt{(r_n - r_0)^2 + f^2(r_0 + (r_n - r_0)t)(\theta_n - \theta_0)^2 + (z_n - z_0)^2}dt \\
\leq& e^{z_0+1}\sqrt{(r_n - r_0)^2 + (\theta_n - \theta_0)^2 + (z_n - z_0)^2}
\end{aligned}
$$

So the result follows. □

Now we are ready to prove prove Theorem 8.1.

Proof. Let $\{P_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. By Lemma 8.4, we have either $\lim z_n = -\infty$ or $\lim z_n = z_0$ for some $z_0 \in \mathbb{R}$. In the first case, we have

$$d(P_n, (0, 0, -n)) \leq e^{z_n} + e^{-n} \to 0.$$

Therefore Cauchy sequence $\{P_n\}$ and $\{(0, 0, -n)\}$ are equivalent to each other.

In the second case that $z_0 = \lim z_n \in \mathbb{R}$, we can assume $z_n \in [z_0 - 1, z_0 + 1]$ for any $n \in \mathbb{Z}^+$. By Lemma 8.5, we know that $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. Let $r_0 = \lim r_n$. We have two sub-cases, either $r_0 = 0$ or $r_0 > 0$. When $r_0 = 0$, take a smooth curve l from $(0, 0, z_0)$ to P_n as $l(t) = (r_n t, \theta_n t, z_0 + (z_n - z_0)t)$. Then

$$
\begin{aligned}
d((0, 0, z_0), P_n) \leq \|l\| =& \int_0^1 e^{z(t)}\sqrt{r_n^2 + f^2(r_n t)\theta_n^2 + (z_n - z_0)^2}dt \\
\leq& e^{z_0+1}\int_0^1 \sqrt{r_n^2 + 4\pi^2 f^2(r_n t) + (z_n - z_0)^2}dt \\
\to& 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore $P_n \to (0, 0, z_0)$ in M.

In the second sub-case that $r_0 > 0$, by Lemma 8.6, we have that $\lim \theta_n = \theta_0$ for some $\theta_0 \in \mathbb{R}$. Then by Lemma 8.7, we have that P_n converges to $P_0 = (r_0, \theta_0, z_0)$ in M. So the result follows. □

Theorem 8.2. *The capacity of $\partial_c M = \{\infty\} \subset \overline{M}_c$ is zero.*

Proof. Define a decreasing function $\varphi \in \mathcal{C}^{\infty}(\mathbb{R})$ such that

$$\varphi(z) = \begin{cases} 1 & z \leq 0 \\ 0 & z \geq 1. \end{cases}$$

For any $a \in \mathbb{R}$, define $\varphi_a \in \mathcal{C}^{\infty}(M)$ as $\varphi_a(P) = \varphi(z - a)$ for any $P = (r, \theta, z) \in M$. Then $\varphi = 1$ on $H_{(-\infty,a)} = B(\infty, e^a)$ and $\varphi = 0$ outside $H_{(-\infty,a+1)} =$

$B(\infty, e^{a+1})$. Then

$$\begin{aligned}\int_M \varphi_a^2 dV_g &\le \int_{H_{(-\infty,a+1)}} dV_g \\ &= \int_{-\infty}^{a+1}\int_0^{2\pi}\int_0^\infty e^{3z} f(r) dr d\theta dz \\ &= 2\pi e^{3a+3}\int_0^\infty f(r)dr \\ &\to 0, \quad \text{as } a \to -\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_M |\nabla\varphi_a|_g^2 dV_g &= \int_{H_{(a,a+1)}} |\varphi'(z-a)|^2 e^{-2z} dV_g \\ &= \int_a^{a+1}\int_0^{2\pi}\int_0^\infty |\varphi'(z-a)|^2 e^z f(r) dr d\theta dz \\ &\le 2\pi(e^{a+1}-e^a)\sup_{\mathbb{R}}|\varphi'|\int_0^\infty f(r)dr \\ &\to 0, \quad \text{as } a \to -\infty.\end{aligned}$$

Therefore the result follows. □

PROPOSITION 8.8. *Let $o = (0,0,0)$. Then $\overline{B(o,2)} \setminus B(\infty, e^{-1})$ is not compact in M.*

PROOF. By Corollary 8.2, we have

$$\overline{B(o,2)} - B(\infty, e^{-1}) \supset H_{(-\infty,0]} - H_{(-\infty,-1)} = H_{(-1,0]}.$$

Consider the sequence $P_n = (n,0,0)$ in $H_{(-1,0]}$. We claim

$$d(P_m, P_n) \ge \min(e^{-1}, 1-e^{-1}) \quad \text{for any } m \ne n. \tag{8.8}$$

Let $l : [0,1] \to M$ be an arbitrary smooth curve from P_m to P_n. Then either $l \subset H_{(-1,+\infty)}$ or l will hit the plane $z = -1$. In the first case, we have

$$\|l\| \ge \int_0^1 e^{z(t)}|\dot r(t)|dt \ge e^{-1}|r_m - r_n| \ge e^{-1}$$

In the second case, take $t = t_0$ be the first time l hit the plane $z = -1$. Then

$$\|l\| \ge \int_0^{t_0} e^{z(t)}|\dot z(t)|dt \ge e^{z(0)} - e^{z(t_0)} = 1 - e^{-1}.$$

Combining these two cases, we have $\|l\| \ge \min(e^{-1}, 1-e^{-1})$ for any piecewise smooth curve from P_m to P_n. So the claim follows. Therefore, there is no convergent subsequence of $\{P_n\}$ and thus the result follows. □

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