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Orbit classification and asymptotic constants for d-symmetric covers

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Abstract. We show that the parameter space of cyclic degree d covers of a marked Riemann surface (X, x_0) fully branched over two distinguished points is $\mathcal{C}_d(X) = (X_{ab} - H_1(X;\mathbb{Z})\widetilde{x}_0)/dH_1(X;\mathbb{Z})$, where $\pi_{ab} : X_{ab} \to X$ is the universal abelian cover. Pullback of a holomorphic 1-form ω on X gives 1-forms and translation structures on all covers of X. We call degree d cyclic covers with the pull back 1-form d-symmetric covers and use the induced flat geometry on $\mathcal{C}_d(X, x_0, \omega) = (\mathcal{C}_d(X), \pi^*_{ab}\omega)$ to study the geometry of individual d-symmetric covers. The geometry on $\mathcal{C}_d(\mathbb{C}/\mathbb{Z}[i])$ allows a straightforward $\mathrm{SL}_2\mathbb{Z}$ orbit classifiaction for d symmetric torus covers and en route a classification of their Teichmüller curves. For general d-symmetric covers we present formulas for asymptotic quadratic growth rates (Siegel-Veech constants) of geodesic loops and other geodesic segments in terms of the parameter space geometry. Combining the orbit classification and formulas for Siegel-Veech constants, we carry out nearly complete calculations for d-symmetric covers.

1. Introduction and Results

This paper is concerned with geometric problems and classification problems for cyclic covers and studies those from a global and geometric viewpoint. Instead of looking at a particular cover over a fixed base surface, or the (finite) set of, say cyclic torus covers of fixed degree and branching, we view a cover as point in a parameter surface, obtained by varying the relative branching loci. Under our assumptions the parameter surface will itself be a cover of the base surface. A flat metric, induced by a 1-form on the base surface, pulls back to any cover, in particular to the parameter curve. Once equipped with a pull back 1-form we call a cyclic cover of degree d a d-symmetric differential. Using the parameter curves, it is almost elementary to classify Teichmüller curves defined by d-symmetric differentials that cover tori $\mathbb{C}/\Lambda$ equipped with the standard 1-form dz inherited from $\mathbb{C}$. The covering and translation structure of the parameter space has been used in [**S2**, **EMS**] and more recently by Duryev [**D18**] to study Teichmüller curves for genus 2 torus covers.

In general both discovery and classification of Teichmüller curves is challenging. As for the classification part, this is even more true for torus covers since those are abundant. In fact, covers of $\mathbb{T} := \mathbb{C}/\mathbb{Z}[i]$ branched only over rational

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points give Teichmüller curves. Despite recent progress by Eduard Duryev [**D18**] the Teichmüller curve classification is still open in genus 2. Duryev uses the flat geometry of the moduli curve paired with some topological properties. Our initial motivation was to give a class of examples, where the asymptotic growth rate can be can be determined for all points of the parameter curve and not only for generic covers , i.e. those that are not Teichmüller curves, as in [**EMS**]. There is a good amount of literature devoted to Teichmüller curves, such as the Teichmüller curves stemming from the billiard in a regular polygon found by Veech [**V1**], and some induced by genus two surfaces described by Calta [**C**] and independently McMullen [**McM1**],...,[**McM5**]. More examples were found by Bouw and Möller [**BM**] and Hooper [**H**]. Cyclic covers on the other hand can be constructed easily and have been used to built examples with interesting properties and as a periscope to study flat surfaces in higher genus. For results in this direction see [**EKZ11**], [**FMZ1**] and for an application [**FS**].

1.1. Results. Let X be a compact Riemann surface. A cyclic cover $Y \to X$ of degree d is called *d-symmetric*, if it is fully branched over two *distinct* points $x_0 \neq x_1$ in X. A cover is *fully branched*, if all branch points have maximal order d. Let us call two d-symmetric covers $Y_1 \to X$ and $Y_2 \to X$ *isomorphic*, if there exists an orientation preserving homeomorphism $Y_1 \to Y_2$ that is $\mathbb{Z}_d$ equivariant. In addition we identify $\mathbb{Z}_d$ covers that agree up to a change of the $\mathbb{Z}_d$ action by a $\mathbb{Z}_d$ homomorphism, given by multiplication with a unit $\mathbb{Z}_d^*$. Let $\pi_{ab} : (X_{ab}, x_{ab}) \to (X, x_0)$ be the universal abelian cover of X marked in $x_{ab} \in \pi_{ab}^{-1}(x_0)$. The universal abelian cover is the regular cover with deck group $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, g the genus of X. For an integer $d > 1$ we denote the $\mathbb{Z}$-submodule $dH_1(X,\mathbb{Z}) \subset H_1(X,\mathbb{Z})$ by dH_1 and $H_1(X,\mathbb{Z})$ by H_1. Let $\pi_d : X_{ab}/dH_1 \to X$ be the quotient cover and $x_d = x_{ab} + dH_1$. We show:

THEOREM 1. *Fix $x_0 \in X$, then isomorphy classes of d-symmetric covers $\pi : Y \to X$ branched over x_0 and $x \in X \backslash \{x_0\}$ are parameterized by $((X_{ab} - H_1 \cdot x_{ab})/dH_1, x_d)$. If the class of the cover π is given by the point $z_\pi \in X_{ab}/dH_1$, then $\pi_d(z_\pi) = x$.*

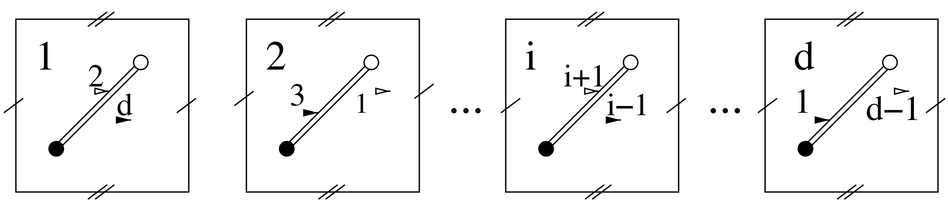

FIGURE 1. A d-symmetric cover of the standard torus.

Torus covers. If X is a complex torus, we may identify it with its Jacobian $Jac(X) \cong \mathbb{C}/\Lambda$, where Λ is the lattice $\{\int_\gamma \omega : \gamma \in H_1(X;\mathbb{Z})\}$ generated by a nontrivial holomorphic one form $\omega \in \Omega(X)$. In particular $H_1(X;\mathbb{Z}) \cong \Lambda$. Put $\mathbb{T}_\Lambda := \mathbb{C}/\Lambda$, and recall the subset of points $\mathbb{T}_\Lambda[n] \subset \mathbb{T}_\Lambda$ that become $[0]$ when multiplied with $n \in \mathbb{N}$ are the *n-torsion points* of $\mathbb{T}_\Lambda$. Denote the n-torsion points, that are not already m-torsion points for some $m|n$ by $\mathbb{T}_\Lambda(n)$ and call them the *primitive n-torsion* points. If $\pi : Y \to \mathbb{C}/\Lambda$ is a cover branched over two distinct points $[z_1]$ and $[z_0]$ define $b_\pi = [z_1 - z_0] \in \mathbb{T}_\Lambda$.

COROLLARY 1 (Torus covers). *Let $\Lambda \subset \mathbb{C}$ be a lattice, then any d-symmetric torus cover $\pi : Y \to \mathbb{C}/\Lambda$ is represented by a point $z_\pi \in (\mathbb{C} - \Lambda)/d\Lambda$, so that $\pi_d(z_\pi) = b_\pi$.*

Tori, that are a quotient of $\mathbb{C}$ by a lattice inherit a natural euclidean metric from $\mathbb{C}$. Since the metric tensor $|dz|^2 = dz \otimes d\bar{z}$ depends only on the differential dz, we obtain a metric on Y via the pullback $\pi^* dz = \omega$. In fact any non trivial holomorphic differential $\omega \in \Omega(X)$ on a Riemann surface X determines a flat metric away from the zeros of X, this pair is denoted by $(X, |\omega|)$. The Riemannian metric given by a 1-form is an euclidean metric with cone point type singularities at the zeros of ω, for more see [**Zo**]. Returning to tori, let $\mathrm{SL}(\Lambda) \subset \mathrm{SL}_2\mathbb{R}$ be the group of orientation preserving real linear maps of $\mathbb{C}$ that stabilize the lattice Λ. These maps descend to $\mathbb{C}/\Lambda$ and define orientation preserving affine homeomorphisms of $\mathbb{C}/\Lambda$ fixing $[0] \in \mathbb{C}/\Lambda$. The group $\mathrm{SL}(\Lambda)$ acts on branched covers of $\mathbb{C}/\Lambda$ by post-composition and acts on the torus $(\mathbb{C} - \Lambda)/d\Lambda$, since the set of removed points is $\mathrm{SL}(\Lambda)$ invariant. The $\mathrm{SL}(\Lambda)$ action on covers and on their parameter space are compatible in the sense:

$$A \cdot \#_z^d(\mathbb{C}/\Lambda, dz) = \#_{Az}^d(\mathbb{C}/\Lambda, dz). \tag{1}$$

Here $\#_z^d(\mathbb{C}/\Lambda, dz)$ denotes a d-symmetric torus cover in the isomorphy class given by $z \in (\mathbb{C} - \Lambda)/d\Lambda$. The dotted $\mathrm{SL}(\Lambda)$-action is the one on covers and Az denotes the action on $\mathbb{C}/\,\mathrm{d}\,\Lambda$ induced from the real linear action of $A \in \mathrm{SL}(\Lambda)$ on $\mathbb{C}$. The following is well known:

PROPOSITION 1. *For any $n \in \mathbb{N}$ the primitive n-torsion points lie on a single* $\mathrm{SL}(\Lambda)$ *orbit. Those are all finite* $\mathrm{SL}(\Lambda)$ *orbits. All other* $\mathrm{SL}(\Lambda)$ *orbits are dense in* $\mathbb{T}_\Lambda$.

One can show this statement about finite orbits by successively applying parabolic elements from $\mathrm{SL}(\Lambda)$ by moving a general primitive n-torsion point into a particular primitive n-torsion point. One can interpret this fact in terms of flat geometry: The complex linear bijective map *multiplication by* $n \in \mathbb{N}$, that is $z \mapsto nz$, maps the integer lattice $\Lambda = \mathbb{Z}[i]$ to the lattice $n\mathbb{Z}[i]$, is $\mathrm{SL}_2\mathbb{Z}$ equivariant and hence induces an $\mathrm{SL}_2\mathbb{Z}$-equivariant bijective map $(\mathbb{C}/\mathbb{Z}[i], dz) \to (\mathbb{C}/n\mathbb{Z}[i], dz)$. Since there is a map in $\mathrm{SL}_2\mathbb{Z}$ mapping $z \in \mathbb{Z}[i]$ to 1, if and only if $\gcd(\mathrm{Re}\, z, \mathrm{Im}\, z) = 1$ the orbit classification is equivalent to the existence of a line segment from 0 to some $z + n\mathbb{Z}[i]$ that does not contain a point of $n\mathbb{Z}[i]$ in its interior. Thus on the torus, there is a line segment from $[0]$ to $[z]$, that is, besides its endpoints, completely contained in $(\mathbb{C} - n^{-1}\mathbb{Z}[i])/\mathbb{Z}[i]$, or in other words: $[z]$ can be *illuminated* from $[0]$ on $(\mathbb{C} - n^{-1}\mathbb{Z}[i])/\mathbb{Z}[i]$. For the integer lattice $\mathbb{Z}[i]$ we note:

COROLLARY 2. *For $d \in \mathbb{N}$ every finite* $\mathrm{SL}_2\mathbb{Z}$ *orbit on $\mathbb{C}/d\mathbb{Z}[i]$ is a set of primitive n torsion points for some $n \in \mathbb{N}$. The stabilizer of the point $[d/n]$ is the congruence group* $\Gamma_1(n) \subset \mathrm{SL}_2\mathbb{Z}$.

These rather elementary observations on the $\mathrm{SL}_2\mathbb{Z}$ action on tori become powerful statements when we interpret those tori as parameter spaces of d-symmetric covers. Using Corollary 2 for instance, one easily obtains the $\mathrm{SL}_2\mathbb{Z}$ orbit decomposition for d-symmetric covers branched over a given rational point. For $n \in \mathbb{Z}$, let

$D(n)$ denote the *number of positive divisors of* n. If further $d \in \mathbb{N}$ let d_n be the maximal divisor of d that is coprime to n, i.e. $d_n|d$ is maximal with $\gcd(d_n, n) = 1$.

THEOREM 2. *[Covers with torsion branching] The set of d-symmetric covers π with $b_\pi \in \mathbb{T}_\Lambda(n)$ consists of d^2 covers that lie on $D(d_n)$ different* $\mathrm{SL}(\Lambda)$ *orbits.*

This statement is shown as part of Proposition 6 on page 211 and Corollary 5 on page 211. The following corollary generalizes the appearance of either two or one $\mathrm{SL}(\Lambda)$-orbits for torsion torus covers in genus 2, see [**Ka1**, **HL**] and [**McM2**], for d-symmetric torus covers:

COROLLARY 3. *If d is prime and $n \in \mathbb{N}$, then the d-symmetric covers with relative branching $b_\pi \in \mathbb{T}_\Lambda(n)$ are contained in one* $\mathrm{SL}(\Lambda)$*-orbit if $d|n$, otherwise* $\gcd(d, n) = 1$ *and the covers lie on two* $\mathrm{SL}(\Lambda)$*-orbits.*

In particular the four genus 2-symmetric covers with relative branching $b_\pi \in \mathbb{T}_\Lambda(n)$ are on one $\mathrm{SL}(\Lambda)$ *orbit, if n is even and on two* $\mathrm{SL}(\Lambda)$ *orbits, if n odd.*

Each finite $\mathrm{SL}(\Lambda)$ orbit of d-symmetric covers is (roughly speaking) the intersection of a single Teichmüller curve with the covers of a fixed torus. For more on Teichmüller curves see section 6. Reformulating Corollary 3 and Theorem 3 in terms of Teichmüller curves gives:

COROLLARY 4. *For any $d, n \in \mathbb{N}$ the Teichmüller curve determined by any primitive n-torsion point of* $(\mathbb{C} - \Lambda)/d\Lambda$ *is isometric to* $\mathbb{H}/\Gamma_1(n)$. *If d is prime and n a multiple of d, then every d-symmetric cover with branching $b_\pi \in \mathbb{T}[n]$ is on one Teichmüller curve. If d is prime and n is not a multiple of d, then there are* 2 *Teichmüller curves containing the covers with branching $b_\pi \in \mathbb{T}(n)$.*

In summary, for d-symmetric torus covers the $\mathrm{SL}(\Lambda)$ orbit classification is equivalent to an, in this case elementary, illumination principle. Consider a set $A \subset X$ of a translation surface (X, ω). We say that a point $p \in \overline{A}$ is *visible* or *illuminated* from another point in $q \in \overline{A}$, if there is a regular line segment, i.e. one that does not contain zeros of ω, from p to q that is contained in A. In words, we require the segment away from its endpoints to be in A. The following states that some torsion points in the space of d-symmetric covers cannot be illuminated.

PROPOSITION 2. *If $d \in \mathbb{N}$, the only points on* $(\mathbb{C} - \Lambda)/d\Lambda$ *that are not illuminated from* $[0]$ *are those primitive n torsion points with $n < d$ so, that n does not divide d.*

To explain the relevance of illumination, we look at the above spaces as parameter spaces of certain covers of a surface, say X. In this case a given point z in parameter space represents a cover $X_z \to X$ and a line illuminating z projects to a line segment $s \subset X$ in the complement of the zero set of $\omega \in \Omega(X)$. This regular line segment allows us to construct the translation cover X_z via a cutting, copy and paste construction along s. The existence of such an illuminating segment allows us to apply maps from $\mathrm{SL}(X) \subset \mathrm{SL}_2\mathbb{R}$, if non trivial, to move the cover around in parameter space and shorten, in the finite orbit case even minimize, the length of the line segment.

1.2. Counting line segments on d-symmetric torus covers. The standard foliation of $\mathbb{R}^2$ by oriented parallel lines tangent to $\theta \in S^1$ descend to line foliations $\mathcal{F}_\theta(\mathbb{C}/\Lambda)$ on the translation torus $\mathbb{C}/\Lambda$. By pullback we obtain direction foliations on every (translation) cover of a translation torus. Direction foliations can be defined on any translation surface, see section 3. We are mainly interested in two types of leaves of direction foliations, the *compact leaves* and the *saddle connections.* A saddle connection is a leaf that does not contain any zero of ω, or marked point, but is bounded at both ends by those. Compact leaves on the other hand appear in families of parallel, isotopic loops that cover a maximal open cylinder bounded by saddle connections. The *length* of a compact leaf equals the circumference, or *width* of the cylinder the leaf determines with respect to the induced euclidean metric.

Below we restrict our presentation to d-symmetric covers $\#_z^d(\mathbb{T}, dz)$ of $\mathbb{T}$. Let us consider the counting functions:

$$N_C(\#_z^d(\mathbb{T}, dz), T) := \#\{\text{iso. classes of cpt. leaves on } \#_z^d(\mathbb{T}, dz) \text{ of length} \leq T\}$$

and its equivalent $N_{SC}(\#_z^d(\mathbb{T}, dz), T)$ for saddle connections. For the torus $\mathbb{T}$ itself one has the elementary $\lim_{T\to\infty} \frac{N_C(\mathbb{T},T)}{T^2} = 6/\pi$ obtained by counting integer lattice points visible from the origin. This quadratic asymptotic constant exists for d-symmetric covers as well, and can be expressed using flat geometric data of the parameter space equipped with the pullback flat structure from $\mathbb{T}$. The general formalism is presented and applied to genus 2 torus covers in [**EMS**]. Below we calculate the normalized quadratic constants $\mathbf{c}_C(S) := \frac{\pi}{6} \lim_{T\to\infty} \frac{N_C(S,T)}{T^2}$ for d-symmetric torus covers. Those constants are generally known as Siegel-Veech constants, see [**V3**, **EMZ**] and [**EMS**]. Foundational work on the asymptotic growth rates, such as quadratic estimates, has been done by Howard Masur [**Ma1**, **Ma2**, **Ma3**] and William Veech [**V1**, **V2**, **V3**].

Since the Siegel-Veech constants are independent of the (unimodular) lattice, we restrict some of the following statements to the integer lattice. If $\Lambda = \mathbb{Z}[i]$ the horizontal direction on $\mathbb{T}$ is periodic and so the horizontal direction of $(\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$ decomposes into d maximal cylinders foliated by horizontal leaves. More precisely, denote by $\mathscr{C}_k \subset (\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$ the maximal open cylinder containing the image of the point $(k - 1/2)i \subset \mathbb{C}$ under the covering map $\mathbb{C} - \mathbb{Z}[i] \to (\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$. Further denote the upper boundary of the cylinder $\mathscr{C}_k$ by $\partial^{top}\mathscr{C}_k$. Each boundary contains d saddle connections. Attached to each cylinder and boundary is a datum, say $(w_{k1}, ..., w_{kn_k}) \in \mathbb{N}^{n_k}$ for $\mathscr{C}_k$ and $(b_{k1}, ..., b_{km_k}) \in \mathbb{N}^{m_k}$ for $\partial\mathscr{C}_k$, that records the length of every horizontal cylinder on each d-symmetric differential $\#_z^d(\mathbb{T}, dz)$ whenever $z \in \mathscr{C}_k$ and $z \in \partial\mathscr{C}_k$ respectively. This definition uses, that the circumference of horizontal cylinders on $\#_z^d(\mathbb{T}, dz)$ only depend on the cylinder $\mathscr{C}_k$, or saddle connection $\partial\mathscr{C}_k$ that contains z. Any periodic direction on $(\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$ has a rational slope, so that there is an element of $\mathrm{SL}_2\mathbb{Z}$ mapping it to the horizontal direction.

THEOREM 3. **[Cylinders]** *Under the previous assumptions and conventions, if* $z \in (\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$ *has infinite* $\mathrm{SL}_2\mathbb{Z}$ *orbit, the Siegel-Veech constant for maximal cylinders for* $\#_z^d(\mathbb{T}, dz)$ *is:*

$$\mathbf{c}_C(\#_z^d(\mathbb{T}, dz)) = \frac{1}{d} \sum_{k=1}^{d} \sum_{j=1}^{n_k} \frac{1}{(w_{jk})^2}. \tag{2}$$

If $z \in (\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$ *has finite* $\mathrm{SL}_2\mathbb{Z}$ *orbit, say* $\mathscr{O}_z := \mathrm{SL}_2\mathbb{Z} \cdot z$*, then:*

$$\mathbf{c}_C(\#_z^d(\mathbb{T}, dz)) = \frac{1}{|\mathscr{O}_z|} \sum_{k=1}^{d} \left(\sum_{j=1}^{n_k} \frac{|\mathscr{O}_z \cap \mathscr{C}_k|}{(w_{kj})^2} + \sum_{j=1}^{m_k} \frac{|\mathscr{O}_z \cap \partial^{top}\mathscr{C}_k|}{(b_{kj})^2} \right). \tag{3}$$

We show these formulas for general d-symmetric differentials in section 5.

To calculate the Siegel-Veech constants in Theorem 3 for d-symmetric torus covers we need the circumferences and multiplicities of horizontal cylinders for each surface in a cylinder $\mathscr{C}_k$.

PROPOSITION 3. *The d-symmetric surface* $\#_z^d(\mathbb{T}, dz)$ *has the following horizontal cylinder decomposition:*

$$\left.\begin{array}{l} \gcd(k,d) \textit{ cylinders of width } \frac{d}{\gcd(k,d)} \textit{ and} \\ \gcd(k+1,d) \textit{ cylinders of width } \frac{d}{\gcd(k+1,d)} \end{array}\right\} \quad \textit{if } z \in \mathscr{C}_k,$$

and

$$\gcd(k,d) \textit{ cylinders of width } \frac{d}{\gcd(k,d)}, \qquad \textit{if } z \in \partial^{top}\mathscr{C}_k$$

It remains to determine how many points of each finite $\mathrm{SL}_2\mathbb{Z}$ orbit happen to be in a particular horizontal cylinder, or on a particular saddle connection in $(\mathbb{C}/d\mathbb{Z}[i], d\mathbb{Z}[i])$. This question is considered in section 10.

Weighted counting formulas. In [**EKZ11**,**EKZ14**] it was shown, that *Lyapunov exponents* of the Teichmüller geodesic flow and Siegel-Veech constants are related. For this relation one needs to count cylinders weighted by their area:

$$N_{\mathrm{C},\alpha}(S,T) := \sum_{\substack{C \in \mathrm{C}(S) \\ |C| < T}} \mathrm{area}^{\alpha}(C),$$

where $\mathrm{C}(S)$ denotes the set of cylinders on S. Note, that for $\alpha = 0$ this is the counting problem that Siegel-Veech constant given by formula 2 in Theorem 3. The normalized Siegel-Veech constant is again defined as:

$$\mathbf{c}_{\mathrm{C},\alpha}(S) = \frac{\pi}{6} \lim_{T \to \infty} \frac{N_{\mathrm{C},\alpha}(S,T)}{T^2}.$$

Below we use Euler's totient: $\varphi(p) = \#\{j \in \mathbb{Z}_d : \gcd(j,p) = 1\}$.

THEOREM 4. *The Siegel-Veech constants for the weighted asymptotic quadratic growth rate of cylinders on* $\#_z^d(\mathbb{T}_\Lambda, dz)$ *with* $z \in \mathbb{C}/d\Lambda$ *of infinite* $\mathrm{SL}(\Lambda)$*-orbit is:*

$$\mathbf{c}_{C,\alpha}(\#_z^d(\mathbb{T}, dz)) = \frac{2}{1+\alpha} \sum_{p|d} \frac{\varphi(p)}{p^{3-\alpha}}. \tag{4}$$

Note, that $\alpha \mapsto \mathbf{c}_{C,\alpha}$ is continuous. For $\alpha = 0$, suppressing its dependence on d, we obtain the standard cylinder Siegel-Veech constant:

$$\mathbf{c}_C = \mathbf{c}_{C,0} = 2 \sum_{p|d} \frac{\varphi(p)}{p^3}. \tag{5}$$

If on the other hand $\alpha = 1$, that is weighing with the cylinder area, we obtain the sum of Lyapunov exponents [**EKZ11**]:

$$\sum_{i=1}^{d} \lambda_i = \mathbf{c}_{C,1} + \frac{1}{6}\frac{d^2-1}{d} = \sum_{p|d} \frac{\varphi(p)}{p^2} + \frac{1}{6}\frac{d^2-1}{d}. \tag{6}$$

During the time this manuscript went through the review process, the author learned that David Aulicino has independently calculated these and other *area Siegel-Veech constants*. For a general evaluation of generic area Siegel-Veech constants for branched cyclic torus covers see the forthcoming article [**AS**]. For a background on Lyapunov exponents, see [**EKZ11**]. In section 10 and section 11 we evaluate quadratic constants for (some) d-symmetric differentials with finite orbit.

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2. Background on translation surfaces

Below we recall briefly some standard terminologies in the theory of translation surfaces. For a more comprehensive background this topic see [**MT**] and [**Zo**]. In this section we assume X is a Riemann surface, either compact, or compact after adding finitely many points. As in the introduction we consider a metric on X defined by a holomorphic 1-form $\omega \in \Omega(X)$. More generally the 1-form induces a *translation structure* on $\mathring{X} := X \backslash Z(\omega)$, where $Z(\omega)$ is the zero set of ω, as follows. Around every point $x_0 \in \mathring{X}$ $\zeta_0(x) := \int_{x_0}^{x} \omega$ defines *natural* charts. The maximal atlas associated to those charts is called *translation structure*. Indeed, in natural charts the coordinate changes are translations $\zeta_0 = \zeta_1 + c$, $c \in \mathbb{C}$, because $d\zeta_0 = \omega = d\zeta_1$. A translation structure allows us to pull back all translation invariant geometric objects of $\mathbb{R}^2$ to $\mathring{X}$. First, the euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$ is translation invariant and defines a flat metric $|\omega|$ on $\mathring{X}$. Next, the translation invariant line field on $\mathbb{C}$ in a given direction $\theta \in S^1 \subset \mathbb{C}$, viewed as points on the unit circle, defines a directed line field $\mathcal{F}_\theta(\omega)$ on $\mathring{X}$. Note that θ is well-defined on the surface since chart changes are translations. In particular we have a canonical trivialization of the (unit) tangent bundle on $\mathring{X}$. Finally, dz is invariant under translations and defines a 1-form on $\mathring{X}$. That form is ω, since $dz = d\zeta = \omega$. All pullback structures extend to the whole surface X with some care. The metric's $|\omega|$ continuation has cone point singularities in $Z(\omega)$: An order n zero of ω is a metric cone point of total angle $2(n+1)\pi$. Near a zero of order n, ω looks like $z^n dz$. All objects, including the line fields can be defined directly using ω, for example $\mathcal{F}_\theta(\omega) = \ker \operatorname{Im} e^{-i\theta}\omega$ with leaves l directed so that $\int_l \operatorname{Re} e^{-i\theta}\omega > 0$. A pair (X, ω) is called a translation surface. The leaves of any direction foliation of (X, ω) are traces of geodesics and it is convenient to identify them with the respective geodesic. A leaf of a direction foliation is called *regular*, if it does not contain a singular point, or removed point. The leaves that start and terminate at singular points, are called *saddle connections*. For our purpose it is useful to extend that definition to leaves that start and terminate at marked points or removed points of (X, ω). Closed regular leaves on (X, ω)

appear in families of isotopic parallel lines. A maximal family occupies a region on (X, ω) that is isometric to an open cylinder $\mathbb{R}/w\mathbb{Z} \times (0, h)$, where the leaves are represented by the loops given by the level sets of the projection from $\mathbb{R}/w\mathbb{Z} \times (0, h)$ on $(0, h)$. In particular the *width* w of the cylinder is the $|\omega|$-length of any leaf in the family. The boundary of a maximal cylinder consists of saddle connections. The pull back ω along any (branched) covering map $\pi : Y \to X$ of Riemann surfaces, gives a translation structure $(Y, \pi^*\omega)$ on Y. We call the map of pairs a translation covering. Note, that if a direction on (X, ω) contains a cylinder of closed leaves, then this is true for the same direction on all translation covers of (X, ω). The cylinder width on a cover is an integer multiple of the width of its image cylinder on (X, ω). By eventually considering the maximal cylinders of $(X, \{\text{branch points}\})$ with branch points marked, the height of a cylinder and of any of its preimages on a cover are the same.

2.1. Affine group and Veech group. Let us consider the group $\mathrm{SL}_2\mathbb{R}$ acting real linearly on $\mathbb{C}$. Then it acts by post-composition of charts on translation structures to give a new translation structure on X. The translation structure obtained by postcomposition with $A \in \mathrm{SL}_2\mathbb{R}$ is characterized by the 1-form $A\omega$. An *affine map* $\phi : (X, \omega) \to (X, \omega)$ is orientation preserving homeomorphism of X that is affine linear in natural charts (defined by the translation structure). If X is connected, than its derivative $D\phi$ is necessary constant and it is not hard to see $D\phi \in \mathrm{SL}_2\mathbb{R}$. The affine maps of (X, ω) are a group denoted by $\mathrm{Aff}^+(X, \omega)$, its image $\mathrm{SL}(X, \omega) := D\,\mathrm{Aff}^+(X, \omega) \subset \mathrm{SL}_2\mathbb{R}$ is commonly called the *Veech group* of (X, ω).

Two translation structures defined by (X, ω_1) and (X, ω_2) are *equivalent*, if there is an affine map $\phi : X \to X$, with respect to natural charts in each 1-form, so that $\phi^*\omega_2 = D\phi \cdot \omega_1$. Here $D\phi \cdot \omega_1 := (1, i) D\phi(\mathrm{Re}\,\omega, \mathrm{Im}\,\omega)^T$ with $D\phi$ in matrix representation. If in particular $\omega_1 = \omega_2 = \omega$ then we have that (X, ω) and $(X, A\omega)$ induce the same translation structure whenever $A \in \mathrm{SL}(X, \omega)$. The affine group of a fixed translation surface (X, ω) acts on covers of (X, ω) by postcomposition.

We call two translation covers of equivalent, if the diagram

$$
\begin{array}{ccc}
(Y_1, \tau_1) & \xrightarrow{\widetilde{\psi}} & (Y_2, \tau_2) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
(X, \omega_1) & \xrightarrow{\psi} & (X, \omega_2)
\end{array}
\tag{7}
$$

commutes. Here $\psi : (X, \omega_1) \to (X, \omega_2)$ is an affine map and consequently $\widetilde{\psi}$ is an affine lift of ψ. All covering maps are translation maps. In other words, the 1-forms in the upper row of the diagram are pull backs. If $\omega = \omega_1 = \omega_2$, we must have $D\psi = D\widetilde{\psi} \in \mathrm{SL}(X, \omega)$ and $D\widetilde{\psi} \cdot \tau_1 = \widetilde{\psi}^*\tau_2$.

The group $\mathrm{SL}_2\mathbb{R}$ acts on the translation structure of a cover of (X, ω) by post-composing cover and base with the same element of $\mathrm{SL}_2\mathbb{R}$. If $A \in \mathrm{SL}_2\mathbb{R}$ is so that $A = D\psi$ for some $\psi \in \mathrm{Aff}^+(X, \omega)$, then post-composition of the deformed translation structure gives a new translation covering. It is equivalent to the old one in the sense above, if ψ lifts.

Lattice surfaces and optimal dynamics. A translation surface (X, ω) is called a *lattice surface*, or *Veech surface*, if $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2\mathbb{R}$. This

property is equivalent for the volume of $\mathrm{SL}_2\mathbb{R}/\mathrm{SL}(X,\omega)$ being finite, or the hyperbolic area of $\mathcal{H}/\mathrm{SL}(X,\omega)$ being finite. A branched cover $f : Y \to X$ of a lattice surface (X,ω) is itself a lattice surface $(Y, f^*\omega)$, if f is ramified over points with finite $\mathrm{Aff}^+(X,\omega)$ orbit, such as the zeros of ω. It is rather generally difficult to find lattice surfaces, that are not already covers of lattice surfaces, for the description of several families see [**V1**, **C**, **McM1**] and [**BM**].
Example. If $\Lambda \subset \mathbb{C}$ is a lattice then the torus $\mathbb{C}/\Lambda$ is a lattice surface with Veech group conjugate to $\mathrm{SL}_2\mathbb{Z}$. The group $\mathrm{SL}_2\mathbb{Z}$ is the Veech group of the standard torus $\mathbb{C}/\mathbb{Z}[i]$. The action of the group $\mathrm{SL}_2\mathbb{Z}$ on the torus is induced by its linear action on $\mathbb{R}^2$, in particular all rational points are periodic.

Lattice surfaces have a property known as *optimal dynamics*. That is, in a given direction on a lattice surface (X,ω) all leafs of the foliation $\mathcal{F}_\theta(X,\omega)$ are either compact, or dense. In fact, for a dense direction the directional (speed 1) flow is ergodic with respect to the flow invariant measure induced by Lebesgue measure on $\mathbb{C}$. In a direction with compact leaves those form maximal open cylinders of parallel periodic leaves bounded by saddle connections. It is common to call those directions completely periodic.

For the calculation of quadratic growth rates it is an important property of a lattice surface (X,ω), that the set of completely periodic directions decomposes into *finitely* many $\mathrm{SL}(X,\omega)$ orbits. The *holonomy vector* $\mathrm{hol}(C) \in \mathbb{R}^2$ is the vector in direction of a cylinder of periodic leaves, that has modulus the width of the cylinder C. Because $\phi \in \mathrm{Aff}^+(X,\omega)$ maps cylinders to cylinder and acts locally linear on $\mathbb{R}^2$ via their constant derivative we have $\mathrm{hol}(\phi C) = D\phi\,\mathrm{hol}(C)$.

For covers it is convenient to use a relative version of the coverings Veech group. For a given cover $(Y,\tau) \to (X,\omega)$, the *relative Veech group* $\mathrm{SL}(Y/X,\tau) \subset \mathrm{SL}(X,\omega)$ is the group of derivatives of affine maps that lift to affine maps of (Y,τ), i.e. lifts so that diagram 7 with $(Y,\tau) = (Y_1,\tau_1) = (Y_2,\tau_2)$ commutes in the sense described.

Most important for our considerations is, if (X,ω) is a *lattice surface*, a quadratic asymptotics $\lim_{T\to\infty} \frac{N_V((X,\omega),T)}{T^2}$ exists for virtually all relevant leaves of finite length and can be calculated by Veech's formula, see [**V2**] and [**GJ**]. More generally, quadratic constants exist for all branched covers of lattice surfaces and can be calculated as in [**EMS**].

To evaluate quadratic growth rates of d-cyclic covers of a lattice surface (X,ω) equipped with the pullback 1-forms and metrics, which we will call *d-symmetric covers* (of (X,ω)), we involve the parameter space of d-cyclic covers and note:

Finite $\mathrm{SL}_2\mathbb{Z}$ orbits on tori

To calculate quadratic growth rates for d-symmetric torus covers we need an $\mathrm{SL}_2\mathbb{Z}$ orbit classification of their parameter space. Granting the statements in the introduction this $\mathrm{SL}_2\mathbb{Z}$ orbit classification can be extracted from the standard $\mathrm{SL}_2\mathbb{Z}$ action on the translation torus $\mathbb{T}_{[0]} = (\mathbb{C}/\mathbb{Z}[i], [0], dz)$, marked in $[0]$. The discussion generalizes to tori $\mathbb{C}/\Lambda$ defined by other unimodular lattices Λ without adding essential new ideas.

The group of orientation preserving affine diffeomorphisms $\mathrm{Aff}^+(\mathbb{T}, dz)$ of $(\mathbb{T}, dz)$ is isomorphic to the group $\mathrm{SL}_2\mathbb{Z} \ltimes \mathbb{C}/\mathbb{Z}[i]$ with composition rule

$$(A,[a]) \circ (B,[b]) = (A \cdot B, [b + Aa]) \quad \text{where } a,b \in \mathbb{C} \;\text{ and } A,B \in \mathrm{SL}_2\mathbb{Z}.$$

The two parts of $\mathrm{Aff}^+(\mathbb{T}, dz)$ are seen in

$$(8) \qquad 0 \longrightarrow \mathbb{C}/\mathbb{Z}[i] \xrightarrow{\ i\ } \mathrm{SL}_2\mathbb{Z} \ltimes \mathbb{C}/\mathbb{Z}[i] \xrightarrow{\ \mathrm{D}\ } \mathrm{SL}_2\mathbb{Z} \longrightarrow 1$$

We eliminate the continuous subgroup of translations $\mathrm{Aut}(\mathbb{T}, dz) \cong \mathbb{C}/\mathbb{Z}[i]$ by making the origin $[0] \in \mathbb{T}$ a fixed point.

Let us denote the subgroup $\mathrm{SL}(\mathbb{T}_{[x]}, dz) \subset \mathrm{SL}_2\mathbb{Z}$ to be the set of linear maps that stabilize the point $[x] \in \mathbb{T}$. While in many cases the projection of $\mathrm{SL}(\mathbb{T}_{[x]}, dz)$ to $\mathrm{PSL}_2(\mathbb{R})$ is the called the Veech group of the two marked torus $\mathbb{T}_{[x]} = (\mathbb{C}/\mathbb{Z}[i], [0], [x])$, we will regard $\mathrm{SL}(\mathbb{T}_{[x]}, dz)$ as the Veech group of $(\mathbb{T}_{[x]}, dz)$. So we *distinguish* the marked points. The group $\mathrm{SL}(\mathbb{T}_{[x]}, dz)$ is a lattice in $\mathrm{SL}_2\mathbb{R}$, if and only if $[x] = x + \mathbb{Z}[i]$ is rational. More precisely:

PROPOSITION 4. *Given* $a, b, n \in \mathbb{N}_0$ *with* $\gcd(a, b, n) = 1$ *then the stabilizer* $\mathrm{SL}(\mathbb{T}_{[x]}, dz)$ *of* $[x] = [\frac{a}{n}, \frac{b}{n}]$ *is conjugate to*

$$\Gamma_1(n) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\mathbb{Z} : \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mod n \right\} \subset \mathrm{SL}_2\mathbb{Z}.$$

In particular for $[x] = [\frac{1}{n}, 0]$ *we have* $\mathrm{SL}(\mathbb{T}_{[x]}, dz) = \Gamma_1(n)$.

This is an easy exercise, see **[S1]**.

For $[\frac{1}{n}, 0] \in \mathbb{T}$ we have

$$(9) \qquad \mathrm{SL}_2\mathbb{Z} \cdot \left[\frac{1}{n}, 0\right] = \left\{ \left[\frac{a}{n}, \frac{b}{n}\right] \in \mathbb{T} : \ a, b, n \in \mathbb{Z} \text{ with } \gcd(a, b, n) = 1 \right\}$$

in particular

$$[\Gamma_1(n) : \mathrm{SL}_2\mathbb{Z}] = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) = \varphi(n)\psi(n).$$

The last product is taken over all prime divisors p of n. The two functions on the right are the well known *Euler* φ *function* and the *Dedekind* ψ *function*:

$$(10) \qquad \varphi(n) := n \prod_{p|n} \left(1 - \frac{1}{p}\right), \quad \psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

2.2. Torsion points on tori. For the torus $\mathbb{T}_d = \mathbb{C}/d\mathbb{Z}[i]$ any $m \in \mathbb{N}$ defines a group homomorphism $\mathbb{T}_d \to \mathbb{T}_d$ by $[z]_d \mapsto m \cdot [z]_d$ ($[z]_d := z + d\mathbb{Z}[i]$). The kernel of this m-homomorphism is

$$\mathbb{T}_d[m] := \ker(\mathbb{T}_d \xrightarrow{\ m\ } \mathbb{T}_d) = \frac{d}{m}\mathbb{Z}[i]/d\mathbb{Z}[i].$$

The point in the kernel are called *torsion points*. The *order* of a torsion point $[z]_d \in \mathbb{T}_d$ is the smallest $m \in \mathbb{N}$ such that $m \cdot [z]_d = 0 \in \mathbb{T}_d$. Denote the set of torsion points of order m on $\mathbb{T}_d$ by $\mathbb{T}_d(m)$. On the standard torus $\mathbb{T} = \mathbb{T}_1$ we have by equation 9

$$(11) \qquad \mathbb{T}(m) = \mathrm{SL}_2\mathbb{Z} \cdot [1/m, 0].$$

The rescaling map $\mathbb{T}_d \xrightarrow{\ d^{-1}\ } \mathbb{T}$, given by

$$[z]_d = z + d\mathbb{Z}[i] \mapsto d^{-1}(z + d\mathbb{Z}[i]) = d^{-1}z + \mathbb{Z}[i] = [d^{-1}z]$$

is $\mathrm{SL}_2\mathbb{Z}$ equivariant and identifies torsion points of order m.

Let $n = p_1^{l_1} \cdot p_2^{l_2} \dots p_r^{l_r}$ be the *prime-factor decomposition* for $n \in \mathbb{N}$. Then the number of positive *divisors* of n is

$$D(n) := (l_1 + 1) \cdots (l_r + 1).$$

PROPOSITION 5. *The real linear action of* $\mathrm{SL}_2\mathbb{Z}$ *on* $\mathbb{T}_d$ *restricted to* $\mathbb{T}_d[m] = \frac{d}{m}\Lambda/d\Lambda \subset \mathbb{T}_d$ *contains precisely* $D(m)$ *orbits.*

PROOF. After taking the $\mathrm{SL}_2\mathbb{Z}$ equivariant map from $\mathbb{T}_d$ to $\mathbb{T}$, we need to count the $\mathrm{SL}_2\mathbb{Z}$ orbits on the m-torsion points $\mathbb{T}[m]$. Since by the $\mathrm{SL}_2\mathbb{Z}$-orbit classification for $\mathbb{T}$, $\mathrm{SL}_2\mathbb{Z} \cdot [1/n, 0] = \mathbb{T}(n)$ for any $n \in \mathbb{N}$ and $\mathbb{T}[m] = \bigsqcup_{n|m} \mathbb{T}(n)$ the statement follows. □

For every $d > 1$ we consider the map $\pi_d : \mathbb{T}_d \to \mathbb{T}$ defined by taking $[z]_d \in \mathbb{T}_d$ modulo $\mathbb{Z}[i]$. We want to classify the $\mathrm{SL}_2\mathbb{Z}$ orbits containing the preimage $\pi_d^{-1}[1/m, 0] \subset \mathbb{T}_d$ of the point $[1/m, 0] \in \mathbb{T}$.

PROPOSITION 6. *For* $m \in \mathbb{N}$ *consider* $[1/m, 0] \in \mathbb{T}$. *Then for given* $d \in \mathbb{N}$ $\mathrm{SL}_2\mathbb{Z}(\pi_d^{-1}[1/m, 0]) \subset \mathbb{T}_d$ *is the union of* $D(d_m)$ *orbits.*

PROOF. Represent the set $\pi_d^{-1}[1/m, 0]$ by $\{(k, l + 1/m) \in \mathbb{R}^2 : 0 \le k, l < d\}$. Rescaling with d^{-1} transforms this becomes $\{(km/dm, (lm + 1)/dm) \subset \mathbb{R}^2 : 0 \le k, l < d\}$ this represents points on $\mathbb{T}$ and we need the $\mathrm{SL}_2\mathbb{Z}$ orbits through these points on $\mathbb{T}$. First let us look which values $\gcd(km, lm + 1, dm)$ will take when the integers k and l range between 0 and $d - 1$. Since the middle term of $\gcd(km, lm + 1, dm)$ leaves remainder 1 modulo any divisor of m, $\gcd(km, lm + 1, dm)$ is relatively prime to m. We claim that any divisor of d_m is attained in the set $\{\gcd(km, lm + 1, dm) : 0 \le k, l \le d - 1\}$. Indeed, given $p|d_m$, take $k = p$. Since $\gcd(p, m) = 1$, we get a full set of remainders $lm + 1 \mod p$ for $0 \le l \le d - 1$. Thus, for some $l = l_p$ we have $p|l_p m + 1$ By the $\mathrm{SL}_2\mathbb{Z}$ orbit classification for $\mathbb{T}$ $[p/d, (l_p m + 1)/dm] \in \mathrm{SL}_2\mathbb{Z}[p/dm, 0]$ and there are $D(d_m)$ such orbits. □

It follows from this proposition that the maximal number of $\mathrm{SL}_2\mathbb{Z}$ orbits generated by $\pi_d^{-1}[1/m, 0] \subset \mathbb{T}_d$ is achieved when $\gcd(d, m) = 1$ and it is $D(d)$.

COROLLARY 5. *If* $\gcd(d, m) = 1$, *then the set* $\pi_d^{-1}[1/m, 0] \subset \mathbb{T}_d$ *lies on* $D(d)$ $\mathrm{SL}_2\mathbb{Z}$*-orbits. If* $d|m^n$ *for some* $n \in \mathbb{N}$ *the set* $\pi_d^{-1}[1/m, 0]$ *lies on one* $\mathrm{SL}_2\mathbb{Z}$ *orbit. In particular if* $d = 2$, *then the points in* $\pi_2^{-1}[1/m, 0] \subset \mathbb{T}_2$ *lie on one* $\mathrm{SL}_2\mathbb{Z}$ *orbit if and only if* m *is even, otherwise there are two* $\mathrm{SL}_2\mathbb{Z}$ *orbits through this set.*

Note, that $d|m^n$ for some $n \in \mathbb{N}$ is equivalent to: For any prime p, so that $p|d$ also $p|m$.

3. Parameter spaces of cyclic covers

Cyclic covers defined by homology classes. Given a group G, we call a cover $Y \to X$ *G-cover* if G acts properly discontinuously on Y with quotient space $X = Y/G$. With $\mathring{X} = X \backslash \{x_0, ..., x_n\}$ we consider $\mathbb{Z}$-covers of $\mathring{X}$, X a (compact) Riemann surface. Those covers are parameterized by $\mathrm{Hom}(\pi_1(\mathring{X}, x), \mathbb{Z})$. Using commutativity of $\mathbb{Z}$ and the Hurewicz isomorphism

$$\pi_1(\mathring{X}, x)/[\pi_1(\mathring{X}, x), \pi_1(\mathring{X}, x)] \cong H_1(\mathring{X}; \mathbb{Z}),$$

one finds $\mathrm{Hom}(\pi_1(\mathring{X}, x), \mathbb{Z})$ is $\mathrm{Hom}(H_1(\mathring{X}; \mathbb{Z}), \mathbb{Z})$ as a set. Algebraic intersection defines a non-degenerate pairing

$$\langle \cdot\ , \cdot \rangle\ : H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}) \times H_1(\mathring{X}; \mathbb{Z}) \to \mathbb{Z}$$

using that we can represent any $\mathbb{Z}$-cover by a relative homology class. The same remains true if we replace $\mathbb{Z}$ by $\mathbb{Z}_d$ in the above homology groups and take the intersection modulo d. One assigns an (eventually disconnected) $\mathbb{Z}_d$ (respectively $\mathbb{Z}$) cover to an element of $H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d)$ as follows.

A class $\gamma \in H_1(X, \{x_0, ..., x_n\}; \mathbb{Z})$ characterizes a cover $p : X_\gamma \to X$ with deck group $\mathbb{Z}$ by prescribing how loops lift from X to X_γ through intersection with γ. In fact, a lift $\widetilde{\sigma}$ of a loop $\sigma : [t_0, t_1] \to X$ to X_γ is determined by its deck shift $\widetilde{\sigma}(t_1) = \langle \gamma, [\sigma] \rangle \cdot \widetilde{\sigma}(t_0)$. Here $[\sigma] \in H_1(X; \mathbb{Z})$ denotes the homology class defined by σ and $\cdot$ denotes the action of $\mathbb{Z}$ as group of deck-transformations. For such a cover on the other hand, if $\widetilde{\rho} : [t_0, t_1] \to X_\gamma$ is a curve with $\widetilde{\rho}(t_1) = n \cdot \widetilde{\rho}(t_0)$ for some $n \in \mathbb{Z}$, then $n = \langle \gamma, [\pi \circ \widetilde{\rho}] \rangle$ where $[\pi \circ \widetilde{\rho}] \in H_1(X; \mathbb{Z})$ is the homology class of the loop $\pi \circ \widetilde{\rho}$ on X. Covers with deck group $\mathbb{Z}_d$ are obtained as quotients from covers with deck group $\mathbb{Z}$. Equivalently $\mathbb{Z}_d$ covers are defined by considering deck-shifts of lifted curves modulo d. Not all covers characterized by (relative) homology classes in this fashion are connected. The cover associated to $2[\gamma] \in H_1(X; \mathbb{Z})$, where $[\gamma] \in H_1(X; \mathbb{Z})$ is a non-trivial class of a simple loop, for example, is not connected.

Proposition 7. *Let*

$$\{[\gamma_i] : i = 1, ..., 2g\} \cup \{[\gamma_j^r] : j = 1, ..., n\} \subset H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d)$$

be a basis. Then the cover defined by the class

$$[\gamma] = \sum_{j=1}^{n} r_j [\gamma_j^r] + \sum_{i=1}^{2g} a_i [\gamma_i] \in H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$$

is connected, if and only if $\gcd(r_1, ..., r_n, a_1, ..., a_{2g}, d) = 1$. *A relative homology class with coefficients in $\mathbb{Z}$ determines a connected cover, if and only if*

$$\gcd(r_1, ..., r_n, a_1, ..., a_{2g}) = 1.$$

For $\mathbb{Z}_d$ classes that means, there is no proper divisor $k|d$ with $k \cdot [\gamma] \equiv 0 \mod d$. If, on the other hand $\gcd(r_1, ..., r_n, a_1, ..., a_{2g}, d) = k > 1$ then $(d/k) \cdot [\gamma] \equiv 0 \mod d$.

Recall that $\mathbb{Z}_d$-covers up to $\mathbb{Z}_d$ equivariant isomorphisms are in one-to-one correspondence to classes in $H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d)$. To parameterize d-cyclic covers we will not distinguish covers that have distinct $\mathbb{Z}_d$ actions on the fibers of the cover. In order to preserve the cyclic structure of the cover the maps between fibers must be isomorphisms of $\mathbb{Z}_d$. Those are given by the elements of $\mathbb{Z}_d^*$, the group of units in the ring $\mathbb{Z}_d$, applied multiplicatively. Let $\mathbb{P}H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d)$ be the set of $\mathbb{Z}_d^*$ orbits in $H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d) \backslash \{0\}$, i.e. the quotient $(H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d) \backslash \{0\}) / \mathbb{Z}_d^*$.

Proposition 8. *Isomorphy classes of d-cyclic covers are in one-to-one correspondence to $\mathbb{Z}_d^*$ orbits in $H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d) \backslash \{0\}$.*

Proof. We show that the action of $\mathbb{Z}_d^*$ on homology is induced by the multiplicative action of $\mathbb{Z}_d^*$ on the fibers of the cover associated to the class. Given the cover $\pi : X_\gamma \to X$ defined by the class $\gamma \in H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d) \backslash \{0\}$, we refer to the elements of a fiber $\pi^{-1}(x) \subset X_\gamma$ as decks. Note that a fiber $\pi^{-1}(x)$ is identified with $\mathbb{Z}_d$. Suppose a lift $\widetilde{c}$ of the curve $c : I \to X \backslash \{x_0, ..., x_n\}$ starts on deck $i \mod d$

of $\pi^{-1}(x) \subset X_\gamma$ and ends on deck $(\langle\gamma, c\rangle + i) \mod d$. Shuffling the d elements of the fiber by a permutation σ, the lift of c that starts on deck $j = \sigma(i)$ will end on deck $\sigma((\langle\gamma, c\rangle + \sigma^{-1}(j)) \mod d)$. In our case σ is multiplication with $k \in \mathbb{Z}_d^*$ and the previous expression becomes:

$$k(\langle\gamma, c\rangle + k^{-1}j) \equiv \langle k\gamma, c\rangle + j \mod d.$$

So the cover $X_{k\gamma}$ defined by the class $k\gamma$ is the cover obtained from X_γ by rearranging the decks via k. □

Consequently the classes in

$$\mathbb{P}H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d) := H_1(X, \{x_0, ..., x_n\}; \mathbb{Z}_d)\backslash\{0\}/\mathbb{Z}_d^*$$

parameterize d-cyclic covers. Since the ramification points of a d-symmetric cover are assumed to have maximal order d, the weights $r_k \in \mathbb{Z}_d$ of the relative classes $[\gamma_k^r]$ need to be in $\mathbb{Z}_d^*$. This condition alone ensures connectedness independent from the remaining absolute homology part defining the cover. So we can always assume, that one relative cycle has weight 1. In particular, since d-symmetric covers have two branch points, we pick a class with relative cycle that has weight 1. Let us denote the set of those classes by $\mathbb{P}_1H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$. As before $[n]_d \in \mathbb{Z}_d$ denotes the residue class of an integer n modulo d, then using $a_0 \equiv 1$ for d-symmetric covers. The following decomposition is obvious:

Proposition 9. *For any $x_1 \in X\backslash\{x_0\}$*

$$\mathbb{P}H_1(X, \{x_0, x_1\}; \mathbb{Z}_d) = \sqcup_{n|d}(H_1(X; \mathbb{Z}_d)) \times \{[n]_d[\gamma_0]\}$$

and

$$\mathbb{P}_1H_1(X, \{x_0, x_1\}; \mathbb{Z}_d) = H_1(X; \mathbb{Z}_d) \times \{[\gamma_0]\}.$$

So after choosing a basis class $[\gamma_0]$ for the relative homology to be a simple curve $\gamma_r(t),\ t \in [0, 1]$ with $\gamma_0(0) = x_0$ and $\gamma_0(1) = x_1$, we can write any class $[\gamma] \in \mathbb{P}_1H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$ as

$$[\gamma] = [\gamma_0] + \sum_{i=1}^{2g} a_i[\gamma_i], \quad a_i \in \mathbb{Z}_d$$

where the $[\gamma_i], i = 1, ..., 2g$ are a basis of $H_1(X, \mathbb{Z}_d)$. In particular, as sets

$$\mathbb{P}_1H_1(X, \{x_0, x_1\}; \mathbb{Z}_d) \cong (\mathbb{Z}_d)^{2g} \cong H_1(X; \mathbb{Z}_d).$$

The cover associated to a projective homology class. Let us now construct a branched cover $X_{[\gamma]} \to X$ associated to the class $[\gamma] \in \mathbb{P}H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$. First pick a homology class in $H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$ that represents the chosen projective class, here also denoted by $[\gamma]$. Recall that topologically a cover is given by the way (oriented) loops $\beta : [0, 1] \to X\backslash\{x_0, x_1\}$ lift. Suppose $X_{[\gamma]} \to X$ is given by the class $[\gamma] = a_0[\gamma_0] + \sum_{i=1}^{2g} a_i[\gamma_i]$ with $a_i \in \mathbb{Z}_d$ and simple closed and oriented curves γ_i, then every time β crosses γ_i with *positive orientation*, its lift, say $\widetilde{\beta}$, will move a_i decks *up*.

In order to construct $X_{[\gamma]}$ from the marked surface (X, x_0), realize each class $[\gamma_i]$ as simple oriented curve starting in x_0. Then cut X along all γ_i. Denote the resulting surface X^{cut}, along each cut there are two oriented strands, we label those by $+a_i$ and $-a_i$. To get $X_{[\gamma]}$ take d copies $\sqcup_{k=1}^d X_k^{cut}$ of X^{cut} and identify the $+a_i$ strand on X_k^{cut} with the $-a_i$ strand on X_l^{cut}, when $l = k + a_i \mod d$. We denote the surface obtained from $[\gamma]$ by $\#^d_{[\gamma]}X$. Identifying all copied points defines the covering map to X. The group $\mathbb{Z}_d$ acts on $\#^d_{[\gamma]}X$, because identifications are done cyclically.

3.1. Cyclic covers over the universal abelian cover. The *universal abelian cover* $\pi_{ab} : X_{ab} \to X$ of a Riemann surface, also called *homology cover*, is the regular cover associated to the commutator subgroup $K := [\pi_1(X, x_0), \pi_1(X, x_0)]$ of the fundamental group $\pi_1(X, x_0)$ of X. The homology cover has deck group $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ and $H_1(X_{ab};\mathbb{Z}) \cong K/[K, K]$. In particular the π_{ab} image of any loop γ in X_{ab} is trivial in homology:

$$[\pi_{ab} \circ \gamma] = 0 \in H_1(X;\mathbb{Z}).$$

If, as before, $x_0 \in X$ denotes the locus of the first branch point, we pick

$$x_{ab} \in \pi_{ab}^{-1}(x_0) \subset X_{ab},$$

obtaining a marked cover

$$\pi_d : (X_{ab}, x_{ab}) \to (X, x_0).$$

For $\widetilde{x}_1 \in X_{ab}$ consider a path γ from x_{ab} to $\widetilde{x}_1$, and the following map

$$\widetilde{x}_1 \mapsto [\pi_{ab} \circ \gamma] \in \begin{cases} \mathbb{P}_1 H_1(X, \{x_0, x_1\};\mathbb{Z}), & \text{if } \ x_0 \neq x_1 := \pi_{ab}(\widetilde{x}_1) \\ H_1(X;\mathbb{Z}), & \text{if } \ x_0 = \pi_{ab}(\widetilde{x}_1) \end{cases} \tag{12}$$

PROPOSITION 10. *The homology class associated to $\widetilde{x}_1 \in X_{ab}$ is well-defined.*

PROOF. If $\widetilde{x}_1 \notin \pi^{-1}(x_0)$, then $x_0 \neq x_1 := \pi_{ab}(\widetilde{x}_1)$. We have to show that two different curves γ_1 and γ_2 connecting x_{ab} with $\widetilde{x}_1 \in X_{ab}$ project to the same homology class in $H_1(X, \{x_0, x_1\};\mathbb{Z})$. This is true, because $\pi_{ab} \circ (\gamma_2^{-1} * \gamma_1)$ is a loop whose homotopy class lies in the commutator subgroup of $\pi_1(X, x_0)$ and therefore defines the trivial class in $H_1(X;\mathbb{Z})$. The argument is the same, if we assume $\widetilde{x}_1 \in \pi^{-1}(x_0)$. ☐

Let us now denote the above map (into the union of relative homology groups) by π_{ab,x_0}. So, π_{ab,x_0} assigns a $\mathbb{Z}$-cover to each point $\widetilde{x}_1 \in X_{ab}$ in the marked surface (X_{ab}, x_{ab}). This cover is connected, if $\widetilde{x}_1 \notin \pi_{ab}^{-1}(x_0)$ and eventually degenerate, if $\widetilde{x}_1 \in \pi_{ab}^{-1}(x_0)$. In both these cases π_{ab,x_0} is bijective since the points of $\pi_{ab}^{-1}(x_1)$ are in one-to-one correspondence to $\mathbb{P}_1 H_1(X, \{x_0, x_1\};\mathbb{Z})$ if $x_0 \neq x_1$, or to $H_1(X;\mathbb{Z})$ in the other case.

A family of $\mathbb{Z}$ covers over X_{ab}. The construction of covers associated to a relative homology class defines a family $\mathfrak{X}_d$ of $\mathbb{Z}_d$ covers on X_{ab} for each $d \in \mathbb{Z}$. This construction works for $\mathbb{Z}$ covers and the associated family over X_{ab} is denoted by $\mathfrak{X}_\infty$. The alternative construction of the family $\mathfrak{X}_\infty$ below may give a better idea of its regularity.

For $x_1 \in X$ consider a directed simple path, say ϕ, from $x_{ab} \in X_{ab}$ to any point $\widetilde{x}_1 \in \pi_{ab}^{-1}(x_1) \in X_{ab}$. The path can be chosen so that it is disjoint to any of its deck-translates under the deck group $H_1(X, \mathbb{Z})$. We remove all the deck translates of ϕ and fit in two strands, labeled right and left, for each side in each translate. Call the resulting surface $X_{ab}^{cut}(\widetilde{x}_1)$ take $\mathbb{Z}$ labeled copies and identify the copy of a particular right strand on the i-th deck in $\mathbb{Z} \times X_{ab}^{cut}(\widetilde{x}_1)$ with the left strand of the same origin in the $(i + 1)$st deck. We obtain a connected surface $\#^\infty_{\widetilde{x}_1} X_{ab}$ with a $\mathbb{Z} \times H_1(X;\mathbb{Z})$ action. Here the $\mathbb{Z}$ action induced by $m \cdot (i, x) = (i + m, x)$ and the

$H_1(X;\mathbb{Z})$ action induced by $\gamma \cdot (i,x) = (i, \gamma x)$ for $(i,x) \in \mathbb{Z} \times X_{ab}^{cut}(\widetilde{x}_1)$ descend to $\#_{\widetilde{x}_1}^{\infty} X_{ab}$. Note that both actions commute. So $\#_{\widetilde{x}_1}^{\infty} X_{ab}$ is a $\mathbb{Z}$-cover of X_{ab} and its quotient by the deck group $H_1(X;\mathbb{Z})$ is the $\mathbb{Z}$-cover $\#_{\widetilde{x}_1}^{\infty} X$ of X.

Varying $\widetilde{x}_1$ over X_{ab} we obtain the family $\mathfrak{X}_\infty \to X_{ab}$ of infinte cyclic covers $\#_{\widetilde{x}_1}^{\infty} X \to X$. The group of integers acts on that family by deck-transformations on each individual surface. For $d > 1$ consider the quotient family $\mathfrak{X}_\infty / d\mathbb{Z} \to X_{ab}$ with respect to the subgroup $d\mathbb{Z}$.

Proposition 11. *If* $\#_{\widetilde{x}_1}^{\infty} X \in \mathfrak{X}_\infty$*, then* $\#_{\widetilde{x}_1}^{\infty} X / d\mathbb{Z} \cong \#_{[\widetilde{x}_1]_d}^{\infty} X \in \mathfrak{X}_d$.

Proof. If $[\gamma] \in \mathbb{P}_1 H_1(X, \{x_0, x_1\}; \mathbb{Z})$ is the projective class defining the cover $\pi_{\widetilde{x}_1} : \#_{\widetilde{x}_1}^{\infty} X \to X$, then $\gamma = \pi_{\widetilde{x}_1} \widetilde{\gamma}$ where $\widetilde{\gamma}$ is a lift of γ to X_{ab} that starts in x_{ab} and ends in $\widetilde{x}_1$. Now consider $\#_{\widetilde{x}_1}^{\infty} X / \mathbb{Z}_d$. The lift $\widetilde{\sigma}$ of any loop $\sigma : [0,1] \to X \backslash \{x_0, x_1\}$ to the original $\mathbb{Z}$ cover $\#_{\widetilde{x}_1}^{\infty} X$ projects to the loop $\widetilde{\sigma}_d$ on $\#_{\widetilde{x}_1}^{\infty} X / \mathbb{Z}_d$ given by identifying points whenever their decks differ by an element in $d\mathbb{Z}$. Since deck changes of $\widetilde{\sigma}$ are given by the intersection $\langle \gamma, \sigma \rangle \in \mathbb{Z}$ the deck changes of its projection $\widetilde{\sigma}_d$ to the quotient surface are given by $\langle \gamma, \sigma \rangle \mod d$. Since this is true for any loops γ on X we see that the d cover $\#_{\widetilde{x}_1}^{\infty} X / d\mathbb{Z}$ is given by the class $[\sigma]_d \in \mathbb{P}_1 H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$. This class is in $\mathbb{P}_1 H_1(X, \{x_0, x_1\}; \mathbb{Z}_d)$ since it is the image of a class in $\mathbb{P}_1 H_1(X, \{x_0, x_1\}; \mathbb{Z})$, so particularly the cover $\#_{\widetilde{x}_1}^{\infty} X / d\mathbb{Z}$ is fully branched. The claim follows. $\square$

As a consequence we have the parameter space of d symmetric covers.

Corollary 6. *The points of* $(X_{ab} - H_1 x_{ab})/dH_1$ *parameterize d-symmetric covers.*

The covers defined by the points $H_1 x_{ab}/dH_1$ are used obtain a family over the compact parameter space. We use polygonal representations for those.

3.2. Polygonal representation of d-symmetric covers. Any (compact and connected) translation surface (X, ω) can be represented by a planar polygon, with pairs of parallel edges that are identified by translations. This allows us to do the previous copy, cut and glue construction of $\mathbb{Z}_d$ covers using representations of homology classes by straight line segments, or concatenations of straight line segments. This is a special case of the *slit construction* that utilizes a cut along a single regular line segment.

3.3. Degenerate covers via absolute homology classes. The space and family of d-symmetric surfaces over (X, x_0, ω) has a natural compactification by adding covers for the lattice points $\pi_d^{-1}(x_0) \subset X_{ab}/(d\mathbb{Z})^{2g}$. Since $\pi_d^{-1}(x_0)$ and $H_1(X;\mathbb{Z}_d)$ can be identified as sets, we may represent the respective points by "d-cyclic covers" defined by absolute homology classes in $H_1(X;\mathbb{Z}_d)$. This can be done as before just using the polygon representation in the plane: By gluing d copies of X cut along line segments representing an absolute homology basis. Along the cuts the copies are cyclically identified according to the values of the homology cycle $[\gamma] \in H_1(X;\mathbb{Z}_d) \cong \mathbb{Z}_d^{2g}$ for the respective classes. These surfaces are not always connected, but nevertheless carry a $\mathbb{Z}_d$ action with quotient (X, ω). The number of connected components is a divisor of d. We obtain a connected surface by identifying the d preimages of x_0 to one point. Using our usual convention, we denote the connected surface by $\#_z^d(X, x_0, \omega)$ for the cover given by $[\gamma] \in H_1(X;\mathbb{Z}_d)$, so that $z = [\gamma] \cdot x_d \in X_{ab}/dH_1$. We call these covers *degenerate covers*. Figure

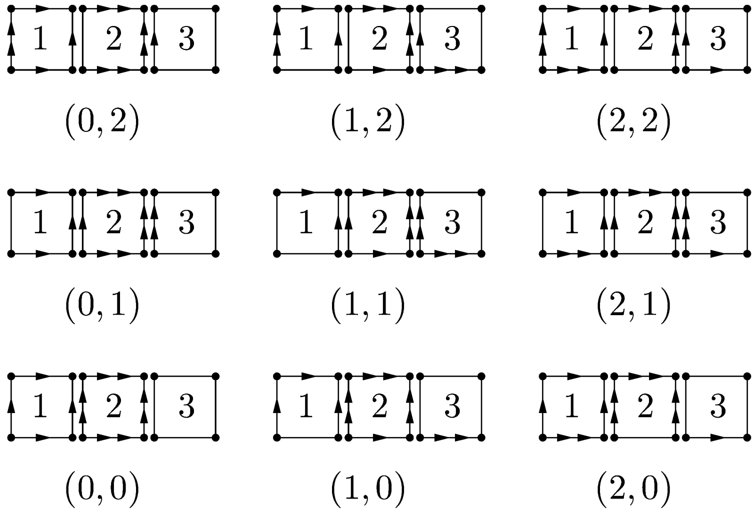

FIGURE 2. Degenerate degree 3 differentials at lattice coordinates of parameter space.

2 shows the nine $\mathbb{Z}_3$ covers of $\mathbb{T}$ defined by absolute homology classes. The coordinates below each cover correspond to their (real) lattice point coordinate in $\mathbb{Z}[i]/3\mathbb{Z}[i] \subset \mathbb{C}/3\mathbb{Z}[i]$. Parallel sides of the squares with the same symbol pattern in each figure are considered identified by a translation. The degenerate covers have a singularity that is not induced by a zero of the pullback of ω, nevertheless they have a natural interpretation as limits obtained from collapsing the two cone points of a d-symmetric differential to one point. Together with the results of the previous section we have:

THEOREM 5. *Let (X, x_0, ω) be a marked abelian differential of genus g. Then the d-symmetric forms over (X, x_0, ω), degenerate forms included, are parameterized by the points in $(X_{ab}/dH_1, x_d, \pi^{-1}\omega)$. The point $x_d \in \pi_d^{-1}(x_0)$ corresponds to the trivial class in $H_1(X; \mathbb{Z}_d)$, i.e. to the degenerate cover $\#_{[0]}^d(X, x_0, \omega)$ defined by the trivial homology class.*

Since the degenerate d symmetric forms represented by points in $\pi_d^{-1}(x_0)$ have only one (artificial) branch point they may have symmetries and so the compactification $(X_{ab}/dH_1, x_d)$ does not necessary classify covers up to isomorphy. This can be seen particularly in the case of torus covers. Tori are hyperelliptic and we can consider a hyperelliptic involution that exchanges the branch points of the cover. That in turn induces an involution of the parameter torus $(\mathbb{C}/dH_1, x_d)$, that is the hyperelliptic involution fixing x_d. Since the hyperelliptic involution fixes the preimage of x_0 on degenerate surfaces, those are symmetric. On the other hand the point in $(\mathbb{C}/dH_1, x_d)$ that represents this cover may not be fixed by the hyperelliptic involution of $(\mathbb{C}/dH_1, x_d)$. So we have two isomorphic copies for those among the d^2 degenerate surfaces in $(\mathbb{C}/dH_1, x_d)$ that are not already fixed by the hyperelliptic involution.

4. Asymptotic formulas for branched covers

Here we recall a formalism to calculate asymptotic quadratic growth rates for (branched covers of) lattice surfaces. In case the covering is itself a lattice surface we derive our formula from a variant of the asymptotic formula presented in Gutkin and Judge [**GJ**] and independently Vorobetz [**Vrb**]. Throughout this section we assume (X, ω) is a lattice surface with Veech-group $\mathrm{SL}(X, \omega)$.

Growth rate of vector distributions. Take a discrete and countable distribution of vectors, say $V \subset \mathbb{R}^2$, and let $B_T \subset \mathbb{R}^2$ be the ball of radius T. If existent, we want to calculate the quadratic asymptotic of $\#(V \cap B_T)$, that is $\pi c(V) := \lim_{T\to\infty} \frac{\#(V\cap B_T)}{T^2}$. For the vector distribution $V = \mathbb{Z}^2$ it is easy to see $c(\mathbb{Z}^2) = 1$. Indeed for $T = N \in \mathbb{N}$ considering the rescaled lattice $\frac{1}{N}\mathbb{Z}^2$ we count the number of rational points in the unit disk with denominator N. This converges to the area of the unit disk. So $c(V)$ measures the quadratic growth rate relative to the integer lattice.

Asymptotic quadratic constants have some obvious properties: They are invariant under translations of V and if $A \in \mathrm{GL}_2(\mathbb{R})$, then $c(AV) = \frac{1}{\det(A)} c(V)$. In particular $c(AV) = c(V)$, if $A \in \mathrm{SL}_2\mathbb{R}$. For a given distribution V the *primitive distribution* $\mathbb{P}V \subseteq V$ contains the points of V *visible from the origin* and its *completion* $\mathbb{Z}V := \{mv \in \mathbb{R}^2 : m \in \mathbb{Z}, v \in V\}$. We call a vector-distribution V *complete*, if $V = \mathbb{Z}V$. Given a vector distribution V that has defined quadratic asymptotics for $\mathbb{Z}V$ and $\mathbb{P}V$ then there is the following universal relation:

$$c(\mathbb{Z}V) = \zeta(2)c(\mathbb{P}V) = \frac{\pi^2}{6}c(\mathbb{P}V).$$

Distributions generated by Veech groups. We now take a look at the asymptotic growth rate of the vector distribution $V = \Gamma \cdot v$ that is the orbit of a *lattice* subgroup $\Gamma \subset \mathrm{SL}_2\mathbb{R}$ applied to a vector $v \in \mathbb{R}^2\backslash\{0\}$. If for example $\Gamma = \mathrm{SL}_2\mathbb{Z}$ and $v = (1, 0)^T$, then

$$\mathrm{SL}_2\mathbb{Z}(1, 0)^T = \{(a, b) \in \mathbb{Z}^2 : \gcd(a, b) = 1\} \subset \mathbb{Z}^2 = \mathrm{GL}_2(\mathbb{Z})(1, 0)^T$$

is the set of *visible points* and so is the primitive distribution associated to $\mathbb{Z}^2$. If $v = \lambda(1, 0)^T$, where $\lambda \in \mathbb{R}\backslash\{0\}$, the identity

$$c(\mathrm{SL}_2\mathbb{Z}v) = \frac{1}{\|v\|^2} c(\mathrm{SL}_2\mathbb{Z}(1, 0)^T) = \frac{1}{\|v\|^2}\frac{6}{\pi^2} c(\mathbb{Z}^2) = \frac{1}{\|v\|^2}\frac{6}{\pi^2}$$

follows because the distribution under consideration is $D_\lambda\mathbb{Z}^2$ where D_λ is the diagonal matrix with diagonal entries λ.

The next result gives the asymptotic constant of a distribution generated by a general *lattice group* $\Gamma \subset \mathrm{SL}_2\mathbb{R}$. Recall that Γ is a lattice, if it is discrete and $\mathbb{H}/\Gamma$ has finite hyperbolic area. Since distributions generated by subgroups of $\mathrm{SL}_2\mathbb{R}$ contain only visible points, we normalize by multiplying with $\pi^2/6$, so that $c(\mathrm{SL}_2\mathbb{Z}(1, 0)^T) = 1$. A lattice $\Gamma \subset \mathrm{SL}_2\mathbb{Z}$ is called symmetric, if $-\mathrm{id} \in \Gamma$, or equivalently $-\Gamma = \Gamma$. The following fundamental formula ties the asymptotic growth rate for Γ orbits of vectors to geometric properties of its action on the hyperbolic plane. Statement equivalent to the following can be found in Gutkin and Judge [**GJ**], Vorobetz [**Vrb**], but also in [**EMM**].

PROPOSITION 12. *Assume the subgroup of linear maps $N_v \subset \mathrm{SL}_2\mathbb{R}$ stabilizing the vector $v \in \mathbb{R}^2$ has nontrivial intersection with a symmetric lattice $\Gamma \subset \mathrm{SL}_2\mathbb{R}$ and let A be a generator of $N_v \cap \Gamma$, then*

$$c(\Gamma v) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\Gamma)} \frac{|\langle A u_v^\perp, u_v\rangle|}{|v|^2}, \tag{13}$$

where u_v is the unit vector in direction v and $u_v^\perp \in \mathbb{R}^2\backslash\{0\}$ is a unit vector perpendicular to v.

We apply formula 13 to a holonomy vector v defined by a saddle connection, or by a maximal cylinder on a lattice surface (X,ω). In this case the group Γ is the Veech group $\mathrm{SL}(X,\omega)$ and formula 13 gives the asymptotic quadratic constant of the set of holonomy vectors in the $\mathrm{SL}(X,\omega)$ orbit of v. One easily applies the formula to tori, particularly to $\mathbb{T} := \mathbb{C}/\mathbb{Z}[i]$ marked at $[0]$. This torus has $\mathrm{SL}(X,\omega) = \mathrm{SL}_2\mathbb{Z}$ as Veech group. The horizontal saddle connection and any horizontal loop have holonomy $v = (1,0)^T$. The set $\mathrm{SL}_2\mathbb{Z}\cdot(1,0)^T \subset \mathbb{Z}^2$ is the set of integer points *visible from the origin* for which we calculated $c(\mathrm{SL}_2\mathbb{Z}\cdot(1,0)^T) = 1$. The right hand side of formula 13 gives the same value. Since $(1,0)^\perp = (0,1)$ and the stabilizer of $(1,0)^T$ in $\mathrm{SL}_2\mathbb{Z}$ is generated by (the powers of) $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$, one obtains the above quadratic constant by taking $\mathrm{vol}(\mathbb{H}/\mathrm{SL}_2\mathbb{Z}) = \pi/3$ into account.

If we assume the horizontal foliation of the Veech surface (X,ω) is periodic, taking $v^\perp = \left[\begin{smallmatrix}0\\h\end{smallmatrix}\right]$ and $v = \left[\begin{smallmatrix}w\\0\end{smallmatrix}\right] \in \mathbb{R}^2$ to be the height h and circumference w of a maximal cylinder, then we have Veech's asymptotic formula **[V1, V2]**:

$$c(\mathrm{SL}(X,\omega)\cdot v) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\Gamma)} \frac{l}{w^2}, \tag{14}$$

where $l \in \mathbb{Q}\frac{w}{h}$ is the positive minimal, so that $\left[\begin{smallmatrix}1&l\\0&1\end{smallmatrix}\right] \in \mathrm{SL}(X,\omega)$.

Veech's asymptotic formula. We recall Veech's asymptotic formula given formula 14. Assume (X,ω) is a Veech surface with Veech group $\mathrm{SL}(X,\omega)$. Generally picking any periodic direction on a Veech surface we can associate a holonomy vector $\mathrm{hol}(C) = \int_{l_C}\omega \subset \mathbb{C}$ to a maximal periodic cylinder C by integration of any of its core leaves. The length of a cylinder equals the modulus of its holonomy vector. Since A acts real linearly by postcomposition of natural charts taking holonomy commutes with the action of the affine group in the following sense:

$$\mathrm{hol}(\phi C) = D\phi\,\mathrm{hol}(C).$$

Here ϕC denotes the image of C under the affine map ϕ. With regard of the above counting formula we need to know the orbit of a cylinder with respect to $\mathrm{Aff}^+(X,\omega)/\mathrm{Aut}(X,\omega) \cong \mathrm{SL}(X,\omega)$. It is known that the group of parabolic matrices in $\mathrm{SL}_2\mathbb{R}$ with $\mathrm{hol}(C)$ as eigendirection has nontrivial intersection with $\mathrm{SL}(X,\omega)$. Let us denote this subgroup by $N_1 \subset \mathrm{SL}(X,\omega)$. Then $\mathrm{SL}(X,\omega)/N_1$ parameterizes the slopes obtained from $\mathrm{hol}(C)$. Those correspond to cusps of $\mathbb{H}/\mathrm{SL}(X,\omega)$, in group theoretical terms the conjugacy classes of parabolic subgroups in $\mathrm{SL}(X,\omega)$. Since there are only finitely many cusps, we can write down an asymptotic constant as follows: For each cusp, labeled with $j = 1,...,n$ pick a representative totally periodic direction on (X,ω). The j-th direction has n_j cylinders C_k^j, $k = 1,...,n_j$.

Let us put $w_k^j := |\operatorname{hol}(C_k^j)|$. Then we have

$$c(X,\omega) = \frac{\pi}{3\operatorname{vol}(\mathbb{H}/\Gamma)} \sum_{j=1}^{n} i_j \sum_{k=1}^{n_j} \frac{1}{(w_k^j)^2}. \tag{15}$$

Here $i_j = \operatorname{lcm}(m_{j1}^{-1}, ..., m_{jn_j}^{-1})$ where $m_{jk} = h_k^j (w_k^j)^{-1}$ and h_k^j is the height of cylinder C_k^j.

Asymptotic formula for covers of lattice surfaces. Consider a branched cover $\pi : (Y,\tau) \to (X,\omega)$ of a lattice surface (X,ω). The Veech-group $\mathrm{SL}(X,\omega)$ of (X,ω) acts on covers and the stabilizer is the relative group $\mathrm{SL}(Y/X,\tau) \subseteq \mathrm{SL}(X,\omega) \cap \mathrm{SL}(Y,\tau)$. If $\mathrm{SL}(Y/X,\tau)$ is a lattice, then (Y,τ) is a Veech surface and $\mathrm{SL}(Y/X,\tau)$ has finite index in both groups $\mathrm{SL}(X,\omega)$ and $\mathrm{SL}(Y,\tau)$. In particular the orbit

$$\mathscr{O}_Y := \mathrm{SL}(X,\omega)[Y \to X] = \mathrm{SL}(X,\omega)/\mathrm{SL}(Y/X,\tau)$$

on covers is finite and has order $|\mathscr{O}_Y| = [\mathrm{SL}(X,\omega) : \mathrm{SL}(Y/X,\tau)]$. To calculate quadratic asymptotics, it is sufficient to use the relative Veech group.

Pick a set of periodic directions labeled by $j = 1, ..., n$ on (X,ω) representing the cusps of $\mathbb{H}/\mathrm{SL}(X,\omega)$. Pick one periodic direction. Without restrictions of generality we can assume this direction is *horizontal*, by rotating the translation structure if necessary.

Proposition 13. *The cusps of* $\mathbb{H}/\mathrm{SL}(Y/X,\tau)$ *in the preimage of the horizontal cusp with respect to the map* $\mathbb{H}/\mathrm{SL}(Y/X,\tau) \to \mathbb{H}/\mathrm{SL}(X,\omega)$ *are in one-to-one correspondence to the* $N_h(X,\omega)/N_h(Y/X,\tau)$ *orbits on* $\mathrm{SL}(X,\omega)/\mathrm{SL}(Y/X,\tau)$.

Proof. Cusps of $\mathbb{H}/\mathrm{SL}(Y/X,\tau)$, or $\mathrm{SL}(Y/X,\tau)$ are in one-to-one correspondence to $\mathrm{SL}(Y/X,\tau)$ orbits of slopes of periodic directions on (Y,τ). Directions on (Y,τ) are periodic if and only if they are periodic on (X,ω), we fix a periodic direction, say the horizontal direction, and need to see how many $\mathrm{SL}(Y/X,\tau)$ orbits the set of directions $\mathrm{SL}(X,\omega)/N_h$ has. As for the action of $\mathrm{SL}(Y/X,\tau)$ on $\mathrm{SL}(X,\omega)$ the cosets $\mathrm{SL}(X,\omega)/\mathrm{SL}(Y/X,\tau)$ parameterize the orbits. We note that the orbits of $N_h(X,\omega)$ on the cosets correspond to the image of the (horizontal) cusps of $\mathrm{SL}(Y/X,\tau)$. Those orbits are the claimed orbits of $N_h(X,\omega)/N_h(Y/X,\tau)$. □

In a geometric way, one can think of the abstract orbit $\mathrm{SL}(X,\omega)/\mathrm{SL}(Y/X,\tau)$ of cover classes as an actual coset $\mathrm{SL}(Y/X,\tau)\backslash\mathrm{SL}(X,\omega)(Y,\tau)$. Then any periodic direction in the horizontal $\mathrm{SL}(X,\omega)$ cusp on a surface in $\mathrm{SL}(Y/X,\tau)\backslash\mathrm{SL}(X,\omega)(Y,\tau)$ is horizontal on another surface, say S, in this orbit. The horizontal foliations of all surfaces in the $N_h(X,\omega)/N_h(Y/X,\tau)$ orbit of S stay horizontal. So a direction in the $\mathrm{SL}(Y/X,\tau)$ orbit of the horizontal one on any of $N_h(X,\omega)/N_h(Y/X,\tau)S$ belongs to the same cusp. And if there is a surface in that cusp on which the horizontal direction is the image of the horizontal direction on S, then the map from S to that surface has to lie in $N_h(X,\omega)/N_h$ because horizontal leafs are preserved. A similar characterization has been applied in [**HL**].

Let us now consider a general periodic direction $\theta \in S^1$ on (X,ω). By $N_\theta(Y/X,\tau) := \mathrm{SL}(Y/X,\tau) \cap N_\theta(X,\omega)$ we denote the parabolic stabilizer of the direction θ on (Y,τ). Its index

$$i_\theta(Y,X) := [N_\theta(X,\omega) : N_\theta(Y/X,\tau)] = |N_\theta(X,\omega) \cdot [Y \to X]| \tag{16}$$

depends only on the cusp defined by θ and we call it the *relative width* of the cusp.

Proposition 14. *Let $v \in \mathbb{C}\backslash\{0\}$ be a vector parallel to a periodic direction on (X,ω) and $(S,\alpha) \in \mathrm{SL}(X,\omega)(Y,\tau)$, then*

$$c(\mathrm{SL}(S,\alpha)v) = \mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))^{-1}\frac{i_v(X)\cdot i_v(S,X)}{[\mathrm{SL}(X,\omega):\mathrm{SL}(Y/X,\tau)]}\frac{1}{|v|^2}. \tag{17}$$

Proof. We apply formula 14 to the (relative) lattice $\mathrm{SL}(S/X,\alpha)$. This group is a conjugate of $\mathrm{SL}(Y/X,\tau)$, in particular it has the same index in $\mathrm{SL}(X,\omega)$. Now use the following facts: The index of the parabolic stabilizer of v in $\mathrm{SL}(S/X,\alpha)$ is $i_v(X)\cdot i_v(S,X)$ and

$$\mathrm{vol}(\mathbb{H}/\mathrm{SL}(Y/X,\tau)) = \mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))[\mathrm{SL}(X,\omega):\mathrm{SL}(Y/X,\tau)].$$

□

To apply this formula to the counting of closed cylinders, let us consider a positive real vector $\vec{\lambda} := (\lambda_1,....,\lambda_n) \in \mathbb{R}^n_+$ and let $\theta \in \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$ be a unit vector. Then $\vec{\lambda}\theta \in \mathbb{C}^n$ and we think of this vector as a set of n holonomy vectors of cylinders or saddle connections in direction θ on any translation cover $(S,\alpha) \in \mathrm{SL}(X,\omega)(Y,\tau)$. Let us now evaluate the quadratic asymptotics of the set

$$\mathrm{SL}(S,\alpha)\cdot\vec{\lambda}\theta := \{\vec{\lambda}A\theta \in \mathbb{C}^n : A \in \mathrm{SL}(S,\alpha)\}$$

and describe it in terms of the orbit space of covers $\mathrm{SL}(X,\omega)/\mathrm{SL}(Y/X,\tau)$. Using the orbit description of a relative cusp $\mathscr{U}_S := N_\theta(X,\omega)\cdot[S \to X]$ we obtain from the previous formula:

$$c(\mathrm{SL}(S,\alpha)\vec{\lambda}\theta) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))}\frac{i_\theta(X)}{|\mathscr{O}_Y|}\sum_{(S,\alpha)\in\mathscr{U}_S}\sum_{i=1}^{n_\theta}\frac{1}{\lambda_i^2} \tag{18}$$

Now consider all directions in the $\mathrm{SL}(X,\omega)$ orbit of the periodic direction θ on (Y,τ). Here we need to sum over the relative cusps on (Y,τ) that appear in $\mathrm{SL}(X,\omega)\theta$. If $C_Y(\theta)$ denotes this set of relative cusps we have

$$c_C(Y,\tau,\theta) = \sum_{\vartheta\in C_Y(\theta)} c(\mathrm{SL}(S,\alpha)\vec{\lambda}_\vartheta\vartheta)$$

where we specialize and take the positive vector to be the holonomy vector of cylinders for each cusp. Putting $\mathscr{O}_\tau := \mathrm{SL}(X,\omega)/\mathrm{SL}(Y/X,\tau)$, the asymptotic constant can be written using the orbit decomposition of cusps:

$$c_C(Y,\tau,\theta) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))}\frac{i_\theta(X)}{|\mathscr{O}_\tau|}\sum_{(S,\alpha)\in\mathscr{O}_\tau}\sum_{i=1}^{n_\alpha}\frac{1}{|\lambda_{\alpha,i}|^2}. \tag{19}$$

Note that for lattice surfaces saddle connection are always in the boundaries of cylinders. So the asymptotic constant for saddle connections is obtained by changing the entries of the last sum of 19 using saddle connection holonomy vectors.

The Siegel-Veech constants including all periodic directions of (X,ω) is now obtained by summing the previous expression over the cusps C_X of (X,ω), that is the cusps of $\mathrm{SL}(X,\omega)$ here represented by a periodic direction.

$$c_C(Y,\tau) = \sum_{\theta\in C_X} c_C(Y,\tau,\theta) \tag{20}$$

To summarize: Besides the *index and length of cylinders in a particular direction* the only quantities required to evaluate the constants are a *finite set of directions on the base surface corresponding to the cusps of its Veech group*, the number $i_X(\theta)$ associated to those directions and the hyperbolic area (of $\mathbb{H}/\mathrm{SL}(X,\omega)$).

Examples – Torus covers branched over one point. Recall that arithmetic surfaces are (representable as) torus covers branched over one point. Taking the lattice defined by holonomy vectors of the absolute periods on an arithmetic surface defines a lattice Λ and a translation covering $(Y,\tau) \to (\mathbb{C}/\Lambda, dz)$ (up to translation). Assuming Λ is unimodular, i.e. $\mathbb{C}/\Lambda$ has area 1 with respect to the natural metric, we can use the $\mathrm{SL}_2\mathbb{R}$ action on (Y,τ) to deform the lattice to be $\mathbb{Z}[i]$. This does not change asymptotic quadratic growth rates. The Veech group of $\mathbb{T} = \mathbb{C}/\mathbb{Z}[i]$ is $\mathrm{SL}_2\mathbb{Z}$, having only one cusp, here represented by by the horizontal direction. Because the modulus of the horizontal cylinder is 1, we have $i_h(\mathbb{T}) = 1$. So,

$$c_C(Y,\tau) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}_2\mathbb{Z})} \frac{1}{|\mathscr{O}_\tau|} \sum_{(S,\alpha)\in\mathscr{O}_\tau} \sum_{i=1}^{n_\alpha} \frac{1}{|\lambda_{\alpha,i}|^2}. \tag{21}$$

From this the general orbit formula for covers of Veech surfaces can be developed.

One takes into account, that given a periodic direction θ of (X,ω), there are exactly i_θ different surfaces in the orbit of the stabilizer $N_\theta \subset (X,\omega)$. Since elements in N_θ map cylinders (and saddle connections) in direction θ to cylinders (and saddle connections) of the same length we find

$$c_C(Y,\tau) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))} \frac{1}{|\mathscr{O}_\omega|} \sum_{(S,\alpha)\in\mathscr{O}_\omega} \sum_{i=1}^{n_\alpha} \frac{1}{|\lambda_{\alpha,i}|^2}. \tag{22}$$

Here $\mathscr{O}_\omega$ denotes the orbit of the cover $(Y,\tau) \to (X,\omega)$ under the full Veech group $\mathrm{SL}(X,\omega)$.

One cusp surfaces. Let us further mention the class of lattice surfaces having properties very similar to arithmetic ones. Those are Veech surfaces with *one cusp*, more precisely:

- *all periodic directions are in the orbit of a direction with k cylinders and*
- *the moduli $w_1/h_1 = ... = w_k/h_k$ of the cylinders are identical.*

As before w_i denotes the width and h_i the height of the i^{th} cylinder. So $i_v(X) = m$ and the Siegel-Veech constant for cylinders is

$$c_C(X,\omega) = \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,\omega))} \sum_{i=1}^{k} \frac{1}{h_i w_i}. \tag{23}$$

A paper by Eskin, Marklof and Morris **[EMM]** contains a discussion of the Veech surfaces X_n obtained by linearization of the billiard in the triangle with angles $(\pi/n, \pi/n, (n-2)\pi/n)$. Veech showed in **[V3]** that X_n has one cusp that stabilizes a direction whose moduli are all the same. Thus formula 23 applies, for mere see **[EMM]** pages 26-28. In this case the w_i and h_i are explicit, see **[V2, EMM]**. Both of the above examples were used to study branched covers. The asymptotic growth rates for 2-fold branched covers of X_n for example have been evaluated in **[EMM]**.

Flat geometry of parameter spaces and counting. We apply the asymptotic formula 20 in combination with the parameter space for d-symmetric covers. To do that we generally assume (X, ω) is a Veech surface, even though some of the following statements will hold in more generality. Let us fix a base point $x_0 \in Z(\omega) \subset X$ that will serve as the basic branch point of any d-symmetric to be considered. Then the set of d-symmetric covers branched over x_0 and $x \in X \backslash \{x_0\}$ is given by the points in the translation cover

$$\pi_d : (X_{ab}/dH_1, \pi_d^{-1}(x_0), \pi_d^*\omega) \to (X, \omega).$$

PROPOSITION 15. *Any affine map* $\phi : (X, x_0, \omega) \to (X, x_0, \omega)$ *lifts to* $(X_{ab}/dH_1, x_d, \pi_d^*\omega)$ *where* $x_d \in \pi_d^{-1}(x_0)$.

PROOF. We need to check the homotopy lifting criterion. Any homomorphism of groups $\psi : G \to H$ maps the commutator subgroup of G into the commutator subgroup of H. Since $\pi_1(X_{ab}) = [\pi_1(X, x_0), \pi_1(X, x_0)]$ we conclude $(\phi \circ \pi_{ab})_* \pi_1(X_{ab}, x_0) = C = (\pi_{ab})_* \pi_1(X_{ab}, x_0)$ for any affine map $\phi \in \mathrm{Aff}^+(X, x_0, \omega)$. Here $C = [\pi_1(X, x_0), \pi_1(X, x_0)] = \pi_1(X_{ab}, x_0)$. If we consider the lift $\widetilde{\phi}$ of ϕ to X_{ab} that fixes $x_{ab} \in \pi^{-1}(x_0)$, then the points in $\pi^{-1}(x_0)$ can be canonically identified with homology classes x_{ab} being the trivial class. Then the action of $\widetilde{\phi}$ restricted to $\pi^{-1}(x_0)$ is the action of ϕ on homology. Since the homology action is a $\mathbb{Z}$-module homomorphism it preserves the submodule $dH(X;\mathbb{Z})$ for any $d \geq 1$. That means $\widetilde{\phi}$ descends to an affine map, say ϕ_d of X_{ab}/dH_1 fixing $x_d := x_{ab} + dH_1$. By construction the cover $\pi_{ab} : (X_{ab}, x_{ab}) \to (X, x_0)$ factors over $(X_{ab}/dH_1, x_d)$ and ϕ_d is a lift of ϕ. □

Under the assumption that (X, x_0, ω) is a Veech surface all the covers

$$\pi_d : (X_{ab}/dH_1, \pi_d^{-1}(x_0), \pi_d^*\omega) \to (X, x_0, \omega).$$

are Veech surfaces and all their Veech groups contain the lattice $\mathrm{SL}(X, x_0, \omega)$ of (X, x_0, ω). Whenever a point on a translation surface is stabilized by all elements of a group of affine maps, we identify the affine maps with their derivatives. With respect to this identification we can make the following statement.

PROPOSITION 16. *For all* $z \in (X_{ab}/dH_1, \pi_d^*\omega)$ *and* $A \in \mathrm{SL}(X, x_0, \omega)$

$$A \cdot \#_z^d(X, x_0, \omega) = \#_{Az}^d(X, x_0, \omega).$$

PROOF. The dotted product is the action of $\mathrm{SL}(X, x_0, \omega)$ on d-symmetric covers. Since $A \in \mathrm{SL}(X, x_0, \omega)$ there is an affine map ϕ of (X, ω) that fixes x_0 with derivative A. The cover $A \cdot \#_z^d(X, x_0, \omega)$ is the cover associated to the homology class $\phi(\gamma)$, if γ is a path representing the relative class $[\gamma]$. This γ has starting point x_0 and by definition lifts to a path $\widetilde{\gamma}$ on X_{ab}/dH_1 that terminates in z. Because ϕ lifts to an affine map $\widetilde{\phi}$ that stabilizes x_d. By construction $\widetilde{\phi}$ maps the lift of $\widetilde{\gamma}$ to a lift of $\phi(\gamma)$ that terminates in $\widetilde{\phi}(z)$. Since all the affine maps have derivative A and fix the initial points of the respective curves, we have $Az = \widetilde{\phi}(z)$ and so the claim for d-symmetric covers follows. □

Any translation cover of (X, x_0, ω) is completely periodic in every direction where (X, x_0, ω) is completely periodic. Let us consider a periodic direction $\theta \in S^1$ on (X, x_0, ω). Then θ is a completely periodic direction for any d-symmetric cover $\#_x^d(X, x_0, \omega)$ given by a point $x \in (X_{ab} - \pi_{ab}^{-1}(x_0))/dH_1$. In particular this x lies on a saddle connection or in an open cylinder of the cylinder decomposition of $(X_{ab} - \pi_{ab}^{-1}(x_0))/dH_1$ in direction θ. The holonomy vectors (with multiplicity) of all cylinders in a given completely periodic direction on a translation surface is called cylinder datum.

LEMMA 1. *Given a completely periodic direction $\theta \in S^1$ on (X, x_0, ω). Then the cylinder datum on a d-symmetric cover in direction θ depends only on the cylinder, or saddle connection of $(X_{ab} - \pi_{ab}^{-1}(x_0))/dH_1$ that contains the point representing the cover.*

PROOF. Again we fix a periodic direction, say $\theta \in S^1$, on (X, x_0, ω). Let us consider a maximal cylinder $\mathscr{C} \subset (X_{ab} - \pi_{ab}^{-1}(x_0))/dH_1$ and its closure $\overline{\mathscr{C}} := \mathscr{C} \cup \partial\mathscr{C}$. Then $\overline{\mathscr{C}_X} := \pi_d(\overline{\mathscr{C}}) \subset X$ is a single closed cylinder on (X, x_0, ω). Indeed $\mathscr{C}_X$ is a cylinder, since regular leaves are mapped to regular leaves. Because $\partial\mathscr{C}$ contains preimages of $Z(\omega)$ or x_0 on each connected component, $\mathscr{C}_X$ has one of those points on its boundary and hence is maximal.

Let us take two points $z_i \in \mathscr{C}$ $i = 1, 2$, and path $\widetilde{\gamma}_{x_d}^{z_i}$ on $(X_{ab} - \pi^{-1}(x_0))/dH_1$ from x_d to z_i. Since both points $z_i \in \mathscr{C}$ lie in the same cylinder we can connect z_1 with z_2 with a line segment l and obtain a loop $\widetilde{\gamma}$ on $(X_{ab} - \pi^{-1}(x_0))/dH_1$. Since loops in $(X_{ab} - \pi^{-1}(x_0))/dH_1$ map to homologically trivial loops on X the intersection number $\pi_d\widetilde{\gamma}$ with any loop on X is trivial. This is particularly true for all loops defined by regular closed leaves in cylinders parallel to $\mathscr{C}$ on (X, ω). Consequently as long as those loops do not intersect $\pi_d(l)$ they have the same intersection number with both relative classes $[\gamma_{x_0}^{x_i}] \in H_1(X, \{x_0, x_i\}; \mathbb{Z}_d)$ defined by $\gamma_{x_0}^{x_i} = \pi_d \circ \widetilde{\gamma}_{x_d}^{z_i}$, $(x_i = \pi_d(z_i),\ i = 1, 2)$. This is trivially true for all leaves that do not intersect $\mathscr{C}_X$. Since the straight line segment $\pi_d(l)$ connects the points x_i in the interior of $\mathscr{C}_X$ there are cylinder leaves of $\mathscr{C}_X$ that do not intersect $\pi_d(l)$. Indeed both points x_i have positive distance from the boundary and any cylinder leaf that is closer to the boundary than the minimum distance of the two points does not intersect $\pi_d(l)$. So the intersections of such leaves with either class $[\gamma_{x_0}^{x_i}]$ are the same.

Now the intersection numbers of the cylinder leaves in a cylinder decomposition of (X, ω) in direction θ with $\gamma_{x_0}^{x_i}$ determine the circumferences of the direction θ cylinders in $\#_{z_i}^d(X, x_0, \omega)$. Since the covers are cyclic that also determines the multiplicity of the cylinders of $\#_{z_i}^d(X, x_0, \omega)$. That shows the claim for all cylinders on either surface $\#_{z_i}^d(X, x_0, \omega)$ in the preimage of any direction θ cylinder on (X, ω) besides $\mathscr{C}_X$. For $\mathscr{C}_X$, that contains the branching points x_i, let us look at x_1 and pretend the cylinder $\mathscr{C}_X$ is horizontal, in order to use geometric phrases. Then the lifts of cylinder loops from $\mathscr{C}_X$ to $\#_{z_1}^d(X, x_0, \omega)$ are determined by a loop below x_1 and by a second loop above x_1. Both can be represented by loops that avoid $\pi_d(l)$ since all cylinder loops of $\mathscr{C}_X$ above, respectively below x_1 have the same lift.

If two points $z_i \in \partial\mathscr{C}$ lie in the same boundary components of $\mathscr{C}$ we can connect them using part of a direction θ leaf in $\mathscr{C}$ as close as we wish to $\partial\mathscr{C}$ together with two appropriate line segments perpendicular to θ that will have z_i as endpoints. Then we apply the same intersection argument as before. □

Because holonomy data are constant on cylinders and saddle connections of the parameter space the orbit type asymptotic formulas can be simplified.

5. Asymptotic constants and parameter space geometry.

The following discussion together with the asymptotic formulas 19 and 20 will provide Theorem 6 and Theorem 7.
Consider the parameter space $(X_{ab}/dH_1, \pi_d^{-1}(x_0), \pi_d^*\omega)$ as the parameter space of d-symmetric covers and assume for simplicity the horizontal direction is completely periodic. Let $\mathscr{C}_i$, $i = 1, ..., n_h$ denote the maximal horizontal cylinders on the marked surface $(X_{ab}/dH_1, \pi_d^{-1}(x_0), \pi_d^*\omega)$ and their top boundary components by $\partial^{top}\mathscr{C}_i$. The previous Lemma shows, that the length spectrum of horizontal cylinders with multiplicity is constant on cylinders, such as $\mathscr{C}_i$ and their (top) boundaries. Each top boundary $\partial^{top}\mathscr{C}_i$ is a union of saddle connections and the Lemma shows that the d-symmetric covers parameterized by $\partial^{top}\mathscr{C}_i$ have the same length spectrum.

Associated to each horizontal cylinder $\mathscr{C}_i$ there is a datum $w_{i,1}, ..., w_{i,n_i}$ for the length of the horizontal cylinders on any d-symmetric cover defined by a point on $\mathscr{C}_i$. The same holds for the length of horizontal cylinders on d-symmetric surfaces parameterized by $\partial^{top}\mathscr{C}_i$, let us denote their total number by m_i and their length datum by $c_{i,1}, ..., c_{i,m_i}$. If $\mathscr{O}_z$ denotes the $\mathrm{SL}(X, x_0, \omega)$-orbit of a $z \in X_{ab}/dH_1$ with finite orbit, then the asymptotic formula for the horizontal cusp θ_h is:

$$(24)\qquad c_{\mathrm{C}}(\theta_h) = C \cdot i_h \sum_{i=1}^{n_h} \left[\frac{|\mathscr{O}_z \cap \mathscr{C}_i|}{|\mathscr{O}_z|} \sum_{k=1}^{n_i} \frac{1}{w_{i,k}^2} + \frac{|\mathscr{O}_z \cap \partial^{top}\mathscr{C}_i|}{|\mathscr{O}_z|} \sum_{k=1}^{m_i} \frac{1}{c_{i,k}^2} \right].$$

Here $C := \frac{\pi}{3\,\mathrm{vol}(\mathbb{H}/\mathrm{SL}(X,x_0,\omega))}$ denotes the constant in formula 18. Recall that i_h is the upper right entry of the minimal parabolic stabilizer of the (periodic) horizontal direction. For tori we have $i_h = 1$. To obtain the general version of the formula stated in the introduction assume $\{\theta_1, ..., \theta_{n_c}\}$ is a set of directions corresponding to the cusps of $\mathrm{SL}(X, x_0, \omega)$. Pick a cylinder decomposition $\mathscr{C}_j^l$ $j = 1, ..., m_l$ of $(X_{ab} - dH_1 x_{ab})/dH_1$ in direction θ_l. Then any $\#_z^d(X, x_0, \omega)$ with $z \in \mathscr{C}_j^l$ has precisely n_j^l maximal cylinders of circumference $w_{j1}^l, ..., w_{jn_j^l}^l$ and height $h_{j1}^l, ..., h_{jn_j^l}^l$ in direction θ_l. Any $\#_z^d(X, x_0, \omega)$ with $z \in \partial^{top}\mathscr{C}_j^l$ has s_j^l maximal cylinders of circumference $c_{j1}^l, ..., c_{js_j^l}^l$ in direction θ_l.

THEOREM 6. **[Cylinders]** *Under the previous assumptions and conventions, if $\#_z^d(X, x_0, \omega)$ has infinite $\mathrm{SL}(X, x_0, \omega)$ orbit, the asymptotic constant for periodic cylinders in the $\mathrm{SL}(X, x_0, \omega)$ orbit of the direction θ_l is:*

$$(25)\qquad \mathbf{c}_C^l(z) = \frac{1}{d^2\,\mathrm{area}(X, \omega)} \sum_{j=1}^{m_l} \sum_{k=1}^{n_j^l} \frac{\mathrm{area}(\mathscr{C}_j^l)}{(w_{jk}^l)^2}.$$

If $\#_z^d(X, x_0, \omega)$ has a finite $\mathrm{SL}(X, x_0, \omega)$ orbit $\mathscr{O}_z$, then:

$$(26)\qquad \mathbf{c}_C^l(z) = \frac{C \cdot i_l}{|\mathscr{O}_z|} \sum_{j=1}^{m_l} \left(\sum_{k=1}^{n_j^l} \frac{|\mathscr{O}_z \cap \mathscr{C}_j^l|}{(w_{jk}^l)^2} + \sum_{k=1}^{s_j^l} \frac{|\mathscr{O}_z \cap \partial^{top}\mathscr{C}_j^l|}{(c_{jk}^l)^2} \right).$$

In either case the asymptotic formula for cylinders on (S, α) is given by

$$\mathbf{c}_C(z) = \sum_{l=1}^{n_c} \mathbf{c}_C^l(z).$$

Saddle connections. The asymptotic formula for saddle connections on $\#_z^d(X, x_0, \omega)$ is analogous to the one for cylinders as long as $z \in X_{ab}/dH_1$ has a finite $\mathrm{SL}(X, x_0, \omega)$-orbit, i.e. if the d-symmetric surface $\#_z^d(X, x_0, \omega)$ is a Veech surface. In fact, one only needs to replace the holonomy vector datum corresponding to widths of horizontal cylinders with the one for the length of the saddle connections s_j bounding the horizontal cylinders of $\#_z^d(X, x_0, \omega)$ in formula 24.

Saddle connections between the two branch points. Let $V(\widehat{x}_0, \widehat{z}_0)$ denote the set of saddle connections on the d-symmetric cover $\pi_z : \#_z^d(X, x_0, \omega) \to (X, x_0, \omega)$ connecting the two cone-points $\widehat{z}_0 = \pi^{-1}\pi_d(z) \in \#_z^d(X, x_0, \omega)$ and $\widehat{x}_0 = \pi^{-1}(x_0) \in \#_z^d(X, x_0, \omega)$. The covering map π_z induces a surjective and isometric map

$$\pi_{d*} : V(\widehat{x}_0, \widehat{z}_0) \to V(x_0, z_0)$$

to the set of regular line segments connecting the marked points x_0 and z_0 on (X, x_0, z_0, ω). Since every regular segment in $V(x_0, z_0)$ has d preimages on $\#_z^d(X, x_0, \omega)$, the map π_* has degree d. In particular

$$c(V(\widehat{x}_0, \widehat{z}_0)) = d \cdot c(V(x_0, z_0))$$

and we only need to describe the respective counting formula for (X, x_0, z_0, ω).

Given this, it seems more interesting to look at certain subsets of saddle connections related to the saddle connections of $(X_{ab}/dH_1, \pi_d^{-1}(x_0), \pi_d^*\omega)$, particularly when $\#_z^d(X, x_0, \omega)$ is a lattice surface, or equivalently, $z \in X_{ab}/dH_1$ has finite orbit with respect to the Veech group $\mathrm{SL}(X, x_0, \omega)$ acting on X_{ab}/dH_1. We are mainly interested in the torus case, so let us further assume $x_d \in \pi_d^{-1}(x_0)$ is fixed under the action of $\mathrm{SL}(X, \omega)$ and that every periodic direction is in the $\mathrm{SL}(X, \omega)$ orbit of the periodic horizontal direction on (X, x_0, ω). Then, since z_0 is periodic, the marked surface (X, x_0, z_0, ω) is a Veech surface and so any $s \in V(x_0, z_0)$ is parallel to a completely periodic direction. Given our assumptions, there is $A \in \mathrm{SL}(X, x_0, \omega)$ so that $s_h = As$ is a horizontal saddle connection on (X, x_0, Az_0, ω) and hence all preimages of s_h on $\#_{Az}(X, x_0, \omega)$ are isometric to s_h. Since $s_h \in V_h(x_0, z_0)$ is horizontal, it is a subset of the π_d image of the horizontal saddle connection $(X_{ab}/dH_1, \pi_d^{-1}(x_0))$ that contains Az. This saddle connection, $\widetilde{s}_h$, emanates from, say $x_s \in \pi_d^{-1}(Ax_0)$ and contains the horizontal segment s_{Az}^+ from x_s to Az that maps isometrically onto s_h under π_d. If the base surface X is a torus then $\widetilde{s}_h$ connects two preimages of x_0. Then the set theoretical complement of $s_{Az}^+ \cup \{Az\}$ in $\widetilde{s}_h$ is also a horizontal saddle connection, say s_{Az}^- that corresponds isometrically to d saddle connections on $\#_{Az}(X, x_0, \omega)$.

We now state an orbit version of the counting formula for saddle connections of the type just described for *torus covers.* If $\#_z^d(X, x_0, \omega)$ is a lattice torus cover, any saddle connection in $V(\widehat{x}_0, \widehat{z}_0)$ is up to isometry (and multiplicity by a factor d) in the $\mathrm{SL}(X, x_0, \omega)$ orbit of a horizontal saddle connection on the marked parameter space $(X_{ab}/dH_1, \pi_d^{-1}(x_0), \mathscr{O}_z \cap SC_h(d))$. Here $\mathscr{O}_z := \mathrm{SL}(X, x_0, \omega) \cdot z$ and $SC_h(d)$ is the set of horizontal saddle connections on $(X_{ab}/dH_1, \pi_d^{-1}(x_0))$. For the formula we only need to know the multiplicity of each horizontal saddle connection on the actual covers. For d-symmetric covers it is always d.

Theorem 7 (Saddle connections on d-symmetric torus covers). *Let $SC_h(d)$ denote the horizontal saddle connections on the parameter space* $(\mathbb{C}/d\mathbb{Z}[i], \mathbb{Z}[i], dz)$

of d-symmetric torus covers $(\mathbb{C}/\mathbb{Z}[i],[0],dz)$*. If* $\#_z^d(\mathbb{C}/\mathbb{Z}[i],[0],dz)$ *is a lattice surface, then the asymptotic quadratic growth rate of saddle connections connecting two cone points of* $\#_z^d(\mathbb{C}/\mathbb{Z}[i],[0],dz)$ *is:*

$$c_{\pm}(z) = \frac{d}{|\mathscr{O}_z|} \sum_{s \in SC_h(d)} \sum_{y \in \mathscr{O}_z \cap s} \left[\frac{1}{|s_y^-|^2} + \frac{1}{|s_y^+|^2} \right]. \tag{27}$$

Since on the torus $(\mathbb{C}/d\mathbb{Z}[i],\mathbb{Z}[i],dz)$ the affine involution induced by $-\,\mathrm{id}$ on $\mathbb{C}$ preserves $\mathbb{Z}[i]$, the formula simplifies:

$$c_{\pm}(z) = \frac{2d}{|\mathscr{O}_z|} \sum_{s \in SC_h(d)} \sum_{y \in \mathscr{O}_z \cap s} \frac{1}{|s_y^+|^2}. \tag{28}$$

The formula is easily derived using the same method as for the counting formulas for cylinders. Instead of square reciprocals of cylinder circumferences assigned to each saddle connection in $SC_h(d)$, one considers the pair of (horizontal reciprocal square) distances of a point in $\mathscr{O}_z \cap SC_h(d)$ to the relevant points in $\pi_d^{-1}(x_0)$.

6. Teichmüller curves and moduli space

If (Y,τ) is a lattice surface with Veech group $\mathrm{SL}(Y,\tau)$, then the image of $\mathrm{SL}_2\mathbb{R}(Y,\tau)$ in $\Omega\mathcal{M}_g$ projects to a Teichmüller curve $\mathcal{C} \subset \mathcal{M}_g$ in moduli space of Riemann surfaces of genus $g = g(Y)$. This curve is the image of the algebraic immersion

$$\mathbb{H}/\mathrm{SL}(Y,\tau) \to \mathcal{M}_g,$$

that is an isometry with respect to the Teichmüller metric on $\mathcal{M}_g$. For more on Teichmüller curves particularly in genus 2, see McMullen [**McM1**]–[**McM5**], as well as Bouw and Möller [**BM**]. The first examples of Teichmüller curves that do not arise as torus covers were discovered by Veech [**V1**].

All d-symmetric (X,x_0,ω) covers of a lattice surface (X,x_0,ω) that have finite $\mathrm{SL}(X,x_0,\omega)$ orbit in their respective parameter space define Teichmüller curves. Since we may not use the full group of affine maps the curves we obtain in the global description below may only be covers of the actual Teichmüller curve.

The family of d-symmetric differentials. To obtain Teichmüller curves we consider the family of all d-symmetric torus covers over any (normalized) base torus. That is, we look at all (unimodular) lattices and include degenerate covers for simplicity. All those covers are parameterized by points in

$$\mathrm{SL}_2\mathbb{R} \ltimes \mathbb{C}/\mathrm{SL}_2\mathbb{Z} \ltimes d\mathbb{Z}[i].$$

If $(Y,\widetilde{x},\tau) \to (\mathbb{C}/\Lambda,[0],dz)$ is a d-symmetric cover with lattice stabilizer $\mathrm{SL}(Y,\widetilde{x},\tau)$ then its orbit

$$\mathrm{SL}_2\mathbb{R} \cdot (Y,\widetilde{x},\tau) \hookrightarrow \mathrm{SL}_2\mathbb{R} \ltimes \mathbb{C}/\mathrm{SL}_2\mathbb{Z} \ltimes d\mathbb{Z}[i].$$

is given by $\mathrm{SL}_2\mathbb{R}/\mathrm{SL}(Y,\widetilde{x},\tau)$. The quotient with respect to $\mathrm{SO}_2\mathbb{R}$,

$$\mathrm{SL}_2\mathbb{R}/\mathrm{SL}(Y,\widetilde{x},\tau) \to \mathbb{H}/\mathrm{SL}(Y,\widetilde{x},\tau)$$

is a finite cover of the Teichmüller curve (Y,τ) defines in moduli space.

7. Cylinder decompositions of d-symmetric forms

We calculate the cylinder decompositions for d-symmetric forms over the fixed lattice $\Lambda := \mathbb{Z}[i]$. Note, that the knowledge of the parameter space and its geometry simplifies some arguments. In this section we will write $\mathbb{T}_d := \mathbb{C}/d\Lambda$, $\dot{\mathbb{T}}_d := (\mathbb{C} - \Lambda)/d\Lambda$ and denote marked tori as $\mathbb{T}_{d,m} := (\mathbb{C}/d\Lambda, \Lambda/d\Lambda)$. Further, let us denote the integer part of $y \in \mathbb{R}_+$ by $\lfloor y \rfloor$ and if $[0,z] \subset \mathbb{C}$ is a line segment, then $[\![0,z]\!] \in H_1(\mathbb{T}, \{[0],[z]\};\mathbb{Z})$ denotes the indicated relative homology class of its image on $\mathbb{T}$. Whenever it does not lead to confusion we will use the abbreviation $(a_1,...,a_n) = \gcd(a_1,...,a_n)$.

PROPOSITION 17. *Let $\#_z^d(\mathbb{T}, dz)$ be a d-symmetric form. Then its horizontal foliation is completely periodic and it contains*

- $(\lfloor \operatorname{Im} z \rfloor, d)$ *cylinders of width* $\frac{d}{(\lfloor \operatorname{Im} z \rfloor, d)}$ *and*
- $(\lfloor \operatorname{Im} z \rfloor + 1, d)$ *cylinders of width* $\frac{d}{(\lfloor Im(z) \rfloor + 1, d)}$, *if* $\operatorname{Im} z \notin \mathbb{Z}_d$
- $(\operatorname{Im} z, d)$ *cylinders of width* $\frac{d}{(\operatorname{Im} z, d)}$, *if* $\operatorname{Im} z \in \mathbb{Z}_d$.

PROOF. Given $\#_z^d(\mathbb{T}, dz)$, denote the projection of $z \in \mathbb{T}_d$ to $\mathbb{T}$ by z. First we assume $\operatorname{Im} z \notin \mathbb{Z}_d$. If l is a horizontal leaf on the two marked torus $(\mathbb{T}, [0], [z])$ that lies on the cylinder having $[z]$ in its upper boundary, then we have an intersection number $\langle l, [\![0,z]\!]\rangle \equiv i+1 \mod d$ where $i \equiv \lfloor \operatorname{Im} z \rfloor \mod d$. By cyclicity every preimage of l on $\#_z^d(\mathbb{T}, dz)$ has length $d/(\lfloor \operatorname{Im} z \rfloor + 1, d)$ and so there must be $(\lfloor \operatorname{Im} z \rfloor + 1, d)$ of those. The same way, if l is a horizontal leaf on $(\mathbb{T}, [0], [z])$, that lies on the cylinder with $[z]$ on its lower boundary, then $\langle l, [\![0,z]\!]\rangle = i$ where i is as above. By cyclicity every preimage of l on $\#_z^d(\mathbb{T}, dz)$ has length $d/(\lfloor \operatorname{Im} z \rfloor, d)$ and then there must be $(\lfloor \operatorname{Im} z \rfloor, d)$ of those.

If $\operatorname{Im} z \in \mathbb{Z}_d$, then $(\mathbb{T}, [0], [z])$ has only one horizontal cylinder and a horizontal loop l intersects with $[\![0,z]\!]$ exactly $\operatorname{Im} z$ times. As before we obtain $(\operatorname{Im} z, d)$ lifted loops of length $d/(\operatorname{Im} z, d)$.

Since every horizontal cylinder of $\#_z^d(\mathbb{T}, dz)$ maps to a horizontal cylinder of $(\mathbb{T}, [0], [z])$ the claim follows. $\square$

The proof of the previous proposition allows us to specify the area of each horizontal cylinder on $\#_z^d(\mathbb{T}, dz)$. We record this for later use.

COROLLARY 7. *If* $\operatorname{Im} z \notin \mathbb{Z}_d$ *then the horizontal cylinders of* $\#_z^d(\mathbb{T}, dz)$ *mapped to the cylinder on* $(\mathbb{T}, [0], [z])$ *that has* $[z]$ *on its top boundary have area* $\operatorname{Im} z \cdot d/\gcd(\lfloor \operatorname{Im} z \rfloor + 1, d)$. *The cylinders mapped to the cylinder on* $(\mathbb{T}, [0], [z])$ *that has* $[z]$ *on its lower boundary have area* $(1 - \operatorname{Im} z) \cdot d/\gcd(\lfloor \operatorname{Im} z \rfloor, d)$. *If* $\operatorname{Im} z \in \mathbb{Z}_d$ *then the horizontal cylinders on* $\#_z^d(\mathbb{T}, dz)$ *have (integer) area* $d/\gcd(\operatorname{Im} z, d)$.

Divisibility properties for the numbers of cylinders calculated in Proposition 17 give:

COROLLARY 8. *Assume* $d > 2$ *is prime, then for any* j *the only possible numbers of maximal* horizontal cylinders *of d-symmetric covers parameterized by* $\mathscr{C}_j$ *are* 2 *and* $d+1$.

Forms parameterized by lattice coordinates. Now we describe cylinder decompositions of surfaces with lattice twist coordinates, i.e. those $\#_z^d(\mathbb{T}, dz)$ with

$$z + d\Lambda \in \Lambda/d\Lambda.$$

Because the hyperelliptic involution of $\mathbb{T}_d$ sends the surface $\#_z^d(\mathbb{T}, dz)$ to the translation equivalent surface $\#_{-z}^d(\mathbb{T}, dz)$, $\Lambda/d\Lambda \subset \mathbb{T}_d$ does not provide a classification space for those. Here we study their horizontal cylinder decompositions and their $\mathrm{SL}_2\mathbb{Z}$-orbits in $\mathbb{T}_d$.

With the convention $(a, b) = \gcd(a, b)$, given a Gaussian integer $z \in \Lambda$ we write $(z) := (\mathrm{Re}\, z, \mathrm{Im}\, z)$. If furthermore $d \in \mathbb{N}$ we set $(z, d) := ((\mathrm{Re}\, z, \mathrm{Im}\, z), d)$.

Corollary 9. *For $z \in \Lambda/d\Lambda$ the d-symmetric differential $\#_z^d(\mathbb{T}, dz)\backslash\{\pi^{-1}[0]\}$ is a disjoint union of (z, d) tori tiled by unit squares. Each such torus has area $d/(z, d)$ and decomposes in $(\mathrm{Im}\, z, d)/(z, d)$ horizontal cylinders of width $d/(\mathrm{Im}\, z, d)$ and $(\mathrm{Re}\, z, d)/(z, d)$ vertical cylinders of width $d/(\mathrm{Re}\, z, d)$. For any given divisor $k|d$ the surfaces that are unions of k tori are all on one $\mathrm{SL}_2\mathbb{Z}$ orbit.*

Proof. If $j \in \mathbb{Z}$, the horizontal foliation of $\#_j^d(\mathbb{T}, dz)$ has d cylinders of width 1 while its vertical foliation has (j, d) cylinders of width $d/(j, d)$. Thus $\#_j^d(\mathbb{T}, dz)$ consists of (j, d) tori each of width 1 and height $d/(j, d)$. By the $\mathrm{SL}_2\mathbb{Z}$ orbit classification every form represented by a point in $\Lambda/d\Lambda$ is on the $\mathrm{SL}_2\mathbb{Z}$ orbit of a form $\#_a^d(\mathbb{T}, dz)$ for some $a \in \mathbb{Z}_d$. The shape of those $\mathrm{SL}_2\mathbb{Z}$ orbits imply that $\#_z^d(\mathbb{T}, dz) \in \mathrm{SL}_2\mathbb{Z} \cdot \#_a^d(\mathbb{T}, dz)$ if and only if $(z, d) = (a, d)$. In particular for $z + d\Lambda \in \Lambda/d\Lambda$ the forms $\#_z^d(\mathbb{T}, dz)$ with fixed (z, d) are unions of (z, d) tori, each of area $d/(z, d)$.

It follows from Proposition 17 that $\#_z^d(\mathbb{T}, dz)$ has $(\mathrm{Im}\, z, d)$ horizontal cylinders of width $d/(\mathrm{Im}\, z, d)$ and $(\mathrm{Re}\, z, d)$ vertical cylinders of width $d/(\mathrm{Re}\, z, d)$. ☐

8. Illumination and non illumination

On most d-symmetric torus covers there is a family of parallel saddle connections that, when removed together with its endpoints, leaves exactly d connected components. Since the only degenerate d-symmetric surface that, after removing one point, falls into d connected components is $\#_0^d(\mathbb{T}, dz)$, i.e. the surface determined by the origin of $\mathbb{C}/d\Lambda$, we can formulate that property in terms of the parameter space geometry as follows. Can we see a given point in $\mathbb{C}\backslash\Lambda/d\Lambda$ from the origin? We will see that there are finitely many points for which this is not possible, at least if $d > 2$. Another way to characterize such a d-symmetric cover is, that one cannot represent its defining relative homology class by a straight line segment that does not intersect itself. In case of a d-symmetric torus cover with second branch point $[z]_\Lambda \in \mathbb{C}/\Lambda$ the existence of a homology representing line segment is the same as the existence of an open line segment in $\{(0, w) : w \in z + d\Lambda\} \subset \mathbb{C}$ that lies in $\mathbb{C} - \Lambda$.

Let $[a, b] \subset \mathbb{C}$ denote the (closed) line segment in $\mathbb{C}$ and $[a, b]_\Lambda$ its image (modulo Λ) on $\mathbb{C}/\Lambda$. Recall the isogeny

$$(\mathbb{C}\backslash\Lambda)/d\Lambda \to (\mathbb{C}\backslash d^{-1}\Lambda)/\Lambda = (\mathbb{C}/\Lambda)\backslash T(d)$$

given by multiplication with d^{-1}. Since the set of d-torsion points $T(d) \subset \mathbb{C}/\Lambda$ is invariant under the action of $\mathrm{SL}(\Lambda)$, this action is well-defined on both tori with the respective points removed. The isogeny d^{-1} is $\mathrm{SL}(\Lambda)$-equivariant.

LEMMA 2. *For $d > 0$ fixed let $n \in \{2, ..., d\}$. Then the d-symmetric torus cover $\#_z^d(\mathbb{C}/\Lambda, dz)$ is representable by a regular slit construction, if and only if z is not a torsion point of order $n \nmid d$. In particular those covers appear only for $d > 2$.*

PROOF. By $\mathrm{SL}_2\mathbb{R}$ invariance of geodesic properties it is enough to show the claim for $\Lambda = \mathbb{Z}[i]$, nevertheless (for readability) we continue to denote the integer lattice by Λ. Representability of a regular d-symmetric form $\#_z^d(\mathbb{T}, dz)$ by a line segment $[0, w]$, for some $w \in z + d\Lambda$ follows, if w is visible from the origin, that is if the line segment $[0, w]$ lies in $\mathbb{C}\backslash\Lambda$. If we can find $A \in \mathrm{SL}(\Lambda)$ mapping the point $[z]$ into the open disk

$$D := \{[w] \in (\mathbb{C}\backslash\Lambda)/d\Lambda : |w| < 1\} \subset (\mathbb{C}\backslash\Lambda)/d\Lambda,$$

we are done. First D is a genuine disk since no point of $\Lambda/d\Lambda$ has distance less than one from the origin. But then the interval $[0, Az]_\Lambda$ lies in $(\mathbb{C}\backslash\Lambda)/d\Lambda$, and by $\mathrm{SL}(\Lambda)$ invariance of $(\mathbb{C}\backslash\Lambda)/d\Lambda$ $A^{-1}[0, A[z]]$ is an interval in $(\mathbb{C}\backslash\Lambda)/d\Lambda$ with endpoints $[0]$ and $[z]$. It is always possible to find a transformation $A \in \mathrm{SL}_2\mathbb{Z} = \mathrm{SL}(\Lambda)$ mapping an irrational $z \in \mathbb{C}$ into D, since $\mathrm{SL}_2\mathbb{Z}$ orbits of irrational numbers are dense on $\mathbb{T}$. For rational $[z]$ we use the isogeny to transform the torus $(\mathbb{C}\backslash\Lambda)/d\Lambda$ to the marked torus $\mathbb{T}\backslash(\frac{1}{d}\Lambda)/\Lambda$, which maps rational points to rational points. This is an $\mathrm{SL}_2\mathbb{Z}$-equivariant map, in particular the action of $\mathrm{SL}_2\mathbb{Z}$ is well-defined on $\mathbb{T}\backslash(\frac{1}{d}\Lambda)/\Lambda$. The image of D under this map is

$$D_d := \{[w] \in \mathbb{C}/d\Lambda : |w| < d^{-1}\} \subset (\mathbb{C}\backslash\frac{1}{d}\Lambda)/\Lambda.$$

By the $\mathrm{SL}_2\mathbb{Z}$ orbit classification we there is an $A \in \mathrm{SL}_2\mathbb{Z}$ that maps a given rational coordinate $z \in (\mathbb{Q}\oplus\mathbb{Q}i)/\Lambda \subset \mathbb{C}/\Lambda$ of denominator n to the point $[1/n] \in \mathbb{C}/\Lambda$. Since $[1/n] \in U_d$ if and only if $n > d$, the existence of a regular slit representation follows by transforming back to $(\mathbb{C}\backslash\Lambda)/d\Lambda$.

The remaining points to consider are in $n^{-1}\Lambda/\Lambda$ with $0 < n < d$. If n is a d-torsion point, then n must divide d and $n^{-1}\Lambda/\Lambda$ is a subset of the set of removed points $d^{-1}\Lambda/\Lambda$. If on the other hand $n \nmid d$, consider a line segment $I \subset \mathbb{C}/\Lambda$ between $[0] \in \mathbb{C}/\Lambda$ and some point $[z] \in n^{-1}\Lambda/\Lambda$. Then there is an $A \in \mathrm{SL}_2\mathbb{Z}$, such that $A \cdot I = [0, 1/n]$. Since $n < d$, the point $[1/d]$ lies on the segment $[0, 1/n] = A \cdot I$. Since $\mathrm{SL}_2\mathbb{Z}$ preserves the order of points the d-torsion point $A[1/d]$ lies on I and so $I \nsubseteq (\mathbb{C}\backslash d^{-1}\Lambda)/\Lambda$ showing the claim. □

We close this section with some remarks: For any representation of the cover $\pi : \#_w^d(\mathbb{C}/\Lambda, dz) \to (\mathbb{C}/\Lambda, dz)$ by a slit construction along $[0, w] \subset \mathbb{C} - \Lambda$ the preimage $\pi^{-1}([0, w])$ consists of d *saddle connections* of length $|w|$. If $\#_z^d(\mathbb{C}/\Lambda, dz)$ is on the $\mathrm{SL}_2\mathbb{Z}$ orbit of $\#_w^d(\mathbb{C}/\Lambda, dz)$ then is representable by a slit construction. Because then there is $A \in \mathrm{SL}_2\mathbb{Z}$, so that $A[w] = [z] \in (\mathbb{C} - \Lambda)/d\Lambda$. Viewing $[0, w] \subset (\mathbb{C} - \Lambda)/d\Lambda$ as a line segment in parameter space and noticing $A[0] = [0]$ we obtain a line segment $A[0, w] \subset (\mathbb{C} - \Lambda)/d\Lambda$ from $[0]$ to $[z]$. Now $\#_z^d(\mathbb{T}, dz)$ can be deformed into the trivial d-symmetric cover $\#_{[0]}^d(\mathbb{T}, dz)$ inside $(\mathbb{C} - \Lambda)/d\Lambda$ by moving $[z]$ into $[0]$ along the line segment $A[0, w] \subset (\mathbb{C} - \Lambda)/d\Lambda$.

The lattice $\Lambda \in \mathbb{C}/d\Lambda$ is a *blocking set* for the finite set $\bigsqcup_{n\nmid d}(n^{-1}d\Lambda)/d\Lambda$ relative to the point $[0] \in \mathbb{C}/d\Lambda$. Informally, a rotating laser installed at the point $[0]$ will never illuminate any point in $\bigsqcup_{n\nmid d}(n^{-1}d\Lambda)/d\Lambda$, because the rays that would illuminate will be swallowed by the holes $\Lambda \in \mathbb{C}/d\Lambda$.

9. Asymptotic constants — cylinders, generic case

Siegel-Veech constants for generic forms. We have all information to evaluate the Siegel-Veech constants as function of the point in the space of d-symmetric covers. As before we restrict our considerations to the covers with absolute period lattice Λ.

Proof of Theorem 4. Since we assume $\#_z^d(\mathbb{T}, dz)$ is not arithmetic, i.e. $[z]_d \in \mathbb{T}_d$ is not a torsion point, it has infinite $\mathrm{SL}_2\mathbb{Z}$ orbit. In that case formula 25 of Theorem 6 is applied. This formula is the integral formula in [**EMS**]. It is itself derived from the Siegel-Veech formula in [**V3**] and states that the the $\mathrm{SL}_2\mathbb{Z}$ orbit of a generic surface can be treated as uniform distribution with respect to Lebesgue measure. The only relevant part of the Siegel-Veech formula is the integral over the horizontal cylinders of the surface parameterizing covers over a fixed base surface, here $\mathbb{T} = \mathbb{C}/\mathbb{Z}[i]$. The measure on the parameter space is the standard Lebesgue measure induced by the euclidian structure. Given this, each point z, representing the cover $\#_z^d(\mathbb{T}, dz)$, is weighted by the areas of the covers horizontal cylinders $C = C(z)$. Recall that the numbers of horizontal cylinders and their width is constant for all surfaces $\#_z^d(\mathbb{T}, dz)$ in a horizontal cylinder $\mathscr{C}$ of parameter space. Written in differential form for any horizontal cylinder in parameter space one obtains:

$$dc = \frac{1}{\mathrm{area}(\mathbb{C}/d\mathbb{Z}[i])} \frac{(\mathrm{area}\, C(z))^\alpha}{w_C^2} \frac{i}{2} dz \wedge d\bar{z}$$

Using the data from Proposition 17 and Corollary 7 this differential becomes:

$$dc = d^{\alpha-4}[(\lfloor \mathrm{Im}\, z \rfloor + 1, d)^{3-\alpha} (\mathrm{Im}\, z)^\alpha + (\lfloor \mathrm{Im}\, z \rfloor, d)^{3-\alpha} (1 - \mathrm{Im}\, z)^\alpha] \frac{i}{2} dz \wedge d\bar{z}$$

Putting $h = \mathrm{Im}\, z$, and $l = \mathrm{Re}\, z$, for the i-th cylinder, with respect of our chosen cylinder parameterization of parameter space one has:

$$dc = d^{\alpha-4}[(i+1, d)^{3-\alpha} h^\alpha + (i, d)^{3-\alpha} (1-h)^\alpha] dh \; dl$$

Integrating over a horizontal cylinder in the bounds $0 \le l \le d$ and $0 \le h \le 1$ gives:

$$\mathbf{c} = 2 \frac{d^{\alpha-3}}{1+\alpha} \sum_{i=1}^{d} (i, d)^{3-\alpha}$$

Using the Euler totient $\varphi(p)$, as in the introduction, brings the formula into its final form:

$$\mathbf{c}_{C,\alpha}^d = \frac{2}{1+\alpha} \sum_{i=1}^{d} \left(\frac{(i,d)}{d} \right)^{3-\alpha} = \frac{2}{1+\alpha} \sum_{p|d} \frac{\varphi(p)}{p^{3-\alpha}}. \tag{29}$$

□

10. Asymptotic constants — cylinders, finite orbit case

We calculate the Siegel-Veech constants for d-symmetric forms represented by torsion points of order n in $\mathbb{T}_d$. For fixed $n \in \mathbb{N}$ take a natural number $1 \le a \le n$ and define

$$c_{d,n}(a) := \frac{\pi}{6} \lim_{T \to \infty} \frac{N(V_n(a), T)}{T^2}$$

to be the quadratic growth rate of the vector distribution

$$V_n(a) := \{\mathrm{SL}_2\mathbb{Z}\cdot \mathrm{hol}(l) : l \text{ regular horizontal leaf on } \#_z^d(\mathbb{T}, dz) \text{ with } z \in \mathcal{L}_{a/n} \cap \mathbb{T}_d(n)\}. \tag{30}$$

Here $\mathcal{L}_{a/n}$ denotes the horizontal leaf on $(\mathbb{T}_d, dz)$, that contains the point $[id\cdot a/n]_d$. As before let $\lfloor ad/n \rfloor \in \mathbb{Z}$ be the integer part of ad/n

LEMMA 3. *For* $1 \leq a \leq n$

$$c_{d,n}(a) = \frac{n}{\varphi(n)\psi(n)}\frac{\varphi((a,n))}{(a,n)}\left(\frac{(\lfloor ad/n\rfloor, d)^3}{d^2} + \frac{(\lfloor ad/n\rfloor+1, d)^3}{d^2}\right). \tag{31}$$

PROOF. Recall that $\mathbb{T}_d(n)$ is the number of primitive torsion points of order n. Consider the horizontal leaf $\mathcal{L}_{a/n} \subset \mathbb{T}_d$ as defined already. Then for fixed a and n:

$$|\{b \in \mathbb{Z}_n : (b, a, n) = 1\}| = |\{b \in \mathbb{Z}_n : (b, (a, n)) = 1\}| = \\ = \begin{cases} |\Psi^{-1}((\mathbb{Z}_{(a,n)})^*)|, & \text{if } (a,n) \geq 2 \\ n, & \text{if } (a,n) = 1 \end{cases}$$

where $\Psi : \mathbb{Z}_n \to \mathbb{Z}_{(a,n)}$ is given by taking classes modulo (a, n). Thus

$$|\mathcal{L}_{a/n} \cap \mathbb{T}_d(n)| = n\frac{\varphi((a,n))}{(a,n)}. \tag{32}$$

By Proposition 17 the horizontal foliation of $\#_z^d(\mathbb{T}, dz)$ with $z \in \mathcal{L}_{a/n} \cap \mathbb{T}_d(n)$

- *always has* $(\lfloor ad/n\rfloor, d)$ *cylinders of width* $d/(\lfloor ad/n\rfloor, d)$
- *and it has* $(\lfloor ad/n\rfloor+1, d)$ *cylinders of width* $d/(\lfloor ad/n\rfloor+1, d)$, *if* $ad/n \notin \mathbb{Z}$.

With $|\mathbb{T}_d(n)| = \varphi(n)\psi(n)$ we find the quadratic growth constants above. □

The first part of the proof of the Lemma implies that the Teichmüller disk through the torsion points $\mathbb{T}(n)$ of order n has

$$cu(n) = \frac{1}{2}\sum_{a=1}^{n}\varphi((a,n)) = \frac{1}{2}\sum_{l|n}\varphi\left(\frac{n}{l}\right)\varphi(l) \tag{33}$$

cusps if $n \geq 3$, and 2 cusps if $n = 2$.

Now the Siegel-Veech constant $\mathbf{c}_{d,n} = \sum_{a=1}^{n} c_{d,n}(a)$ for periodic cylinders on d-symmetric differentials $\#_z^d(\mathbb{T}, dz)$ with $[z]_d \in \mathbb{T}_d(n)$ is

$$\mathbf{c}_{d,n} = \frac{n}{\varphi(n)\psi(n)}\left(\sum_{a=1}^{n}\frac{\varphi((a,n))}{(a,n)}\frac{(\lfloor ad/n\rfloor, d)^3}{d^2} + \right. \\ \left. + \sum_{ad/n\notin\mathbb{Z}}\frac{\varphi((a,n))}{(a,n)}\frac{(\lfloor ad/n\rfloor+1, d)^3}{d^2}\right). \tag{34}$$

To simplify this expression further, we *consider torsion points of order* n *with* $(n, d) = 1$. Then all a with $ad/n \in \mathbb{Z}$ are multiples of n and consequently

$$|\mathbb{T}_d(n) \cap \partial^{btm}\mathscr{C}_i| = \begin{cases} \varphi(n) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \tag{35}$$

and:

$$\mathbf{c}_d(n) = \frac{n}{\varphi(n)\psi(n)} \left(d\frac{\varphi(n)}{n} + 2 \sum_{a \not\equiv 0 \bmod n} \frac{\varphi((a,n))}{(a,n)} \frac{(\lfloor ad/n \rfloor, d)^3}{d^2} \right). \tag{36}$$

We always assume $a \in \{-n+1, -n+2, ... -1, 0, 1, ..., n-1\}$ represents a class in $\mathbb{Z}_n$.

Prime d. If in addition to the previous assumptions d *is prime* the numbers of maximal cylinders in a given direction is either $1, 2, d$, or $d+1$ and the quadratic growth rates are:

# of cylinders	$\mathbf{c}_d(n) =$
1	$\frac{n}{\varphi(n)\psi(n)} \frac{1}{d^2} \sum_{da/n \in \{1,...,d-1\}} \frac{\varphi((a,n))}{(a,n)}$
2	$\frac{n}{\varphi(n)\psi(n)} \frac{2}{d^2} \sum_{d-1 > \frac{a}{n} d > 1} \frac{\varphi((a,n))}{(a,n)}$
d	$\frac{n}{\varphi(n)\psi(n)} \frac{d\varphi(n)}{n} = \frac{d}{\psi(n)}$
d+1	$\frac{n}{\varphi(n)\psi(n)} 2\left(d + \frac{1}{d^2}\right) \sum_{0<\lvert a\rvert<n/d} \frac{\varphi((a,n))}{(a,n)}$

In particular there are two possibilities for prime d:
— *either* $(n,d) = 1$, then there are *no* directions with only *one maximal cylinder* on the forms located on the $\mathrm{SL}_2\mathbb{Z}$ orbit of $[1/n]$ and we find

$$\begin{aligned} \mathbf{c}_d(n) = \frac{n}{\varphi(n)\psi(n)} \Bigg(\frac{2}{d^2} \sum_{1 < \frac{a}{n} d < d-1} \frac{\varphi((a,n))}{(a,n)} + \frac{d\varphi(n)}{n} + \\ + 2\left(d + \frac{1}{d^2}\right) \sum_{0 < \frac{\lfloor a \rfloor}{n} d < 1} \frac{\varphi((a,n))}{(a,n)} \Bigg), \end{aligned} \tag{37}$$

— *or* $(d,n) = d$, then the asymptotic constant is the sum of all four terms in the table above. In particular, there are *always* directions containing precisely *one maximal cylinder* on forms contained in the orbit of $[1/n]$.

Explicit constants for 2-symmetric covers. For small d the above formulæ simplify. In particular for $d = 2$ and n *odd*, formula (37) reads

$$\begin{aligned} c_2(n) = \frac{1}{\varphi(n)\psi(n)} \left(2\varphi(n) + \frac{9n}{4} \sum_{a \not\equiv 0 \bmod n} \frac{\varphi((a,n))}{(a,n)} \right) = \\ = \frac{2}{\psi(n)} + \frac{9}{4}\left(1 - \frac{1}{\psi(n)}\right) = \frac{9}{4} - \frac{1}{4\psi(n)}. \end{aligned} \tag{38}$$

The only thing we have used at this point, is the following identity derived from equation (32):

$$\varphi(n)\psi(n) = n\sum_{a=1}^{n}\frac{\varphi((a,n))}{(a,n)} = \varphi(n) + n\sum_{a=1}^{n-1}\frac{\varphi((a,n))}{(a,n)}.$$

If $d = 2$ the d-cylinder directions are simply 2-cylinder directions. The parameter space $\mathbb{T}_{2,m}$ has two horizontal cylinders, and the horizontal foliation of surfaces parameterized by either one of these cylinders contains $3 = 2+1$ maximal cylinders.

For *even* n the parameter choice $a = n/d = n/2$ gives points on horizontal leaf going through $[i]_2 \in \mathbb{T}_2$. This horizontal leaf (see table 10 above) reflects all directions with exactly one closed cylinder, thus:

(39)
$$\begin{aligned}\mathbf{c}_2(n) &= \frac{1}{\varphi(n)\psi(n)}\left(2\varphi(n) + \frac{1}{4}\varphi(\frac{n}{2}) + \frac{9n}{2}\sum_{2a/n\neq 1,2}\frac{\varphi((a,n))}{(a,n)}\right) = \\ &= \begin{cases}\frac{9}{4\psi(n)} + \frac{9}{4}\left(1 - \frac{2}{\psi(n)}\right) = \frac{9}{4} - \frac{9}{4\psi(n)} & \text{if } 4 \nmid n \\ \frac{17}{8\psi(n)} + \frac{9}{4}\left(1 - \frac{3}{2\psi(n)}\right) = \frac{9}{4} - \frac{5}{4\psi(n)} & \text{if } 4 \mid n.\end{cases}\end{aligned}$$

The two cases appear because:

(40)
$$\varphi\left(\frac{n}{2}\right) = \begin{cases}\varphi(n), & \text{if } 4 \nmid n \\ \frac{1}{2}\varphi(n), & \text{if } 4 \mid n.\end{cases}$$

11. Asymptotic constants — saddle connections

A cover $\pi : (Y,\tau) \to (X,\omega)$ is called *balanced*, if

$$\pi^{-1}(\pi(Z(\omega))) = Z(\omega),$$

where $Z(\omega)$ is the set of zeros of ω. Since d-symmetric covers are fully branched they are *balanced*.

Since $\#_z^d(\mathbb{T}, dz)$ is a $\mathbb{Z}_d$ cover every saddle connection $s \in SC(\mathbb{T}, [0], [z])$ on the marked torus $(\mathbb{T}, [0], [z])$ has d saddle connections as preimages

$$\pi^{-1}(s) = \{s_1, ..., s_d\} \subset SC(\#_z^d(\mathbb{T}, dz).$$

Using that it is easy to calculate the quadratic growth rate of saddle connections on $\#_z^d(\mathbb{T}, dz)$ once the growth rates on the two marked torus $(\mathbb{T}, [0], [z])$ are known. Those are calculated in **[S1]**. As the two marked torus can be viewed as a 1-symmetric cover, the results in **[S1]** on the other hand easily follow from the formulas given in the introduction. All we need to do is to determine the quadratic constants for saddle connections between $[0]$ and $[z]$ on $(\mathbb{T}, [0], [z])$. We may use that $(\mathbb{T}, [0], dz)$ is the parameter space for 1-symmetric forms and apply the formula for saddle connections developed earlier.

To consider parameters $[z]$, that are primitive n-torsion points, we only need the pairs of distances to primitive the points in $\mathrm{SL}_2\mathbb{Z} \cdot [1/n] \cap \mathcal{L}_0 = \{[i/n] \in \mathbb{T} :$

$\gcd(i,n)=1\}$ to the origin and add the summed squares of their reciprocals. So

$$\mathbf{c}_{\pm}(n) = \frac{2n^2}{\varphi(n)\psi(n)} \sum_{(i,n)=1} \frac{1}{i^2} \tag{41}$$

and the limit $n \to \infty$ is, up to multiplicity, the Siegel-Veech constant for saddle connections starting at irrational points, i.e. when $[z] \notin \mathbb{Q}[i]/\mathbb{Z}[i]$:

$$\mathbf{c}_{\pm} = \frac{\pi^2}{3}. \tag{42}$$

As a consequence we obtain:

COROLLARY 10. *The quadratic constant* $c_{\pm}^d(n)$ *for saddle connections* $SC_{\pm}(\#_z^d\mathbb{T}, dz)$ *connecting distinct cone points on a d-symmetric cover* $\#_z^d(\mathbb{T}, dz)$ *so that* $z \in \mathbb{T}_d(n)$ *is*

$$\mathbf{c}_{\pm}^d(n) = d\frac{2n^2}{\varphi(n)\psi(n)} \sum_{(i,n)=1} \frac{1}{i^2}. \tag{43}$$

For d-symmetric covers with infinite orbit:

$$\mathbf{c}_{\pm}^d = d\frac{\pi^2}{3}. \tag{44}$$

While this is elementary, there are quadratic growth rates of certain saddle connections between the two cone points that tell us more about the geometry of the surface.

Let us fix a d-symmetric cover $\pi : \#_z^d(\mathbb{T}, dz) \to (\mathbb{T}, dz)$ and remove the d saddle connections including their endpoints, say $\{s_1, ...s_d\}$, in the preimage of a saddle connection on $\mathbb{T}$ connecting the ramification points of the cover. Then the number of connected components of $\#_z^d(\mathbb{T}, dz)\backslash\{\overline{s}_1, ..., \overline{s}_d\}$ is a divisor m of d. Indeed, moving the parameter z of $\#_z^d(\mathbb{T}, dz)$ in (shrinking) direction of any of the d-saddle connection in parameter space gives a path in parameter space. Along that path the d saddle connections get shrink and after going the distance $|s_i|$ in parameter space they disappear. We hit a degenerate differential $\#_{z_0}^d(\mathbb{T}, dz)$, $z_0 \in \Lambda/d\Lambda$, that is homeomorphic to $\#_z^d(\mathbb{T}, dz)\backslash\{s_1, ...s_d\}$ and the number of connected components of $\#_{z_0}^d(\mathbb{T}, dz)\backslash\{\pi^{-1}[0]\}$ divides d. This motivates the following definition.

DEFINITION 1. *Let us call a family* S_d *of d parallel saddle connections between the cone points of the d-symmetric cover* $\#_z^d(\mathbb{T}, dz)$ m-homologous, *if collapsing the cone points of* $\#_z^d(\mathbb{T}, dz)$ *along the family* S_d *gives a degenerate cover* $\#_{z_0}^d(\mathbb{T}, dz)$ *that has* d/m *connected components after its singular point* $[0]$ *is removed.*

The name m-homologous is inspired by the fact, that a chain of m loops, each formed by a pair of saddle connections, bounds a torus tiled by d/m squares. Let us denote the set of m-homologous saddle connections on $\#_z^d(\mathbb{T}, dz)$ by $SC_m(\#_z^d(\mathbb{T}, dz))$. It is clear that in general

$$\bigsqcup_{m|d} SC_m(\#_z^d(\mathbb{T}, dz)) = SC_{cp}(\#_z^d(\mathbb{T}, dz))$$

where SC_{cp} denotes the set of all saddle connections between distinct cone points. Recall, that the divisors m of d are in one-to-one correspondence to orbits

$$\mathscr{O}_m = \mathrm{SL}_2\mathbb{Z} \cdot [\#_{d/m}(\mathbb{T}, dz)] \subset \Lambda/d\Lambda$$

and that

$$\bigsqcup_{m|d} \mathscr{O}_m = \Lambda/d\Lambda.$$

It is possible to reformulate the counting problem as a counting problem on parameter space $(\mathbb{T}_d, \Lambda/d\Lambda, dz)$. Namely on a d-symmetric cover $\#_z^d(\mathbb{T}, dz)$ consider those m-homologous saddle connections that terminate in a particular degenerate cover, say $\#_{z_m}^d(\mathbb{T}, dz)$. A parallel family S_d of d such saddle connections corresponds to a line segment l from z to the point z_m on $(\mathbb{T}_d, \Lambda/d\Lambda, dz)$, of the same length and direction as any saddle connection in S_d. If $SC(z_m, z)$ denotes the set of all line segments connecting the point z with z_m on $(\mathbb{T}_d, \Lambda/d\Lambda, dz)$. Since $\pi_d : \mathbb{T}_d \to \mathbb{T}$ is a translation map of degree d^2 that maps all lattice points on $\mathbb{T}_d$ to the origin of $\mathbb{T}$ the saddle connections in $SC(z_m, z)$ can be viewed as a set of saddle connections on $\mathbb{T}$. In fact, any saddle connection $s \in SC(z_m, z)$ is mapped isometrically to a saddle connection $(\pi_d)_*(s) \in SC([0], [z])$. So we can identify $SC(z_m, z)$ with a subset of $SC([0], [z])$. However, we need to identify $SC(z_m, z)$ in $SC([0], [z])$ and this is not completely trivial. Instead we apply our counting formula and consider the standard problem of finding all saddle connections in

$$SC(\mathscr{O}_m, z) = \bigsqcup_{z_m \in \mathbb{T}_d(m)} SC(z_m, z).$$

Recall, that all primitive m-torsion points are on one $\mathrm{SL}_2\mathbb{Z}$ orbit denoted by $\mathscr{O}_m$. First consider the case z is a torsion point. Then any saddle connection $s \in SC(\mathscr{O}_m, z)$ has horizontal ones in its $\mathrm{SL}_2\mathbb{Z}$-orbit. If As is horizontal than $[Az]_d$ lies on one of the horizontal leaves through the points $[ki]_d$, where $k = 1, ..., d$. In fact, the horizontal saddle connection of the marked torus $(\mathbb{T}_d, \mathbb{Z}[i]/d\mathbb{Z}[i])$ that contains $[Az]_d$ is bounded by at least one element from $\mathscr{O}_m$, since As is a horizontal line segment from $[Az]_d$ to a point in $\mathscr{O}_m$ that cannot cross any integer lattice point on $\mathbb{T}_d$. By the counting formula we need to record the length of As and this is either $|\{\operatorname{Re} Az\}| = |\operatorname{Re} Az| \mod 1$, or $|1 - \{\operatorname{Re} Az\}| = |\operatorname{Re} Az| \mod 1$. Here we representing any class modulo d by $(0, d]^2$. To write down the counting formula let $\mathscr{S}_{h,m} \subset \mathbb{T}_d$ denote the set of horizontal saddle connections that emanate or terminate in a primitive m-torsion point.

Theorem 8. *Assume $\#_z^d(\mathbb{T}, dz) \to (\mathbb{T}, dz)$ is a d-symmetric differential with $z \in (\mathbb{C} - \mathbb{Z}[i])/d\mathbb{Z}[i]$. Then, if $[z]_d \in \mathbb{T}(n)$ the m-homologous saddle connections on $\#_z^d(\mathbb{T}, dz)$ have Siegel-Veech constant:*

$$\mathbf{c}_m^d(n) = \frac{2 \cdot d}{|\mathscr{O}_n|} \sum_{[y]_d \in \mathscr{S}_{h,m} \cap \mathscr{O}_n} \frac{1}{|\{Re(y)\}|^2}. \tag{45}$$

If $[z]_d \in \mathbb{T}$ is not a torsion point we have:

$$\mathbf{c}_m^d(gen) = \frac{\pi^2}{3} \frac{\varphi(m)\psi(m)}{d}.$$

Proof. For torsion points this is a straight forward application of the parameter space growth rate formula. The fact, that only distances of the form $|\{\mathrm{Re}(y)\}|$ appear is because of the hyperelliptic involution of $\mathbb{T}_d$. This causes the factor 2 in the formula. For generic points we have the standard growth rate between two points when one is generic multiplied by the number of terminal points, here $|\mathscr{O}_m|$. ∎

12. Questions and Comments

The flat geometry of parameter spaces of covers has not been used in depth to study and particularly classify lattice covers, with the exception of Duryev's recent work [**D18**]. The growth rate formulas generalize directly to other parameter spaces of covers. The description of d-symmetric covers given in this paper can be generalized to cyclic and abelian covers, see [**AS**].

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