Contemporary Mathematics
Volume **737**, 2019
https://doi.org/10.1090/conm/737/14855

Sequence spaces on Banach lattices

SURVEY

Geraldo Botelho, Ryan Causey, and Khazhak V. Navoyan

Abstract. In this survey we introduce the concept of Köthe dual for the Dedekind complete Riesz space $L_0 = L_0(X, \mu)$ of all $\mu - measurable$ real functions on the non-empty point set X, and then, for a Banach lattice E, we discuss the generalized sequence space $\lambda_\pi(E)$, the definition of which is based on the Köthe dual of the sequence space λ. Further, we consider the Orlicz space $\ell_\phi(E)$ as well. Finally, we introduce a useful thinning construction of a series for the purposes of further investigation.

1. Introduction

The *Köthe dual space* of the ideal A in the Dedekind complete Riesz space $L_0 = L_0(X, \mu)$ of all $\mu - measurable$ real functions on the non-empty point set X takes its name after G. Köthe, who was one of the first to introduce this notion. Initially this space was introduced by him for the case of sequence spaces, when X is the set of natural numbers with discrete measure, and $L_0 = L_0(X, \mu)$ in this case is the space of all real sequences [**11**].

After giving a possibly complete list of definitions and theorems in Section 2, "Preliminaries", which are necessary to read the current work, we pass in Section 3, "*Köthe dual*", to a detailed description of the *Köthe dual space* of the ideal A in the Dedekind complete Riesz space $L_0 = L_0(X, \mu)$, introduced in Zaanen's book [**11**], "Introduction to operator theory in Riesz spaces". We start with the definition of a career $c(A)$ of the ideal A which (the career), by definition, has the property that every member of A vanishes $\mu - almost\ everywhere$ on its complement. While the characteristic function $\chi_{c(A)}$ of $c(A)$ has the same property as well, there is a counterexample, showing that $\chi_{c(A)}$ is not a member of A. Nevertheless, we see that for any measurable subset Y of $c(A)$, there is an increasing sequence of measurable sets of finite measure which converge to Y, and the characteristic functions of each of these sets belong to the ideal A. Exactly this useful property is used for the further construction of the *Köthe dual space* of the ideal A. In this same Section 3, finally it is shown that an order continuous functional ϕ on the ideal A, and the corresponding to it function $t \in L_0(X, \mu)$ determine each other uniquely $\mu - almost\ everywhere$. The set of all those $t \in L_0(X, \mu)$, corresponding to the elements $\phi \in A_n^\sim$, is a vector

2010 *Mathematics Subject Classification.* Primary 46B42, 46B45.

subspace of $L_0(X,\mu)$, and is denoted by K_A. Moreover, K_A is an ideal in $L_0(X,\mu)$, called the *Köthe dual space* of the ideal A.

In Section 4, on the basis of the sequence space λ, introduced as a subspace of $\mathbb{R}^{\mathbb{N}}$, as well as equipped with the vector lattice structure and assumed to be a Banach lattice, the Banach lattice valued sequence space $\lambda_\pi(E)$ is given, where E is a Banach lattice. Together with $\lambda_\pi(E)$ other similar Banach space and Banach lattice valued sequence spaces are introduced as well, and the norm on $\lambda_\pi(E)$ is defined, followed by very used properties of $\lambda_\pi(E)$ from [**1–6**]. In Section 4, the introduction and properties of the Orlicz function ϕ, Orlicz space ℓ_ϕ, Banach space valued Orlicz space $\ell_\phi(X)$ are given as well.

In Section 5, based on the proof of the Schur property for the Banach space ℓ_1 of all summable real number sequences, introduced in Megginson's book [**9**], "An introduction to Banach space theory", an interesting construction of the series used in the definition of $\lambda_\pi(E)$ is given, in a way that the "middle part" of the thinned series satisfies the property of being greater than the $\frac{3}{4}$ of the initial corresponding series. In the book of Megginson we find the series of ℓ_1 described as "thinned" without further details, which gives a good source of further thinking.

For the theory of Banach spaces the reader is refereed to [**9**], for the theory of Banach lattices to [**10**], and for the basic theory of Riesz spaces to [**11**].

2. Preliminaries

In this section we are going to give basic notions necessary to follow the rest of the material.

THEOREM 2.1. *(Radon-Nikodym theorem). If ν is a σ-additive measure on the σ-algebra Γ of subsets of the set X which is μ-absolutely continuous, then there exists a function f in the space $L_1(X,\Gamma,\mu)$ such that $\nu(A) = \int_A f d\mu$ holds for every $A \in \Gamma$.*

DEFINITION 2.2. The mapping ν from the algebra Γ into $\mathbb{R}$ is called a real finitely additive signed measure on Γ if $\nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$ holds for all disjoint A_1 and A_2 in Γ. If Γ is a σ-algebra and $\nu(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \nu(A_n)$ holds for every disjoint sequence $\{A_n\}_{n=1}^\infty$ in Γ, then ν is called a σ-additive signed measure on Γ.

DEFINITION 2.3. A partially ordered set X is called Dedekind complete if every non-empty subset of X that is bounded above has a supremum. X is called Dedekind σ-complete if every non-empty finite or countable subset of X that is bounded above has a supremum. X is called a lattice, if it contains the supremum and the infimum of each of two elements.

DEFINITION 2.4. The real vector space E is called an ordered vector space if E is partially ordered in a way that the vector space structure and the order structure are compatible, that is

(i) $f \le g$ implies $f + h \le g + h$ for every $h \in E$,
(ii) $f \ge 0$ implies $\alpha f \ge 0$ for every $\alpha \ge 0$ in $\mathbb{R}$.

If, in addition, E is a lattice with respect to the partial ordering, then E is called a Riesz space or a vector lattice.

DEFINITION 2.5. The subset A of the Riesz space E is said to be solid if it follows from $f \in A$ and $|g| \le |f|$ $(g \in E)$, that $g \in A$. A is called an ideal (or an order ideal to distinguish it from the algebraic ideal) if A is a solid linear subspace of E.

DEFINITION 2.6. The non-empty subset D of the Riesz space E is said to be downwards directed ($D \downarrow$) if for any two elements f and g in D there exists an element $h \in D$ such that $h \leq f \wedge g$ ($h \leq inf\{f, g\}$). If $D \downarrow$ and D has infimum f_0, then it is written $D \downarrow f_0$.

DEFINITION 2.7. Let E be a real Riesz space, equipped with a norm. The norm in E is called a Riesz norm if $|f| \leq |g|$ in E implies $\|f\| \leq \|g\|$. Any Riesz space, equipped with a Riesz norm, is called a normed Riesz space. If the normed Riesz space E is norm complete, i.e., every norm Cauchy sequence has a norm limit, then E is called a Banach lattice.

DEFINITION 2.8. The Riesz space E is said to be order separable if every set in E possessing an infimum contains a finite or countable subset having the same infimum.

DEFINITION 2.9. The Riesz space E is said to be super Dedekind complete if E is order separable and Dedekind complete, i.e., if every set in E which is bounded above has a supremum and contains a finite or countable subset having the same supremum.

DEFINITION 2.10. The normed Riesz space E is said to have order continuous norm if, for any subset $D \downarrow 0$ in E, we have $inf\{\|f\| : f \in D\} = 0$. The norm is said to be σ-order continuous if, for any sequence $f_n \downarrow 0$ in E, we have $\|f_n\| \downarrow 0$.

DEFINITION 2.11. An order interval in an ordered vector space G is a subset of G of the form $\{g : g_1 \leq g \leq g_2\}$, where g_1 and g_2 are elements of G satisfying $g_1 \leq g_2$. An operator T between ordered vector spaces E and F is called order bounded if T maps every order interval in E into an order interval in F.

DEFINITION 2.12. The linear operator T mapping the Riesz space E into the Riesz space F is said to be order continuous if for any $D \subseteq E$ such that $D \downarrow 0$ in E, we have $inf\{|Tf| : f \in D\} = 0$ in F. T is said to be σ-order continuous if, for any monotone sequence $f_n \downarrow 0$, we have $inf\{|Tf_n| : n \in \mathbb{N}\} = 0$.

3. Köthe dual

In this section we are going to describe the classical theory of the *Köthe dual space* for $L_0(X, \mu)$, defined below.

We assume that μ is a σ-finite (non-negative and σ-additive) measure in the non-empty point set X and $L_0 = L_0(X, \mu)$ is the Dedekind complete Riesz space of all $\mu - measurable$ real functions on X. In other words, μ is defined on a σ-algebra Γ the members of which are called the $\mu - measurable$ subsets of X, and the real function f on X is called $\mu - measurable$ whenever the set $\{x : x \in X,\ f(x) > \alpha\}$ is $\mu - measurable$ for every real α. It follows then that for α, β real $\{x : x \in X,\ f(x) \geq \alpha\}$, $\{x : x \in X,\ f(x) < \alpha\}$, $\{x : x \in X,\ f(x) \leq \alpha\}$, $\{x : x \in X,\ \alpha \leq f(x) \leq \beta\}$ are also $\mu - measurable$. Recall that functions differing only on a set of measure zero are identified, so that the members of $L_0(X, \mu)$ are in fact equivalence classes of measurable functions. This holds similarly for the members of the σ-algebra Γ.

Let A be an ideal in $L_0(X, \mu)$. The career of A, $c(A)$, is a measurable subset of X having the property that on its complement $n(A) = X \backslash c(A)$ every $f \in A$

vanishes $\mu - almost\ everywhere$, while $c(A)$ itself does not have any subset of positive measure on which all $f \in A$ vanish $\mu - almost\ everywhere$. Since the characteristic function $\chi_{c(A)}$ of $c(A)$ also is such that it vanishes outside $c(A)$, it is natural to ask whether $\chi_{c(A)}$ belongs to the ideal A as well. In other words, having an ideal $A \subseteq L_0(X, \mu)$, and its career $c(A)$, which, as we know, is such that all member functions of A vanish outside $c(A)$, is the characteristic function $\chi_{c(A)}$ of the set $c(A)$ necessarily among those member functions? The following counterexample shows that for an ideal $A \subseteq L_0(X, \mu)$ the characteristic function $\chi_{c(A)}$ of its career $c(A)$ maybe not a member of A [**11**].

COUNTEREXAMPLE 3.1. [**11**] *Let μ be a Lebesgue measure in $[0, 1]$ and let A be the ideal of all measurable functions f on $[0, 1]$ having the property that f vanishes on some interval $[0, \alpha_f]$, where $0 < \alpha_f \leq 1$, and α_f depends on f. Then $c(A) = [0, 1]$, and we see that $\chi_{c(A)}$ is not a member of A, because it vanishes nowhere on $[0, 1]$.*

Further, for any measurable subset Y of $c(A)$ we cannot be sure that its characteristic function χ_Y is a member of A. However, for any measurable subset Y of $c(A)$, there exists a sequence of subsets Y_n of Y with $Y_n \uparrow Y$ such that $\chi_{Y_n} \in A$ for $\forall n \in \mathbb{N}$.

THEOREM 3.2. [**11**, Theorem 29.1] *If A is an ideal in $L_0 = L_0(X, \mu)$, and Y is a measurable subset of the career $c(A)$ of A, then there exists an increasing sequence $\{Y_n\}_{n=1}^{\infty}$ of measurable sets of finite measure such that $Y_n \uparrow Y$ and $\chi_{Y_n} \in A$ for $\forall n \in \mathbb{N}$.*

PROOF. Part 1: Let us first show that if P is a measurable subset of $c(A)$ such that $\mu(P) > 0$, then P has a subset Q of positive measure such that $\chi_Q \in A$. To show this, we notice that since P is a measurable subset of $c(A)$ of positive measure, not all of $f \in A$ vanish $\mu - almost\ everywhere$ on P. Hence there exists a function $f \in A$ such that $f(x) \neq 0$ for all x in some $P_0 \subseteq P$ with $\mu(P_0) > 0$. Through replacing f by $|f|$, we may assume that $f(x) > 0$ for all $x \in P_0$. Now let us take a sequence of positive numbers $\epsilon \downarrow 0$ and $P_n = \{x : x \in P_0,\ f(x) \geq \epsilon_n\}$. We see that $\forall n \in \mathbb{N}$, $\epsilon_{n+1} \leq \epsilon_n$ implies that $P_n \subseteq P_{n+1}$. Then $P_n \uparrow P_0$ (this convergence is a result of f being positive on all points of P_0, and of the sets P_n containing more and more points of P_0 with the growth of n. The points of P_n with infinitely large index n have to be such that f on them is greater than a positive number infinitely close to zero which in limit is zero). Therefore, $\mu(P_n) \uparrow \mu(P_0) > 0$, which implies that starting from some $n = n_0$, $\mu(P_n) > 0$. Let us denote $Q = P_{n_0}$. Since for $\forall x \in Q$, the inequality $\epsilon_{n_0} \leq f(x)$ holds, and hence the equivalent to it $\epsilon_{n_0} \chi_Q(x) \leq f(x)$, $\forall x \in X$ (here also, if necessary substitute f by $|f|$, as before) holds as well, it follows that $\epsilon_{n_0} \chi_Q \leq f$. Considering that f is an element of the ideal A, we imply that $\epsilon_{n_0} \chi_Q \in A$, and A being a vector space, implies that $\chi_Q \in A$. This finishes the construction of a subset Q for P, with the required properties of having positive measure and a characteristic function belonging to A.

Part 2: Let us first assume that $\mu(Y)$ is finite. Now let

$$\alpha = sup\{\mu(Z) : Z \subseteq Y,\ \chi_Z \in A\}.$$

There exists a sequence $\{Z_n\}_{n=1}^{\infty}$ of subsets of Y such that $\chi_{Z_n} \in A$ and $\mu(Z_n) \to \alpha$, as $n \to \infty$, where it may be assumed that the sequence is increasing through, if necessary, replacing Z_n by $Z_1 \cup Z_2 \cup \cdots \cup Z_n$ (here the fact that A is an ideal is important). Then $Z := \lim_{n\to\infty} Z_n = \cup_{n=1}^{\infty} Z_n$ satisfies $\mu(Z) = \alpha$. Let us see that Z is almost equal to Y. Suppose it is not. Then the set $Y \backslash Z$ is a measurable subset of $c(A)$ having positive measure, and so, by Part 1, $Y \backslash Z$ has a subset Q of positive measure β such that $\chi_Q \in A$. Having that $\chi_{Z_n} \in A$ and $\chi_Q \in A$ it can be concluded that $\chi_{Q \cup Z_n} = \chi_Q + \chi_{Z_n} \in A$ (Q and Z_n are disjoint for each n). Thus the characteristic function of $Q \cup Z_n \subseteq Y$ is a member of A, and also

$$\mu(Q \cup Z_n) = \mu(Q) + \mu(Z_n) = \beta + \mu(Z_n) \uparrow \beta + \alpha,$$

as $n \to \infty$, so $\mu(Q \cup Z_n) > \alpha$ for n large enough. This contradicts the definition of α. Hence, Z is almost equal to Y, and so $Z_n \uparrow Y$ with $\chi_{Z_n} \in A$ for all n. This may also be expressed by saying that for any $\epsilon > 0$ there exists a set $R \subseteq Y$ such that $\chi_R \in A$ and $\mu(Y \backslash R) < \epsilon$.

Part 3: Now let us prove the general case when $\mu(Y) = \infty$. Using that the measure μ is σ-finite, we can write $Y = \cup_{n=1}^{\infty} Y_n^{\sim}$ with $\mu(Y_n^{\sim})$ finite for all n. We may also assume that $Y_n^{\sim}$ is increasing as n increases, so $Y_n^{\sim} \uparrow Y$. According to the last statement of Part 2, each $Y_n^{\sim}$ has a subset Y_n such that $\chi_{Y_n} \in A$ and $\mu(Y_n^{\sim} \backslash Y_n) < \frac{1}{n}$. One more time, it may be assumed that Y_n increases with n, so $Y_\infty := \lim_{n\to\infty} Y_n = \cup_{n=1}^{\infty} Y_n$ exists and we have $Y_n^{\sim} \backslash Y_\infty \subseteq Y_n^{\sim} \backslash Y_n$, which implies that $\mu(Y_n^{\sim} \backslash Y_\infty) < \frac{1}{n}$ for $n \in \mathbb{N}$. But

$$Y_n^{\sim} \backslash Y_\infty \uparrow Y \backslash Y_\infty, \ so\ \mu(Y \backslash Y_\infty) = \lim_{n\to\infty} (\mu(Y_n^{\sim} \backslash Y_\infty)) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

This shows that $Y = Y_\infty$ almost everywhere, and that except for a set of measure zero, $Y_n \uparrow Y$. Therefore, the constructed sequence $\{Y_n\}_{n=1}^{\infty}$ is the desired one. □

This theorem is important since it helps to precisely obtain the space $A_n^{\sim}$ of all order continuous linear functionals on a given ideal A in $L_0 = L_0(X, \mu)$. The remark after [**11**, Theorem 1.4] shows that $L_0(X, \mu)$ is super Dedekind complete, so in particular $L_0(X, \mu)$ is order separable. It follows that every ideal A in $L_0(X, \mu)$ is order separable, which means that every downwards directed set D in A satisfying $D \downarrow 0$ contains a sequence $\{f_n\}_{n=1}^{\infty}$ satisfying $f_n \to 0$. Consequently, we see that an order bounded linear functional on A is order continuous if and only if it is σ-order continuous.

Theorem 3.3. [**11**, Theorem 29.2] *If A is an ideal in $L_0 = L_0(X, \mu)$ and the order bounded linear functional ϕ on A is order continuous, i.e., if $\phi \in A_n^{\sim}$, then there exists a real μ- measurable function t on X such that*

(i) *for every $f \in A$ the product tf is μ-summable over X,*
(ii) *$\phi(f) = \int_X tf d\mu$ holds for every $f \in A$.*

The function t thus corresponding to ϕ is uniquely determined $\mu -$ almost everywhere on the career $c(A)$ of A. If ϕ is positive, then $t(x) \geq 0$ for $\mu -$ almost every $x \in c(A)$.

Conversely, if t is a real $\mu -$ measurable function on X possessing the property that tf is $\mu -$ summable over X for every $f \in A$, then the linear functional ϕ on A, defined by

$$\phi(f) = \int_X tf d\mu \ for\ every\ f \in A, \tag{3.1}$$

is order bounded and order continuous. Trivially, ϕ is uniquely determined by t. If $t(x) \geq 0$ for $\mu -$ almost every $x \in c(A)$, then ϕ is positive.

Proof. Let us assume first that $\phi \in A_n^{\sim}$. Then ϕ^+ and ϕ^- are positive members of $A_n^{\sim}$, so it is sufficient to show that each of ϕ^+ and ϕ^- has the desired properties, then it will follow that the desired properties also hold for $\phi = \phi^+ - \phi^-$. So, let us take $0 \leq \phi \in A_n^{\sim}$ and prove that there exists a $\mu - measurable$ function t, non-negative on $c(A)$ and such that (i) and (ii) hold. Using Theorem 3.2, a sequence $\{Y_n\}_{n=1}^{\infty}$ of measurable subsets of $c(A)$ is chosen, such that each Y_n has a finite measure, $Y_n \uparrow c(A)$ and $\chi_{Y_n} \in A$ for $\forall n \in \mathbb{N}$. Therefore, if Z is a $\mu - measurable$ subset of some Y_n, we have $\chi_Z \leq \chi_{Y_n} \in A$, and since A is an ideal, $\chi_Z \in A$. So $\phi(\chi_Z)$ is well-defined. $\phi(\chi_Z)$ is briefly denoted by $\nu(Z)$ and let us look at the subsets of one fixed Y_n, say for $n = n_0$. Given such a subset $Z \subseteq Y_{n_0}$, we apply Theorem 3.2 one more time to choose $Z_k \uparrow Z$ as $k \to \infty$, and since ϕ is positive and order continuous, it follows that $\phi(\chi_{Z_k}) \uparrow \phi(\chi_Z)$. In other words, $\nu(Z_k) \uparrow \nu(Z)$, which shows that ν is a finite, non-negative, σ-additive measure on the σ-algebra of $\mu - measurable$ subsets of Y_{n_0}. Furthermore, the measure ν is $\mu - absolutely\ continuous$, i.e., $\mu(Z) = 0$ implies that $\nu(Z) = 0$. Hence, by the Radon-Nikodym theorem, there exists a non-negative, $\mu - measurable$ function t on Y_{n_0} such that $\nu(Z) = \int_Z t d\mu$ holds for every $\mu - measurable$ subset Z of Y_{n_0}. The function t is $\mu - almost$ uniquely determined on Y_{n_0} ([**11**, Exercise 28.4]). By varying n_0, the function t can now be extended $\mu - almost$ uniquely to $\cup_{n=1}^{\infty} Y_n = c(A)$ such that $\nu(Z) = \int_Z t d\mu$ holds for every Z satisfying $Z \subseteq Y_n$ for some n. Setting $t(x) = 0$ for all $x \in X \backslash c(A)$, the following holds:

$$\phi(\chi_Z) = \nu(Z) = \int_X t \chi_Z d\mu$$

for every Z of this particular type. Hence, if s is now a step function $s = \sum_{k=1}^{p} a_k \chi_{Z_k}$ with all Z_k of this type, it is evident that $\phi(s) = \int_X ts d\mu$. Finally, let $0 \leq f \in A$ be given. It is well-known that there exists a sequence $\{s_n^{\sim}\}_{n=1}^{\infty}$ of $\mu - measurable$ step functions such that $0 \leq s_n^{\sim} \uparrow f$. Then, for each n, $s_n = s_n^{\sim} \chi_{Y_n}$ is a step function of the particular type we need and $0 \leq s_n \uparrow f$ still holds. Hence $\phi(s_n) = \int_X ts_n d\mu$ for all n, by what was proved above. Furthermore, $\phi(s_n) \uparrow \phi(f)$, since ϕ is positive and order continuous, and

$$\int_X ts_n d\mu \uparrow \int_X tf d\mu$$

by the well-known theorem on integration of monotone sequences. It follows, as desired, that $\phi(f) = \int_X tf d\mu$. Finally, for any arbitrary $f \in A$, we have

$$\phi(f) = \phi(f^+) - \phi(f^-) = \int_X tf^+ d\mu - \int_X tf^- d\mu = \int_X tf d\mu.$$

Conversely, assume that the real $\mu - measurable$ function t on X has the property that tf is $\mu - summable$ over X for every $f \in A$. Then $|tf|$ is $\mu - summable$ and so $t^+ f$ and $t^- f$ are $\mu - summable$. It is evident that ϕ, as defined in (3.1), is a linear functional on A and

$$\phi(f) = \int_X tf d\mu = \int_X t^+ f d\mu - \int_X t^- f d\mu = \phi_1(f) - \phi_2(f)$$

for every $f \in A$, so ϕ is the difference of the positive linear functionals ϕ_1 and ϕ_2. This shows already that ϕ is order bounded. Furthermore, $f_n \downarrow 0$ in A implies

$\phi_1(f_n) = \int_X t^+ f_n d\mu \downarrow 0$, so ϕ_1 is order continuous. Similarly, ϕ_2 is order continuous. It follows that ϕ is order continuous. $\square$

As before, let A be an ideal in $L_0 = L_0(X, \mu)$ with carrier $c(A)$. Let us assume that $c(A) = X$. As already shown above, to any $\phi \in A_n^\sim$, there corresponds a function $t \in L_0$ such that $\phi(f) = \int_X tf d\mu$ for every $f \in A$. Conversely, if the function $t \in L_0$ has the property that tf is $\mu -$ *summable* over X for every $f \in A$, then the linear functional ϕ on A, defined by $\phi(f) = \int_X tf d\mu$, satisfies $\phi \in A_n^\sim$. The functional $\phi \in A_n^\sim$ and the corresponding $t \in L_0$ determine each other uniquely $\mu -$ *almost everywhere*, as far as it concerns t. The set of all $t \in L_0$ thus corresponding to elements $\phi \in A_n^\sim$ is obviously a linear subspace of L_0 which is denoted by K_A. Let us see that K_A is an ideal in L_0. If $t \in K_A$, then tf is summable over X for every $f \in A$, i.e., $|tf|$ is summable for every $f \in A$, and so $|t|f$ is summable for every $f \in A$. This shows that $|t| \in K_A$. Furthermore, it is evident that if $0 \leq t_2 \leq t_1$, and $t_1 \in K_A$, then $t_2 \in K_A$. In particular, $|t| \in K_A$ implies that t^+ and t^- are members of K_A, and so $t = t^+ - t^- \in K_A$. To summarize, it is now proved that $t \in K_A$ if and only if $|t| \in K_A$ and $0 \leq t_2 \leq t_1 \in K_A$ implies that $t_2 \in K_A$. Hence, K_A is an ideal in L_0. The ideal K_A is sometimes called the *Köthe dual space* of A, named after G. Köthe, who was one of the first to introduce this notion, first for the case of sequence spaces, i.e., when X is the set of natural numbers with discrete measure, so that $L_0 = L_0(X, \mu)$ in this case is the space of all real sequences [**11**].

4. Sequence spaces

For Banach spaces E and F, $\mathcal{L}(E; F)$ will denote the space of bounded linear operators from E to F. In case that E and F are Banach lattices, $\mathcal{L}^r(E; F)$ will denote the space of regular linear operators from E to F with its regular norm $\|\cdot\|_r$.

For a Banach space H, H^* will denote its topological dual and B_H will denote its closed unit ball. For a vector lattice G, the G-valued sequence space $G^{\mathbb{N}}$ is a vector lattice with the following lattice operations

$$\bar{x} \wedge \bar{y} = (x_i \wedge y_i)_i, \;\; \bar{x} \vee \bar{y} = (x_i \vee y_i)_i,$$

$$\bar{x} = (x_i)_i, \;\; \bar{y} = (y_i)_i \in G^{\mathbb{N}}.$$

For each $n \in \mathbb{N}$ and each $\bar{x} = (x_i)_i \in G^{\mathbb{N}}$, let

$$\bar{x}(\geq n) = (0, \cdots, 0, x_n, x_{n+1}, \cdots), \;\; \bar{x}(n) = (0, \cdots, 0, x_n, 0, \cdots).$$

G^+ will denote the positive cone of G and for each $x \in G$, x^+ and x^- will denote the positive part and the negative part of x, respectively.

Let λ be a sequence space, i.e., a subspace of $\mathbb{R}^{\mathbb{N}}$. The Köthe dual of λ is defined by

$$\lambda' = \Big\{(b_i)_i \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |a_i b_i| < +\infty, \; \forall (a_i)_i \in \lambda\Big\}.$$

λ is called Köthe perfect if $\lambda'' = \lambda$. In addition, if λ is an order continuous Banach lattice, then $\lambda' = \lambda^*$. Thus λ' is also a Banach lattice and for each $a = (a_i)_i \in \lambda$

and each $b = (b_i)_i \in \lambda'$,

$$\|a\|_\lambda = sup\Big\{|\sum_{i=1}^{\infty} a_i b_i| : (b_i)_i \in B_{\lambda'}\Big\},$$

and

$$\|b\|_{\lambda'} = \|b\|_{\lambda^*} = sup\Big\{|\sum_{i=1}^{\infty} a_i b_i| : (a_i)_i \in B_{\lambda}\Big\}.$$

Let λ'_0 denote the closed subspace of λ' consisting of all such sequences of λ' whose tails converge to 0, i.e.,

$$\lambda'_0 = \Big\{a = (a_i)_i \in \lambda' : \lim_n \|a(\geq n)\|_{\lambda'} = 0\Big\}.$$

Then $(\lambda'_0)^* = (\lambda'_0)' = \lambda''$.

LEMMA 4.1. [**2**, Lemma 3.4], [**6**, Lemma 4], [**3**, Lemma 1] *If λ is an order continuous Banach lattice, then a bounded subset B of λ is relatively compact if and only if*

$$\lim_n sup\Big\{\|a(\geq n)\|_\lambda : a = (a_i)_i \in B\Big\} = 0.$$

From now on it will be assumed that λ is a Köthe perfect ($\lambda'' = \lambda$) and order continuous Banach sequence lattice with $\|e_i\|_\lambda = 1$ for all $i \in \mathbb{N}$, where $e_i's$ are standard unit vectors in the sequence space λ. Let G be a Banach lattice as before.

For a Banach space H, let

$$\lambda_\omega(H) = \lambda_{weak}(H) = \Big\{\bar{x} = (x_i)_i \in H^{\mathbb{N}} : (x^*(x_i))_i \in \lambda,\ \forall x^* \in H^*\Big\}$$

and

$$\|\bar{x}\|_{\lambda_\omega(H)} = sup\Big\{\|(x^*(x_i))_i\|_\lambda : x^* \in B_{H^*}\Big\},\ \ \forall \bar{x} = (x_i)_i \in \lambda_\omega(H).$$

Then $\lambda_\omega(H)$ is a Banach space.

Let $\lambda_{\omega,0}(H)$ denote the closed subspace of $\lambda_\omega(H)$ consisting of all such elements of $\lambda_\omega(H)$ whose tails converge to 0, i.e.,

$$\lambda_{\omega,0}(H) = \Big\{\bar{x} \in \lambda_\omega(H) : \lim_n \|\bar{x}(\geq n)\|_{\lambda_\omega(H)} = 0\Big\}.$$

REMARK 4.2. We notice that the Banach space $\ell_1^\omega(H)$ of weakly 1-summable H-valued sequences with norm $\|(x_j)_{j=1}^\infty\|_{\omega,1} = \sup_{\phi \in B_{H^*}} \|(\phi(x_j))_{j=1}^\infty\|_1$ is a particular case of $\lambda_\omega(H)$. And $\ell_1^u(H) = \Big\{(x_j)_{j=1}^\infty \in \ell_1^\omega(H) : \lim_k \|(x_j)_{j=k}^\infty\|_{\omega,1} = 0\Big\}$, the Banach space of unconditionally 1-summable sequences, with the norm inherited from $\ell_1^\omega(H)$ is a particular case of $\lambda_{\omega,0}(H)$.

For each $\bar{x} = (x_i)_i \in \lambda_\omega(H)$, each $a = (a_i)_i \in \lambda'_0$, and each $n \in \mathbb{N}$,

$$\|\sum_{i=n}^{\infty} a_i x_i\|_H = sup\Big\{|\sum_{i=n}^{\infty} a_i x^*(x_i)| : x^* \in B_{H^*}\Big\} \leq$$

$$\leq sup\Big\{\|a(\geq n)\|_{\lambda'} \cdot \|(x^*(x_i))_{i=n}^\infty\|_\lambda : x^* \in B_{H^*}\Big\} \leq$$

$$(4.1) \qquad \leq \|a(\geq n)\|_{\lambda'} \cdot \|\bar{x}\|_{\lambda_\omega(H)} \to 0,\ as\ n \to \infty.[\mathbf{3}]$$

Since the above is true for each $n \in \mathbb{N}$, for $n = 1$ it has the following form:

$$\|\sum_{i=1}^{\infty} a_i x_i\|_H = sup\Big\{|\sum_{i=1}^{\infty} a_i x^*(x_i)| : x^* \in B_{H^*}\Big\} \leq$$

$$\leq \|(a_i)_i\|_{\lambda'} \cdot \|(x_i)_i\|_{\lambda_\omega(H)}.$$

Taking supremum when $(a_i)_i \in B_{\lambda'}$, the following is true:

$$sup\Big\{\|\sum_{i=1}^{\infty} a_i x_i\|_H : (a_i)_i \in B_{\lambda'}\Big\} =$$

$$= sup\Big\{|\sum_{i=1}^{\infty} a_i x^*(x_i)| : x^* \in B_{H^*}, (a_i)_i \in B_{\lambda'}\Big\} \leq$$

$$\leq sup\Big\{\|(a_i)_i\|_{\lambda'} \cdot \|(x_i)_i\|_{\lambda_\omega(H)} : (a_i)_i \in B_{\lambda'}\Big\}.$$

Therefore,

$$sup\Big\{\|\sum_{i=1}^{\infty} a_i x_i\|_H : (a_i)_i \in B_{\lambda'}\Big\} =$$

$$= sup\Big\{\|(x^*(x_i))_i\|_{\lambda} : x^* \in B_{H^*}\Big\} \leq$$

$$\leq \|x^*(x_i)\|_{\lambda_\omega(H)}.$$

Middle part of the inequalities above is $\|(x_i)_i\|_{\lambda_\omega(H)}$, hence

$$sup\Big\{\|\sum_{i=1}^{\infty} a_i x_i\|_H : (a_i)_i \in B_{\lambda'}\Big\} =$$

$$= \|x^*(x_i)\|_{\lambda_\omega(H)} \leq \|x^*(x_i)\|_{\lambda_\omega(H)}.$$

By (4.1), and the explanations following (4.1), the following lemma holds:

LEMMA 4.3. [**3**, Lemma 2] $\bar{x} = (x_i)_i \in \lambda_\omega(H)$ *if and only if for each* $(a_i)_i \in \lambda'_0$, *the series* $\sum\limits_i a_i x_i$ *converges in H. In this case, we have:*

$$\|\bar{x}\|_{\lambda_\omega(H)} = sup\Big\{\|\sum_{i=1}^{\infty} a_i x_i\|_H : (a_i)_i \in B_{\lambda'_0}\Big\}.$$

For each $\bar{x} = (x_i)_i \in \lambda_\omega(H)$, define a linear operator $T_{\bar{x}}$ from λ'_0 to H by $T_{\bar{x}}(a) = \sum_{i=1}^{\infty} a_i x_i$, $\forall a = (a_i)_i \in \lambda'_0$.

PROPOSITION 4.4. [**3**, Proposition 3] Let H be a Banach space. Then $\lambda_\omega(H)$ is isometrically isomorphic to $\mathcal{L}(\lambda'_0, H)$ under the mapping $\bar{x} \to T_{\bar{x}}$. Moreover, $T_{\bar{x}}$ is compact if and only if $\bar{x} \in \lambda_{\omega,0}(H)$.

For a Banach lattice G, let

$$\lambda_\epsilon(G) = \Big\{\bar{x} = (x_i)_i \in G^{\mathbb{N}} : (x^*(|x_i|))_i \in \lambda,\ \forall x^* \in G^{*+}\Big\},$$

and

$$\|\bar{x}\|_{\lambda_\epsilon(G)} = sup\Big\{\|(x^*(|x_i|))_i\|_\lambda : x^* \in B_{G^*+}\Big\},\ \forall \bar{x} = (x_i)_i \in \lambda_\epsilon(G).$$

Then $\lambda_\epsilon(G)$ is a Banach lattice (see [**2**]). Let $\lambda_{\epsilon,0}(G)$ denote the closed sublattice of $\lambda_\epsilon(G)$, consisting of all such elements of $\lambda_\epsilon(G)$ whose tails converge to 0, i.e.,

$$\lambda_{\epsilon,0}(G) = \Big\{\bar{x} \in \lambda_\epsilon(G) : \lim_n \|\bar{x}(\geq n)\|_{\lambda_\epsilon(G)} = 0\Big\}.$$

Then $\lambda_{\epsilon,0}(G)$ is an order ideal of $\lambda_\epsilon(G)$. The following lemma is straightforward.

Lemma 4.5. [**3**, Lemma 4] *Let G be a Banach lattice. Then* $\lambda_\epsilon(G) \subseteq \lambda_\omega(G)$ *and for each* $\bar{x} \in \lambda_\epsilon(G)$, $\|\bar{x}\|_{\lambda_\epsilon(G)} \geq \|\bar{x}\|_{\lambda_\omega(G)}$. *In addition, if* $\bar{x} \geq 0$, $\bar{x} \in \lambda_\epsilon(G) \iff \bar{x} \in \lambda_\omega(G)$. *In this case,* $\|\bar{x}\|_{\lambda_\epsilon(G)} = \|\bar{x}\|_{\lambda_\omega(G)}$.

Proposition 4.6. [**3**, Proposition 5] Let G be a Dedekind complete Banach lattice. Then $\lambda_\epsilon(G)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r(\lambda'_0, G)$ under the mapping $\bar{x} \to T_{\bar{x}}$. Moreover, $T_{\bar{x}} \in \mathcal{K}^r(\lambda'_0, G)$ if and only if $\bar{x} \in \lambda_{\epsilon,0}(G)$, where $\mathcal{K}^r(\lambda'_0, G)$ will denote the linear span of positive compact operators from λ'_0 to G.

For a Banach lattice G, define

$$\lambda'_\omega(G^*) = \lambda'_{weak}(G^*) = \Big\{\bar{x}^* = (x_i^*)_i \in G^{*^{\mathbb{N}}} : (x_i^*(x))_i \in \lambda',\ \forall x \in G\Big\}$$

and

$$\|\bar{x}^*\|_{\lambda'_\omega(G^*)} = sup\Big\{\|(x_i^*(x))_i\|_{\lambda'} : x \in B_G\Big\},\ \forall \bar{x}^* = (x_i^*)_i \in \lambda'_\omega(G^*).$$

Then $\lambda'_\omega(G^*)$ is a Banach space (it may not be a Banach lattice).

Define

$$\lambda_s(G) = \lambda_{strong}(G) = \Big\{\bar{x} = (x_i)_i \in G^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i^*(x_i)| < +\infty,\ \forall (x_i^*)_i \in \lambda'_\omega(G^*)\Big\}$$

and

$$\|\bar{x}\|_{\lambda_s(G)} = sup\Big\{|\sum_{i=1}^{\infty} x_i^*(x_i)| : (x_i^*)_i \in B_{\lambda'_\omega(G^*)}\Big\},\ \forall \bar{x} = (x_i)_i \in \lambda_s(G).$$

Then $\lambda_s(G)$ is a Banach space (it may not be a Banach lattice).

Remark 4.7. We notice that the Banach space of cohen strongly 1-summable sequences $\ell_1\langle G\rangle = \Big\{(x_j)_{j=1}^\infty \in G^{\mathbb{N}} : \|(x_j)_{j=1}^\infty\|_{\ell_1\langle G\rangle} := sup\{\|(\phi_j(x_j))_{j=1}^\infty\|_1 : (\phi_j)_{j=1}^\infty \in B_{\ell_\infty^\omega(G^*)}\} < +\infty\Big\}$, is a particular case of $\lambda_s(G)$.

Define

$$\lambda'_\epsilon(G^*) = \Big\{\bar{x}^* = (x_i^*)_i \in G^{*^{\mathbb{N}}} : (|x_i^*|(x))_i \in \lambda',\ \forall x \in G^+\Big\}$$

$$(Compare\ \lambda_\epsilon(G) = \Big\{\bar{x} = (x_i)_i \in G^{\mathbb{N}} : (x^*(|x_i|))_i \in \lambda,\ \forall x^* \in G^{*+}\Big\}.)$$

and

$$\|\bar{x}^*\|_{\lambda'_\epsilon(G^*)} = sup\Big\{\|(|x_i^*|(x))_i\|_{\lambda'} : x \in B_{G^+}\Big\},\ \forall \bar{x}^* = (x_i^*)_i \in \lambda'_\epsilon(G^*).$$

Then $\lambda'_\epsilon(G^*)$ is a Banach lattice.

Define

$$\lambda_\pi(G) = \Big\{\bar{x} = (x_i)_i \in G^{\mathbb{N}} : \sum_{i=1}^{\infty} x_i^*(|x_i|) < +\infty,\ \forall (x_i^*)_i \in \lambda'_\epsilon(G^*)^+\Big\}$$

and

$$\|\bar{x}\|_{\lambda_\pi(G)} = sup\Big\{\sum_{i=1}^{\infty} x_i^*(|x_i|) : (x_i^*)_i \in B_{\lambda'_\epsilon(G^*)^+}\Big\},\ \forall \bar{x} = (x_i)_i \in \lambda_\pi(G).$$

Then $\lambda_\pi(G)$ is a Banach lattice.

Let $\lambda_{\pi,0}(G)$ denote the closed sublattice of $\lambda_\pi(G)$ consisting of all such elements of $\lambda_\pi(G)$ whose tails converge to 0, that is,

$$\lambda_{\pi,0}(G) = \Big\{\bar{x} \in \lambda_\pi(G) : \lim_n \|\bar{x}(\geq n)\|_{\lambda_\pi(G)} = 0\Big\}.$$

Then $\lambda_{\pi,0}(G)$ is an ideal of $\lambda_\pi(G)$.

PROPOSITION 4.8. [**5**, Proposition 1] Let G and λ be as defined above. If λ is σ-order continuous, then $\lambda_{\pi,0}(G) = \lambda_\pi(G)$.

For Archimedean Riesz spaces E and F, let $E\bar{\otimes}F$ denote the Riesz space tensor product of E and F, with the positive cone C_p defined as follows.

$$C_p = \Big\{\sum_{k=1}^{n} x_k \otimes y_k : n \in \mathbb{N}, x_k \in E^+, y_k \in F^+\Big\}.$$

If, in addition, E and F are Banach lattices, then the positive projective tensor norm on $E\bar{\otimes}F$ is defined by

$$\|u\|_{|\pi|} = inf\{\sum_{k=1}^{n} \|x_k\| \cdot \|y_k\| : x_k \in E^+, y_k \in F^+,\ |u| \leq \sum_{k=1}^{n} x_k \otimes y_k\}$$

for every $u \in E\bar{\otimes}F$. Let $E\hat{\otimes}_{|\pi|}F$ denote the completion of $E\bar{\otimes}F$ with respect to $\|\cdot\|_{|\pi|}$. Then $E\hat{\otimes}_{|\pi|}F$ is a Banach lattice, called the positive projective tensor product.

PROPOSITION 4.9. [**5**, Proposition 2][**2**, Theorem 7.3] If λ is σ-order continuous then $\lambda\hat{\otimes}_{|\pi|}G$ is isometrically isomorphic and lattice homomorphic to $\lambda_{\pi,0}(G)$.

Define $\lambda(G) = \{\bar{x} = (x_i)_i \in G^{\mathbb{N}} : (\|x_i\|)_i \in \lambda\}$ and

$$\|\bar{x}\|_{\lambda(G)} = \|(\|x_i\|)_i\|_\lambda.$$

Then $\lambda(G)$ is a Banach lattice. If λ is σ-order continuous, it follows from [**2**, Lemma 3.4] that for every $a = (a_i)_i \in \lambda$ we have $\lim_n \|a(\geq n)\|_\lambda = 0$ and hence, for every $\bar{x} \in \lambda(G)$, $\lim_n \|\bar{x}(\geq n)\|_{\lambda(G)} = 0$.

PROPOSITION 4.10. [**5**, Proposition 3] Let λ be σ-order continuous. Then a subset B of $\lambda(G)$ is relatively compact if and only if for each $i \in \mathbb{N}$, the set $\{x_i : \bar{x} = (x_i)_i \in B\}$ is a relatively compact subset of G, and

$$\underset{n}{limsup}\{\|\bar{x}(\geq n)\|_{\lambda(G)} : \bar{x} \in B\} = 0.$$

An Orlicz function ϕ is a continuous, nondecreasing and convex function, which maps $[0,\infty)$ to $[0,\infty)$ and is zero only at $t = 0$, with $\underset{t\to\infty}{lim}\phi(t) = \infty$. Every Orlicz function ϕ has a right derivative p and

$$\phi(t) = \int_0^t p(u)du,\ t \geq 0.$$

If p satisfies $p(0) = 0$ and $\underset{t\to\infty}{lim}p(t) = \infty$ (after making these restrictions we exclude only the case $\ell_\phi = \ell_1$ when $\phi(t)$ is equivalent to t), then the right inverse q of p,

$$q(s) = sup\{t : p(t) \leq s\},\ s \geq 0,$$

is a right continuous nondecreasing function such that q is zero at zero and $q(s) > 0$ whenever $s > 0$. Define

$$\phi^*(s) = \int_0^s q(u)du,\ s \geq 0.$$

This ϕ^* is also an Orlicz function, and q is its right derivative. ϕ^* is called the function complementary to ϕ. ϕ has its complementary function if its right derivative p satisfies $p(0) = 0$ and $\underset{t\to\infty}{lim}p(t) = \infty$.

DEFINITION 4.11. [**4**] An Orlicz function ϕ is said to satisfy the Δ_2 condition at zero, if there exist $K > 0$ and $t_0 > 0$ such that $\phi(2t) \leq K\phi(t)$ for every $0 < t \leq t_0$.

When we have the definition for the Δ_2 condition, let us look at the definition of the Orlicz sequence space ℓ_ϕ, and notice its similarity to the definition of the Köthe dual λ', given above and appearing in [**3, 5**].

$$\ell_\phi = \Big\{(a_i)_{i=1}^\infty \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^\infty \phi(|\lambda a_i|) < \infty\ for\ some\ \lambda > 0.\Big\}.$$

On the contrary to the general case of λ', the norm for each element $(a_i)_{i=1}^\infty$ of ℓ_ϕ has a given form of

$$\|(a_i)_{i=1}^\infty\|_{\ell_\phi} = inf\Big\{\lambda > 0 : \sum_{i=1}^\infty \phi\Big(\frac{|a_i|}{\lambda}\Big) \leq 1\Big\}.$$

With this norm, ℓ_ϕ is a Banach space.

Next comes the X-valued Orlicz sequence space $\ell_\phi(X)$, defined by

$$\ell_\phi(X) = \Big\{(x_i)_{i=1}^\infty \in X^{\mathbb{N}} : \sum_{i=1}^\infty \phi(\|\lambda x_i\|) < \infty\ for\ some\ \lambda > 0.\Big\}.$$

For each element $(x_i)_{i=1}^\infty$ of $\ell_\phi(X)$, the $\ell_\phi(X)$ norm is given by:

$$\|(x_i)_{i=1}^\infty\|_{\ell_\phi(X)} = inf\Big\{\lambda > 0 : \sum_{i=1}^\infty \phi\Big(\frac{\|x_i\|}{\lambda}\Big) \leq 1\Big\}.$$

Under this norm, $\ell_\phi(X)$ is a Banach space.

COROLLARY 4.12. [**5**, Corollary 4] Let λ be σ- order continuous. Then a subset B of λ is relatively compact if and only if for each $i \in \mathbb{N}$, the set $\{a_i : a = (a_i)_i \in B\}$ is a bounded subset of $\mathbb{R}$, and

$$\underset{n}{lim}sup\{\|a(\geq n)\|_\lambda : a = (a_i)_i \in B\} = 0.$$

REMARK 4.13. An Orlicz function ϕ satisfies the Δ_2- condition if and only if for each $(x_i)_{i=1}^\infty \in \ell_\phi(X)$, $\underset{n}{lim}\|(x_i)_{i=n}^\infty\|_{\ell_\phi(X)} = 0$ [**4**]. This statement allows us to think that the Δ_2- condition plays the role of σ-order continuity of the sequence space λ in only one direction, since in [**5**] we have, that if λ is σ-order continuous then $\lambda_{\pi,0}(X) = \lambda_\pi(X)$, where

$$\lambda_{\pi,0}(X) = \Big\{(x_i)_{i=1}^\infty \in \lambda_\pi(X) : \underset{n}{lim}\|(x_i)_{i=n}^\infty\|_{\lambda_\pi(X)} = 0\Big\},$$

and a Banach lattice Y is called σ-order continuous if $0 \leq x_n \downarrow 0$ in Y when $x_n \to 0$ in Y.

With the proposition that follows the compact subsets of $\ell_\phi(X)$ are characterized, and actually this is the proposition the proof of which contains a useful technique, possibly applicable for the general sequence space case also.

PROPOSITION 4.14. [**4**, Proposition 1] Let ϕ be an Orlicz function satisfying the Δ_2 condition. Then a subset B of $\ell_\phi(X)$ is relatively compact if and only if for each $i \in \mathbb{N}$, the set $\{x_i : (x_i)_{i=1}^\infty \in B\}$ is a relatively compact subset of X, and

$$\limsup\{\|(x_i)_{i=n}^\infty\|_{\ell_\phi(X)} : (x_i)_{i=1}^\infty \in B\} = 0. \tag{4.2}$$

PROOF. Suppose that B is a relatively compact subset of $\ell_\phi(X)$. It can be shown that $\{x_i : (x_i)_{i=1}^\infty \in B\}$ is a relatively compact subset of X for each $i \in \mathbb{N}$. Next assume that (4.2) does not hold. Note that $\underset{n}{lim}\|(x_i)_{i=n}^\infty\|_{\ell_\phi(X)} = 0$ for each $(x_i)_{i=1}^\infty \in \ell_\phi(X)$ since ϕ satisfies the Δ_2 condition.

We want to show that there exists an $\epsilon_0 > 0$, $(x_i^{(k)})_{i=1}^\infty \in B$ for each $k \in \mathbb{N}$, and a subsequence $n_1 < m_1 < n_2 < m_2 < \dots$ such that

$$\Big\|(x_i^{(k)})_{i=n_k}^\infty\Big\|_{\ell_\phi(X)} \geq \epsilon_0,\ k = 1, 2, \dots,$$

$$\Big\|(x_i^{(k)})_{i=m}^\infty\Big\|_{\ell_\phi(X)} \leq \frac{\epsilon_0}{2},\ m > m_k,\ k = 1, 2, \dots.$$

Since condition (4.2) *is not satisfied, there exists* $\epsilon_0 > 0$ *such that for* $\forall N \in \mathbb{N}$, $\exists n \geq N$ *with*

$$sup\Big\{\|(x_i)_{i=n}^\infty\|_{\ell_\phi(X)} : (x_i)_{i=1}^\infty \in B\Big\} \geq \epsilon_0.$$

Having the assumption above, let us take $N = 1$*, and hence* $\exists n_1 \geq 1$ *such that*

$$sup\Big\{\|(x_i)_{i=n_1}^\infty\|_{\ell_\phi(X)} : (x_i)_{i=1}^\infty \in B\Big\} \geq \epsilon_0.$$

For this n_1, $\exists (x_i^{(1)})_{i=1}^\infty \in B$ *with*

$$\|(x_i^{(1)})_{i=n_1}^\infty\|_{\ell_\phi(X)} \geq \epsilon_0.$$

When n_1 *is already chosen, we consider that* $\underset{n}{lim}\|(x_i)_{i=n}^\infty\|_{\ell_\phi(X)} = 0$ *holds for any* $(x_i)_{i=1}^\infty \in \ell_\phi(X)$*, and in particular* $\underset{n}{lim}\|(x_i^{(1)})_{i=n}^\infty\|_{\ell_\phi(X)} = 0$*, and hence, for* $\frac{\epsilon_0}{2}$*,* *there exists* $N_1 \in \mathbb{N}$ *such that for* $\forall m \geq N_1$

$$\|(x_i^{(1)})_{i=m}^{\infty}\|_{\ell_\phi(X)} \leq \frac{\epsilon_0}{2}.$$

Take m_1*, such that* $m_1 > max\{N_1, n_1\}$*. Then for* $\forall m \geq m_1$*,*

$$\|(x_i^{(1)})_{i=m}^{\infty}\|_{\ell_\phi(X)} \leq \frac{\epsilon_0}{2}.$$

Now we return to the assumption that (4.2) *does not hold, and choose the arbitrary* N *to be the already constructed* m_1*. This means that there exists* $n_2 > m_1$ *such that*

$$sup\Big\{\|(x_i)_{i=n_2}^{\infty}\|_{\ell_\phi(X)} : (x_i)_{i=1}^{\infty} \in B\Big\} \geq \epsilon_0.$$

For n_2*,* $\exists (x_i^{(2)})_{i=1}^{\infty} \in B$ *such that*

$$\|(x_i^{(2)})_{i=n_2}^{\infty}\|_{\ell_\phi(X)} \geq \epsilon_0.$$

Then, since $\lim_n \|(x_i^{(2)})_{i=n}^{\infty}\|_{\ell_\phi(X)} = 0$*, we choose* $N_2 \in \mathbb{N}$ *such that for* $\forall m \geq N_2$*,*

$$\|(x_i^{(2)})_{i=m}^{\infty}\|_{\ell_\phi(X)} \leq \frac{\epsilon_0}{2}.$$

Then we take $m_2 > max\{N_2, n_2\}$*, so for* $\forall m \geq m_2$*,*

$$\|(x_i^{(2)})_{i=m}^{\infty}\|_{\ell_\phi(X)} \leq \frac{\epsilon_0}{2}.$$

By induction, we can consider the construction of $n_1 < m_1 < n_2 < m_2 < \dots$ *complete.*

For each $k, j \in \mathbb{N}$ with $k > j$ (and hence $n_k \geq n_j$),

$$\|(x_i^{(k)})_{i=1}^{\infty} - (x_i^{(j)})_{i=1}^{\infty}\|_{\ell_\phi(X)} \geq$$

$$\|(x_i^{(k)})_{i=n_k}^{\infty} - (x_i^{(j)})_{i=n_k}^{\infty}\|_{\ell_\phi(X)} \geq$$

$$\|(x_i^{(k)})_{i=n_k}^{\infty}\|_{\ell_\phi(X)} - \|(x_i^{(j)})_{i=n_k}^{\infty}\|_{\ell_\phi(X)} \geq$$

$$\epsilon_0 - \frac{\epsilon_0}{2} = \frac{\epsilon_0}{2}.$$

Thus the sequence $(x_i^{(k)})_{i=1}^{\infty}$ in B cannot have any limit point in $\ell_\phi(X)$, which shows that B is not a relatively compact subset of $\ell_\phi(X)$. This contradiction shows that (4.2) holds.

On the other hand, suppose that each $\{x_i : (x_i)_{i=1}^{\infty} \in B\}$ is a relatively compact subset of X and (4.2) holds. Take a sequence $((x_i^{(m)})_{i=1}^{\infty})_{m=1}^{\infty}$ in B. Below we show that by the diagonal method, there exists a subsequence $((x_i^{(m_k)})_{i=1}^{\infty})_{k=1}^{\infty}$ of $((x_i^{(m)})_{i=1}^{\infty})_{m=1}^{\infty}$ such that

$$\lim_k x_i^{(m_k)} \tag{4.3}$$

exists in X for each $i \in \mathbb{N}$.

Figure 4.1 (Diagonal method)

$$\begin{array}{cccccc}
x_1^{(1)} & x_1^{(2)} & \cdots\cdots\cdots & x_1^{(n)} & \cdots\cdots\cdots \\
x_2^{(1)} & x_2^{(2)} & \cdots\cdots\cdots & x_2^{(n)} & \cdots\cdots\cdots \\
x_3^{(1)} & x_3^{(2)} & \cdots\cdots\cdots & x_3^{(n)} & \cdots\cdots\cdots \\
x_4^{(1)} & x_4^{(2)} & \cdots\cdots\cdots & x_4^{(n)} & \cdots\cdots\cdots \\
\cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots
\end{array}$$

In Figure 4.1, the ith row is a subset of $\{x_i : (x_i)_{i=1}^{\infty} \in B\}$*, which is given to be relatively compact. In the first row, choose a convergent subsequence* $(x_1^{(m_k^1)})_{k=1}^{\infty}$*. Then look at* $(x_2^{(m_k^1)})_{k=1}^{\infty}$*, and choose a convergent subsequence of* $(x_2^{(m_k^1)})_{k=1}^{\infty}$*, denoted by* $(x_2^{(m_k^2)})_{k=1}^{\infty}$*. By continuing this process at the ith step we choose a convergent subsequence of* $(x_i^{(m_k^{i-1})})_{k=1}^{\infty}$*, denoted by* $(x_i^{(m_k^i)})_{k=1}^{\infty}$*. By induction, the required subsequence is constructed if we denote* $((x_i^{(m_k^i)})_{i=1}^{\infty})_{k=1}^{\infty}$ *by* $((x_i^{(m_k)})_{i=1}^{\infty})_{k=1}^{\infty}$*.*

For each $\epsilon > 0$, there exists, by (4.2), an $n_0 \in \mathbb{N}$ such that

$$\sup\{\|(x_i)_{i=n_0+1}^{\infty}\|_{\ell_\phi(X)} : (x_i)_{i=1}^{\infty} \in B\} < \frac{\epsilon}{4},$$

that is

$$\|(x_i)_{i=n_0+1}^{\infty}\|_{\ell_\phi(X)} < \frac{\epsilon}{4}, \ \forall (x_i)_{i=1}^{\infty} \in B.$$

By (4.3), there exists a $k_0 \in \mathbb{N}$ such that for each $k, j \in \mathbb{N}$ with $k, j > k_0$,

$$\left\| x_i^{(m_k)} - x_i^{(m_j)} \right\|_X < \frac{\epsilon}{2n_0}, \ i = 1, 2, \dots, n_0.$$

Thus for each $k, j \in \mathbb{N}$ with $k, j > k_0$,

$$\begin{gathered}
\left\| (x_i^{(m_k)})_{i=1}^{\infty} - (x_i^{(m_j)})_{i=1}^{\infty} \right\|_{\ell_\phi(X)} \leq \\
\sum_{i=1}^{n_0} \left\| x_i^{(m_k)} - x_i^{(m_j)} \right\|_X + \\
\left\| (x_i^{(m_k)})_{i=n_0+1}^{\infty} \right\|_{\ell_\phi(X)} + \\
\left\| (x_i^{(m_j)})_{i=n_0+1}^{\infty} \right\|_{\ell_\phi(X)} < \\
\frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{gathered}$$

Therefore, $((x_i^{(m_k)})_{i=1}^{\infty})_{k=1}^{\infty}$ is a Cauchy sequence in $\ell_\phi(X)$ and hence a convergent sequence in $\ell_\phi(X)$. This shows that B is a relatively compact subset of $\ell_\phi(X)$. $\square$

When $\mathbb{R}$ is taken instead of X, the following corollary follows, and in this case $\ell_\phi(X)$ is simply ℓ_ϕ.

COROLLARY 4.15. [**4**, Corollary 2] Let ϕ be an Orlicz function satisfying the Δ_2 condition. Then a subset B of ℓ_ϕ is relatively compact if and only if for each $i \in \mathbb{N}$, the set $\{a_i : (a_i)_{i=1}^\infty \in B\}$ is a bounded subset of $\mathbb{R}$, and

$$\lim_n sup\{\|(a_i)_{i=n}^\infty\|_{\ell_\phi} : (a_i)_{i=1}^\infty \in B\} = 0.$$

Before introducing [**4**, Theorem 3], we need to define what it means for a Banach space to have the type I-Λ-complete continuity property (I-Λ-CCP).

Let G be a compact metrizable Abelian group, and let $B(G)$ be the σ-algebra of Borel subsets of G. Let λ be a normalized Haar measure on G, and let Γ be the dual group of G. For a real or complex Banach space X, we denote by $L_1(G,X)$ the Banach space of all equivalence classes of λ-Bochner integrable functions on G with values in X.

If μ is a countably additive X-valued measure on $B(G)$, we say that it is of bounded variation if $sup \sum\limits_{A\in\pi} \|\mu(A)\| < \infty$, where the supremum is taken over all finite measurable partitions of G.

The measure μ is said to be of bounded average range if there is a positive constant c so that $\|\mu(A)\| \leq c\lambda(A)$ for every $A \in B(G)$.

$M_1(G,X)$ denotes the space of all X-valued measures on $B(G)$ that are of bounded variation.

$M_\infty(G,X)$ denotes the space of all X-valued measures on $B(G)$ that are of bounded average range.

For $\mu \in M_1(G,X)$, define the Fourier coefficient of μ at γ by

$$\hat{\mu}(\gamma) = \int_G \bar{\gamma}(t) d\mu(t).$$

Let Λ be a subset of Γ. A measure μ in $M_1(G,X)$ is called a Λ - measure if $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$.

DEFINITION 4.16. [**4**] A Banach space X is said to have the type I-Λ-complete continuity property (I - Λ - CCP) if every Λ - measure μ in $M_\infty(G,X)$ has a relatively compact range.

DEFINITION 4.17. [**4**] A Banach space X is said to have the type II-Λ-complete continuity property (II - Λ - CCP) if every Λ - measure μ in $M_1(G,X)$ has a relatively compact range.

THEOREM 4.18. [**4**, Theorem 3] *Let ϕ be an Orlicz function satisfying the Δ_2-condition. Let G be a compact metrizable Abelian group, Γ the dual group of G and Λ a subset of Γ. Then $\ell_\phi(X)$ has II - Λ - CCP if and only if both ℓ_ϕ and X have II - Λ - CCP.*

PROOF. We need only to show that if ℓ_ϕ and X have II - Λ - CCP then $\ell_\phi(X)$ has II - Λ - CCP.

Let $\mu : B(G) \to \ell_\phi(X)$ be a λ-continuous, Λ-measure of bounded variation. We want to show that $\{\mu(E) : E \in B(G)\}$, the range of μ, is a relatively compact subset of $\ell_\phi(X)$.

For each $i \in \mathbb{N}$, define

$$\mu_i : B(G) \to X, \ \ E \to \mu(E)_i,$$

where $\mu(E)_i$ is the ith coordinate of $\mu(E)$. Note that for each $E \in B(G)$,

$$\|\mu_i(E)\|_X \leq \|\mu(E)\|_{\ell_\phi(X)}.$$

Each μ_i is also a λ - continuous Λ - measure of bounded variation. Since X has II - Λ - CCP,

$$\{\mu(E)_i = \mu_i(E) : E \in B(G)\} \tag{4.4}$$

is a relatively compact subset of X for each $i \in \mathbb{N}$.

Define

$$\tilde{\mu} : B(G) \to \ell_\phi, \ \ E \to (\|\mu(E)_i\|_X)_i.$$

Note that, for each $E \in B(G)$,

$$\|\tilde{\mu}(E)\|_{\ell_\phi} = inf\{\lambda > 0 : \sum_{i=1}^{\infty} \phi\Big(\frac{\|\mu(E)_i\|_X}{\lambda}\Big) \leq 1\} = \|\mu(E)\|_{\ell_\phi(X)}.$$

Thus $\tilde{\mu}$ is a λ - continuous Λ - measure of bounded variation. Since ℓ_ϕ has II - Λ - CCP, $\{\tilde{\mu}(E) : E \in B(G)\}$ is a relatively compact subset of ℓ_ϕ. By Corollary 4.15,

$$\lim_n sup\Big\{\Big\|(0, \ldots, 0, \|\mu(E)_n\|_X, \|\mu(E)_{n+1}\|_X, \ldots)\Big\|_{\ell_\phi} : E \in B(G)\Big\} = 0.$$

That is,

$$\lim_n sup\Big\{\Big\|(0, \ldots, 0, \mu(E)_n, \mu(E)_{n+1}, \ldots)\Big\|_{\ell_\phi(X)} : E \in B(G)\Big\} = 0. \tag{4.5}$$

It follows from (4.4), (4.5) and Proposition 4.14 that the set $\{\mu(E) : E \in B(G)\}$ is a relatively compact subset of $\ell_\phi(X)$.

□

What comes now is of a valuable interest, since the spaces defined below are quite similar to general sequence spaces.

For a Banach lattice X and an Orlicz function ϕ, define

$$\ell_\phi^\epsilon(X) = \Big\{(x_i)_{i=1}^\infty \in X^{\mathbb{N}} : \Big(x^*(|x_i|)\Big)_{i=1}^\infty \in \ell_\phi, for\ all\ x^* \in {X^*}^+\Big\}$$

and

$$\ell_\phi^\pi(X) = \Big\{(x_i)_{i=1}^\infty \in X^{\mathbb{N}} : \sum_{i=1}^\infty x_i^*(|x_i|) < \infty, for\ all\ (x_i^*)_{i=1}^\infty \in \ell_{\phi^*}^\epsilon(X^*)^+\Big\},$$

with the corresponding norms

$$\|(x_i^*)_{i=1}^\infty\|_{\ell_\phi^\epsilon(X)} = sup\Big\{\|(x^*(|x_i|))_{i=1}^\infty\|_{\ell_\phi} : x^* \in \mathcal{B}_{X^{*+}}\Big\}, \ \forall (x_i^*)_{i=1}^\infty \in \ell_\phi^\epsilon(X),$$

and

$$\|(x_i)_{i=1}^\infty\|_{\ell_\phi^\pi(X)} = sup\Big\{\sum_{i=1}^\infty x_i^*(|x_i|) : (x_i^*)_{i=1}^\infty \in \mathcal{B}_{\ell_{\phi^*}^\epsilon(X^*)^+}\Big\}, \ \forall (x_i)_{i=1}^\infty \in \ell_\phi^\pi(X).$$

Then $\ell_\phi^\pi(X)$ and $\ell_\phi^\epsilon(X)$ are Banach lattices under the corresponding norms.

In Proposition 4.19 a sequential representation of the Fremlin projective tensor product [**7**] of ℓ_ϕ and X is given.

PROPOSITION 4.19. [**4**, Proposition] Let X be a Banach lattice and ϕ an Orlicz function that has its complementary function. If ϕ satisfies the Δ_2 - condition, then $\ell_\phi \hat{\otimes}_F X$ is isometrically lattice isomorphic to $\ell_\phi^\pi(X)$.

A Banach space X is said to semi-embed into a Banach space Y if there is a one-to-one continuous linear operator from X to Y such that the image of the closed unit ball of X is closed in Y. A Banach space property $\mathcal{P}$ is said (i) to be separably determined if a Banach space X has $\mathcal{P}$ whenever every separable closed subspace of X has $\mathcal{P}$, and (ii) to be separably semi- embeddably stable if a separable Banach space X has $\mathcal{P}$ whenever X semi-embeds into a Banach space Y with $\mathcal{P}$. Conditions (i) and (ii) imply that if a Banach space X has $\mathcal{P}$, then so does every closed subspace of X. Conditions (i) and (ii) also imply that if X and Y are isomorphic Banach spaces and Y has $\mathcal{P}$, then X has $\mathcal{P}$.

THEOREM 4.20. [**4**, Theorem 5] *Let X be a Banach lattice, and let ϕ be an Orlicz function that has its complementary function and satisfies the Δ_2-condition. Let $\mathcal{P}$ be a Banach space property such that $\mathcal{P}$ is separably determined and separably semi-embeddably stable. If $\ell_\phi(X)$ has $\mathcal{P}$, then $\ell_\phi \hat{\otimes}_F X$ also has $\mathcal{P}$.*

5. Thinning sequences in $\lambda_\pi(X)$

In this section we give a detailed explanation and clarification of the adapted thinning process mentioned in Megginson's book [**9**], on page 219 in Example 2.5.24.

Let us take a sequence $\left(\bar{x}^{(n)}\right) = \left((x_i^{(n)})_i\right)$ of positive elements of $\lambda_\pi(X)$, which is weakly convergent to zero. May we find conditions under which $\|\left(x_i^{(n)}\right)\|_{\lambda_\pi(X)} \to 0$, as $n \to \infty$? Let us see what conclusions we may achieve if we assume that $\|\left(x_i^{(n)}\right)\|_{\lambda_\pi(X)}$ does not tend to zero.

For the sequence $\left(\bar{x}^{(n)}\right) = \left((x_i^{(n)})_i\right)$ to be weakly null means that $\underset{n}{lim} \sum_{i=1}^\infty x_i^*(|x_i^{(n)}|) = 0$ for $\forall \bar{x}^* = (x_i^*)_i \in \lambda_\epsilon'(X^*)$. It may be as well assumed that $x_i^{(n)} = |x_i^{(n)}|$.

Figure 5.1

$$
\begin{array}{cccccc}
x_1^*(|x_1^{(1)}|) & x_1^*(|x_1^{(2)}|) & \cdots\cdots\cdots & x_1^*(|x_1^{(n)}|) & \cdots\cdots\cdots \\
x_2^*(|x_2^{(1)}|) & x_2^*(|x_2^{(2)}|) & \cdots\cdots\cdots & x_2^*(|x_2^{(n)}|) & \cdots\cdots\cdots \\
x_3^*(|x_3^{(1)}|) & x_3^*(|x_3^{(2)}|) & \cdots\cdots\cdots & x_3^*(|x_3^{(n)}|) & \cdots\cdots\cdots \\
x_4^*(|x_4^{(1)}|) & x_4^*(|x_4^{(2)}|) & \cdots\cdots\cdots & x_4^*(|x_4^{(n)}|) & \cdots\cdots\cdots \\
\cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\
\cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\
(i) & (ii) & \cdots\cdots\cdots & (nth) & \cdots\cdots\cdots \\
\in \ell_1 & \in \ell_1 & \cdots\cdots\cdots & \in \ell_1 & \cdots\cdots\cdots
\end{array}
$$

REMARK 5.1. The sequence of elements of ℓ_1, introduced in Figure 4.1, is convergent to zero in ℓ_1, because $\left((x_i^{(n)})_{i=1}^\infty\right)_{n=1}^\infty$ is chosen to be weakly convergent to zero. Let us first suppose that we have the initial series (not the thinned one).

For a sequence in ℓ_1, to be convergent to zero, it means that for $\forall\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $\forall n \geq N$,

$$\|(x_i^*(x_i^{(n)}))_i\|_{\ell_1} = \sum_{i=1}^{\infty} x_i^*(x_i^{(n)}) < \epsilon.$$

Below, when we will thin this sequence, we will keep all the columns (that is, the elements of the sequence) in their places but we will remove some number of elements from each of the members of the sequence. For the resulting series this inequality will still hold, because its $n-th$ element will differ from the $n-th$ element of the initial series by only not having the first, let us say r (r_n, depending on each n) elements. Hence the ℓ_1 norm of the $n-th$ element of the sequence of ℓ_1, resulted after thinning (or throwing away first r elements), will be a formal positive series less than the series, corresponding to the initial sequence (the ℓ_1 norm of the $n-th$ element of the initial series in ℓ_1), which is less than ϵ. Hence the thinned series will also be convergent to zero in ℓ_1.

We want to find conditions under which

$$\lim_n \sup \Big\{ \sum_{i=1}^{\infty} x_i^*(|x_i^{(n)}|) : (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+} \Big\} = 0,$$

which is what $\|\cdot\|_{\lambda_\pi(X)}$-convergence means by the definition.

Let us notice one more time that since

$$\Big(\bar{x}^{(n)}\Big) = \Big((x_i^{(n)})_i\Big) \subseteq \lambda_\pi(X),$$

it follows that each of $(i), (ii), \ldots, (nth), \ldots,$ is an element of the space ℓ_1 of summable sequences.

Moreover, we have to stress that not only

$$\sum_{i=1}^{\infty} x_i^*(|x_i^{(n)}|) < \infty$$

for each fixed $\bar{x}^* = (x_i^*)_i \in \lambda'_\epsilon(X^*)$ and each fixed $n \in \mathbb{N}$, but also

$$\sup \Big\{ \sum_{i=1}^{\infty} x_i^*(|x_i^{(n)}|) : \ \ \forall (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+} \Big\} < \infty,$$

for each fixed $n \in \mathbb{N}$.

The last relation follows because $\lambda_\pi(X)$-norm of $\Big(x_i^{(n)}\Big)_i$ is finite for each fixed $n \in \mathbb{N}$. Besides, the weak nullity of $\Big(\bar{x}^{(n)}\Big) = \Big((x_i^{(n)})_i\Big)$, that is, the fact that

$$\lim_n \sum_{i=1}^{\infty} x_i^*(|x_i^{(n)}|) = 0,$$

for $\forall \bar{x}^* = (x_i^*)_i \in \lambda'_\epsilon(X^*)$, implies that for each fixed $\bar{x}^* = (x_i^*)_i \in \lambda'_\epsilon(X^*)$, $\Big((x_i^*(|x_i^{(n)}|))_i\Big)$ is a norm convergent to zero sequence of l_1.

$$\Big(\lim_n \sum_{i=1}^{\infty} x_i^*(|x_i^{(n)}|) = 0 \Longleftrightarrow \lim_n \Big\| \Big(x_i^*(|x_i^{(n)}|)\Big)_i \Big\|_{\ell_1} = 0\Big).$$

Another observation is that by Proposition 1 of [**5**], since λ is σ-order continuous, it follows that $\lambda_{\pi,0}(X) = \lambda_\pi(X)$, which means that for each fixed $n \in \mathbb{N}$

$$\underset{k\to\infty}{lim} \sup\Big\{ \sum_{i=k}^{\infty} x_i^*(|x_i^{(n)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+} \Big\} = 0$$

("sup" tail converges to zero).

After making the few observations above, let us return to our assumption that $\|\Big(x_i^{(n)}\Big)\|_{\lambda_\pi(X)}$ does not converge to zero as n converges to infinity. May this assumption eventually lead us to a contradiction?

From this assumption it follows that there is a subsequence (n_j) of $\mathbb{N}$ and a positive scalar t such that

$$\sup\Big\{ t\cdot \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_j)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+} \Big\} \geq 1.$$

By thinning $t \cdot \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_j)}|)$ if necessary, it may be assumed that there is a sequence (t_{n_j}) of nonnegative integers such that $0 = t_{n_1} < t_{n_2} < \dots$ and

$$t \cdot \sum_{i=t_{n_j}+1}^{t_{n_{j+1}}} x_i^*(|x_i^{(n_j)}|) > \frac{3}{4} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_j)}|).$$

Let us describe more in detail the process of constructing such a sequence. We take $t_{n_1} = 0$. Having n_1 fixed at this step, we take $t_{n_2} > t_{n_1} = 0$ such that

$$t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_1)}|) > \frac{3}{4} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_1)}|)$$

(tail converges to zero).

Now let us look at $t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|)$.

If

$$t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|) \leq \frac{1}{8} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|),$$

then we choose $t_{n_3} > t_{n_2}$ to be such that also

$$t \cdot \sum_{i=t_{n_3}+1}^{\infty} x_i^*(|x_i^{(n_2)}|) \leq \frac{1}{8} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|).$$

Then

$$t \cdot \sum_{i=t_{n_2}+1}^{t_{n_3}} x_i^*(|x_i^{(n_2)}|) \geq \frac{3}{4} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|).$$

In general, we would have been lucky to have

$$t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|) \leq \frac{1}{8} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|),$$

so we have to consider the case when

$$t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|) > \frac{1}{8} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|).$$

In this case, we throw away the first t_{n_2} members of

$$\left(x_i^*(|x_i^{(n_2)}|) \right)_i,$$

renumber it, so that in the sum below $x_1^*(|x_1^{(n_2)}|)$ was previously $x_{t_{n_2}+1}^*(|x_{t_{n_2}+1}^{(n_2)}|)$, further $x_2^*(|x_2^{(n_2)}|)$ was previously $x_{t_{n_2}+2}^*(|x_{t_{n_2}+2}^{(n_2)}|)$, next $x_3^*(|x_3^{(n_2)}|)$ was previously $x_{t_{n_2}+3}^*(|x_{t_{n_2}+3}^{(n_2)}|)$, etc., and again look at

$$t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|)$$

(this last sum is under the new numbering).

If this last sum is $\leq$ than

$$\frac{1}{8} \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|),$$

then we are satisfied, and if not we repeat the step, by throwing it away.

(The sum from 1 to ∞ is the initial sum, when nothing has been thrown away from it yet).

At most after 8 steps, after having thrown away $7 \cdot t_{n_2}$ members of $\left(x_i^*(|x_i^{(n_2)}|) \right)_i$, and each time renumbering it, we get that

$$t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|) \leq \frac{1}{8} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|),$$

where the series in the right corresponds to the initial numbering (when nothing was thrown away).

Next, we choose $t_{n_3} > t_{n_2}$ such that

$$t \cdot \sum_{i=t_{n_3}+1}^{\infty} x_i^*(|x_i^{(n_2)}|) \leq \frac{1}{8} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|),$$

(where on the right is still the initial series, with none of its members thrown away yet).

Finally we have that

$$t \cdot \sum_{i=t_{n_2}+1}^{t_{n_3}} x_i^*(|x_i^{(n_2)}|) \geq \frac{3}{4} \cdot t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|),$$

where on the right may as well be taken the thinned series (its $7 \cdot t_{n_2}$ many members removed).

By induction, we may consider the required sequence (t_{n_j}), $0 = t_{n_1} < t_{n_2} < \dots$, be constructed.

Now we will show the same "thinning" construction for supremums.

Take $t_{n_1} = 0$. Here n_1 is fixed, and we already stated that for each fixed n

$$\mathop{lim}_{k\to\infty} \sup\Big\{ \sum_{i=k}^{\infty} x_i^*(|x_i^{(n)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+} \Big\} = 0,$$

which is the supremum tail convergence to zero. Therefore, we can choose $t_{n_2} > t_{n_1} = 0$ such that

$$\sup\Big\{t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_1)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\} \geq \frac{3}{4}\cdot \sup\Big\{t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_1)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\}.$$

Next, if

$$\sup\Big\{t \cdot \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\} \leq \frac{1}{8}\cdot \sup\Big\{t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\},$$

then we choose $t_{n_3} > t_{n_2}$ in such a way that

$$\sup\Big\{t \cdot \sum_{i=t_{n_3}+1}^{\infty} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\} \leq \frac{1}{8}\cdot \sup\Big\{t \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\}.$$

If we could have that

$$\sup\Big\{t \cdot \sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\} = \sup\Big\{t \sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\} +$$

$$+ \sup\Big\{t \cdot \sum_{i=t_{n_2}+1}^{t_{n_3}} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\} + \sup\Big\{t \sum_{i=t_{n_3}+1}^{\infty} x_i^*(|x_i^{(n_2)}|) : \ (x_i^*)_i \in \mathcal{B}_{\lambda'_\epsilon(X^*)^+}\Big\},$$

then it would follow that

$$\sup\left\{t\cdot\sum_{i=t_{n_2}+1}^{t_{n_3}} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\} \geq \frac{3}{4}\cdot\sup\left\{t\sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\}.$$

Next, if

$$\sup\left\{t\cdot\sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\} > \frac{1}{8}\cdot\sup\left\{t\sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\},$$

then we throw away at most $7\cdot t_{n_2}$ many terms of

$$\left(x_i^*(|x_i^{(n_2)}|)\right)_i,$$

and after renumbering it, get that

$$\sup\left\{t\cdot\sum_{i=1}^{t_{n_2}} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\} \leq \frac{1}{8}\cdot\sup\left\{t\sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\},$$

where the supremum in the right is taken for the initial series, with no members thrown away. Then we choose $t_{n_3}>t_{n_2}$ with

$$\sup\left\{t\cdot\sum_{i=t_{n_3}+1}^{\infty} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\} \leq \frac{1}{8}\cdot\sup\left\{t\sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\},$$

where the supremum of the series in the right is still for the initial series, with no terms removed.

Finally we get that

$$\sup\left\{t\cdot\sum_{i=t_{n_2}+1}^{t_{n_3}} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\} \geq \frac{3}{4}\cdot\sup\left\{t\sum_{i=1}^{\infty} x_i^*(|x_i^{(n_2)}|):\ (x_i^*)_i\in\mathcal{B}_{\lambda'_\epsilon(X^*)^+}\right\},$$

where the supremum in the right may also be taken for the final renumbered series, with at most $7\cdot t_{n_2}$ many terms removed.

By induction, the construction is finalized.

References

[1] G. Botelho and J. R. Campos, *On the transformation of vector-valued sequences by linear and multilinear operators*, Monatsh. Math. **183** (2017), no. 3, 415–435, DOI 10.1007/s00605-016-0963-4. MR3662075

[2] Q. Bu and G. Buskes, *Schauder decompositions and the Fremlin projective tensor product of Banach lattices*, J. Math. Anal. Appl. **355** (2009), no. 1, 335–351, DOI 10.1016/j.jmaa.2009.01.061. MR2514471

[3] Q. Bu, Y. Li, and X. Xue, *Some properties of the space of regular operators on atomic Banach lattices*, Collect. Math. **62** (2011), no. 2, 131–137, DOI 10.1007/s13348-010-0007-7. MR2792516

[4] Q. Bu, D. Ji, and X. Xue, *Complete continuity properties for the Fremlin projective tensor product of Orlicz sequence spaces and Banach lattices*, Rocky Mountain J. Math. **40** (2010), no. 6, 1797–1808, DOI 10.1216/RMJ-2010-40-6-1797. MR2764221

[5] D. Ji, Y. Li, and Q. Bu, *The complete continuity properties for the positive projective tensor product of atomic Banach lattices*, Positivity **17** (2013), no. 1, 17–25, DOI 10.1007/s11117-011-0141-9. MR3027643

[6] N. Dunford and J. T. Schwartz, *Linear operators. Part I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988. General theory; With the assistance of William G. Bade and Robert G. Bartle; Reprint of the 1958 original; A Wiley-Interscience Publication. MR1009162

[7] D. H. Fremlin, *Tensor products of Archimedean vector lattices*, Amer. J. Math. **94** (1972), 777–798, DOI 10.2307/2373758. MR0312203

[8] D. H. Fremlin, *Tensor products of Banach lattices*, Math. Ann. **211** (1974), 87–106, DOI 10.1007/BF01344164. MR0367620

[9] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York, 1998. MR1650235

[10] P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991. MR1128093

[11] A. C. Zaanen, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, 1997. MR1631533

Email address: Botelho@ufu.br

Email address: CauseyRM@miamioh.edu

Email address: Kh_Navoyan@ufu.br