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Some geometric properties of relative Chebyshev centres in Banach spaces

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Abstract. In this paper we characterize Property-(R_1), a generalization of $1\frac{1}{2}$ ball property. As a necessary and sufficient condition of a subspace Y with Property-(R_1) we derive that $r(y,F) = rad_Y(F) + d(y, cent_Y(F))$ for any bounded subset F and $y \in Y$. We introduce the notion of modulus of relative chebyshev centre and characterize Property-(R_1) in terms of this modulus. It is observed that if Y is a finite co-dimensional strongly proximinal subspace of a L_1 predual space X and F is a finite subset of X then $rad_Y(F) = rad_X(F) + d(F,Y)$. We characterize continuity of $cent_V(.)$ in terms of the modulus of relative chebyshev centre.

1. Introduction

1.1. Notations and Definitions. By X we always mean a Banach space. For $x \in X$ and $r > 0$ $B(x,r)$ and $B[x,r]$ represent the open and closed ball centered at x and radius r respectively. By $CB(X), CL(X), CC(X), \mathcal{F}(X)$ we mean the set of all closed and bounded, closed, closed convex and finite subsets of X respectively. The underlying field for all the spaces is assumed to be Real.

For $x \in X, F \in CB(X)$ we define the following.

Notation. (1) $r(x,F) = \sup\{\|x-y\| : y \in F\}$

(2) $rad_V(F) = \inf\{r(x,F) : x \in V\}$

(3) $cent_V(F) = \{v \in V : r(v,F) = rad_V(F)\}$

(4) $\delta - cent_V(F) = \{v \in V : r(v,F) \leq rad_V(F) + \delta\}$

(5) *For* $B \subseteq X$ $B_\varepsilon = \{x \in X : d(x,B) \leq \varepsilon\}$.

(6) $S_\varepsilon(F) = \{x \in X : r(x,F) \leq \varepsilon\}$.

Note that,

(1) $cent_V(F) = \left\{\bigcap_{y\in F} B[y, rad_V(F)]\right\} \cap V$.

(2) $\delta - cent_V(F) = \left\{\bigcap_{y\in F} B[y, rad_V(F) + \delta]\right\} \cap V$.

(3) $B_\varepsilon = B + \varepsilon B_X$ and $S_\varepsilon(F) = \bigcap_{x\in F} B[x,\varepsilon]$.

Note that if $V \in CC(X)$ and $F \in \mathcal{F}$ the set $cent_V(F)$ may be empty, although the set $\delta - cent_V(F)$ is always nonempty for any $\delta > 0$. A pair $(V, \mathcal{F})$ where

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$V \in CL(X)$ and $\mathcal{F} \subseteq CB(X)$ is said to have *restricted center property* (in short r.c.p.) if for all $F \in \mathcal{F}, cent_V(F) \neq \emptyset$. $rad_V(F)$ represents the radius of the smallest ball (if it exists) in X centered in V which contains F, $cent_V(F)$ represents the possible points of centres of these balls. $\delta - cent_V(F)$ represents the points in V which are also in the δ perturbation of radius these balls.

We now introduce the central character of this paper to the reader.

Definition 1.1. [**9**] Let $V \in CC(X)$ and $\mathcal{F} \subseteq CB(X)$, the triplet $(X, V, \mathcal{F})$ is said to have Property-(R_1) if for $v \in V, F \in \mathcal{F}, r_1, r_2 > 0$ the condition $B(v, r_1) \cap S_{r_2}(F) \neq \emptyset$ and $S_{r_2}(F) \cap V \neq \emptyset$ would imply that $V \cap B[v, r_1] \cap S_{r_2}(F) \neq \emptyset$.

The above definition is a set valued analogue of the following.

Definition 1.2. [**12**] A subspace Y of a Banach space X is said to have $1\frac{1}{2}$ ball property if $y \in Y$ and $x \in X$ the open balls $B(x, r_1), B(y, r_2)$ intersects and the closed ball $B[x, r_1]$ has non empty intersection with the subspace Y then the intersection $Y \cap B[x, r_1] \cap B[y, r_2]$ is non empty.

It is clear that Property-(R_1) implies the property of $1\frac{1}{2}$ ball if $\mathcal{F}$ contains the set of all singletons. A list of examples are given in [**9**] having this property. Let us recall that a Banach space X is said to be L_1 predual if X^* is isometric with $L_1(\mu)$ for some measure space (Ω, Σ, μ).

Let us recall that a subspace Y of X is said to have *Best approximation property* (or *Proximinal*) if for any point $x \notin Y$ there exists $y \in Y$ such that $\|x-y\| = d(x, Y)$, where $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$. For any such subspace Y one can define the set valued mapping $P_Y : X \to CB(Y)$ by $P_Y(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$. P_Y is called the metric projection for the subspace Y.

It is well known that a subspace with $1\frac{1}{2}$ ball property also satisfies Best approximation property and also the metric projection $x \mapsto P_Y(x)$ satisfies a continuity criterian (it is *upper Hausdorff semi continuous*). One can define the following weaker notion than $1\frac{1}{2}$ ball property, subspace Y with which P_Y is upper Hausdorff semi continuous.

For a subspace Y and $\delta > 0$, let us define $P_Y(x, \delta) = \{y \in Y : \|x - y\| \leq d(x, Y) + \delta\}$.

Definition 1.3. [**7**] A subspace Y of a Banach space X is said to be *Strongly proximinal* if given $\varepsilon > 0$ and $x \in X$ there exists a $\delta(\varepsilon, x) > 0$ such that $P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y$.

In [**8**] the author observed that a subspace with $1\frac{1}{2}$ ball property also Strongly proximinal. A set valued analogue of Strong proximinality is introduced in [**9**] by the following.

Definition 1.4. [**9**] Let $V \in CC(X)$ and $\mathcal{F} \subseteq CB(X)$, the triplet $(X, V, \mathcal{F})$ is said to have Property-(P_1) if for given $\varepsilon > 0$ and $F \in \mathcal{F}$ there exists a $\delta(\varepsilon, F) > 0$ such that $\delta - cent_V(F) \subseteq cent_V(F) + \varepsilon B_X$.

It is clear that if V is a subspace and if $\mathcal{F}$ contains all singletons then V is Strongly proximinal if the triplet $(X, V, \mathcal{F})$ has Property-(P_1).

We encounter various continuity of a set valued map in a normed linear space. Let us recall the following definition in this context.

Definition 1.5. Let T be a topological space and $\Gamma : T \to CB(X)$ be a set valued map. Γ is said to be

(a) upper Hausdorff semi-continuous, abbreviated uHsc. (resp. lower Hausdorff semi-continuous, abbreviated lHsc) if for every $t_0 \in T$ and every $\varepsilon > 0$, there is a neighborhood N of t_0, such that $\Gamma(t) \subseteq \Gamma(t_0) + \varepsilon B_X$ (resp. $\Gamma(t_0) \subseteq \Gamma(t) + \varepsilon B_X$) for each $t \in N$.

(b) Γ is Hausdorff continuous, abbreviated H-continuous, if it is both uHsc and lHsc.

Definition 1.6. A real valued map $\varphi : (X, \tau_X) \to \mathbb{R}$ is said to be upper semi continuous (usc) (lower semi continuous (lsc)) if for any real $\alpha \in \mathbb{R}$ the set $\{x \in X : \varphi(x) \geq \alpha\}$ ($\{x \in X : \varphi(x) \leq \alpha\}$) is closed.

1.2. A Brief review of the known results and the important outcomes of this investigation. History of (relative) Chebyshev Centre dates back to the papers [**1**–**4**, **9**] and also many more. A compactness criterion (like norm, weak or w^*-compactness) of the unit ball of the space makes sure of the non emptyness of the (relative) Chebyshev centre of any finite subset of the space. An ultimate result in this direction is available in [**1**] where the author proves that a uniformly convex Banach space always provides a non empty Chebyshev centre for any bounded subset and the set valued map which sends the bounded set to its centre is single valued and also uniformly continuous with respect to the Hausdorff metric. Authors in [**3**] and [**9**] prove this result into its full generalization. Now coming to the cases when the space is not reflexive (or the unit ball has no compactness criterion) we refer the reader to the articles [**6**] and [**11**]. In [**6**] the author observes that a space of type $C(K)$, with its standerd meaning, any of its finite co-dimensional subspace always provide a non empty Chebyshev centre for any compact subset of $C(K)$ if and only if the subspace is proximinal.

In [**11**] the authors establish some geometric properties of Chebyshev centre and relative Chebyshev centre in the space of type $C(K)$, in particular it is observed that $cent_G(F) = cent_{C(K)}(F) + R$, where $R = \inf\{\|x - g\| : x \in cent_{C(K)}(F), g \in G\}$ where G is a closed convex subset of $C(K)$. (see [**11**, Theorem 2.2]). We derive a similar result for a L_1 predual space when the above G is a finite co-dimensional subspace and F is a finite subset. Full generality of this result for the spaces of type L_1 is still unknown.

David Yost introduced the notion of $1\frac{1}{2}$ ball property (see [**12**]) of a closed subspace of a Banach space, a judicial modification of 2 ball property. In [**12**] Yost explored some geometric and analytic properties of subspaces of a Banach space having this property, viz. existence of nearest point property, continuity of metric projection, quasi additivity of metric projection, uniqueness of Hahn-Banach extension map from Y^* to X^* etc. Afterwards few articles appeared following the same line of investigation (see, [**8**, **10**, **12**–**14**]) which explore many interesting consequences of $1\frac{1}{2}$ ball property. Yost defined some other property viz. weak $1\frac{1}{2}$ ball property, which is in fact equivalent to $1\frac{1}{2}$ ball property, also proved in [**12**]. A hidden geometry of this property is that the closed unit ball of the whole space has a *flat face parallel* to the subspace. This refined geometry is established in [**10**, Proposition 1].

Yost's school of thought is adopted into the set-valued optimization technique by Pai and Nowroji in [**9**] in the name of Property-(R_1), see Definition 1.1. We continue this investigation in this paper and derive few characterizations of this Property.

A Section wise illustration of this work is given below. Most of our observations are related to the triplet $(X, V, \mathcal{F})$, where $V \in CC(X)$ and $\mathcal{F} \subseteq \mathcal{F}(X)$.

In Section 2 our prime objective is to establish some geometric properties of subspaces having Property-(R_1) and to find some new examples having this property. Few of these geometric properties are the set valued analogues of what was observed in [**8**] for $1\frac{1}{2}$ ball property.

In Section 3 we introduce the notion of *Modulus of Restricted Chebyshev centre* viz. $\varepsilon_V(F, t)$ corresponding to the triplet $(X, V, \mathcal{F})$ where $F \in \mathcal{F}$ and $0 < t < 1$. We characterize Property-(R_1) in terms of this modulus. We also derive that if Y is a finite codimensional subspace of a L_1 predual X which is strongly proximinal then for any finte subset F of X, $cent_Y(F) = cent_X(F) + d(F, Y)$. The result is known for spaces of type $C(K)$ endowed with the supremum norm, we derive it for the category of L_1 predual spaces.

The paper concludes with the fact that the set valued map $cent_V(.)$ is continuous at F if and only if $\varepsilon(., t)$ are continuous at F for all $t > 0$, provided the subspace V has r.c.p. for all $F \in \mathcal{F}$.

2. Property-(R_1)

Let $V \in CL(X)$, $F \in CB(X)$ then for all $v \in V$ it is always true that $d(v, cent_V(F)) \geq r(v, F) - rad_V(F)$. In fact if $z \in cent_V(F)$, then $r(v, F) - rad_V(F) = r(v, F) - r(z, F) \leq \|v - z\|$, true for all $z \in cent_V(F)$, hence the result.

Definition 2.1. We call the triplet $(X, V, \mathcal{F})$ has weak Property(R_1) if for $v \in V, F \in \mathcal{F}, r_1, r_2 > 0$ the condition $B[v, r_1] \cap S_{r_2}(F) \neq \emptyset$ and $S_{r_2}(F) \cap V \neq \emptyset$ would imply that $V \cap B[v, r_1 + \varepsilon] \cap S_{r_2+\varepsilon}(F) \neq \emptyset$, for all $\varepsilon > 0$.

We first show that the two different conditions viz. Property-(R_1) and weak Property-(R_1) are equivalent.

Theorem 2.2. *For a triplet $(X, V, \mathcal{F})$ where $\mathcal{F} \subseteq CB(X)$, V is closed convex, $0 \in V$ and closed under translation. Then the following are equivalent.*

(a) *$(X, V, \mathcal{F})$ has Property-(R_1).*
(b) *$(X, V, \mathcal{F})$ has weak Property-(R_1).*
(c) *For $F \in \mathcal{F}$ $r > 0$ the condition $B(0, r) \cap S_1(F) \neq \emptyset$ and $S_1(F) \cap V \neq \emptyset$ would imply that $V \cap B[0, r] \cap S_1(F) \neq \emptyset$.*
(d) *For $F \in \mathcal{F}$ and $r > 0$ the condition $B[0, r] \cap S_1(F) \neq \emptyset$ and $S_1(F) \cap V \neq \emptyset$ would imply that $V \cap B[0, r + \varepsilon] \cap S_{1+\varepsilon}(F) \neq \emptyset$, for all $\varepsilon > 0$.*

Proof. It is clear that $(a) \iff (c)$ and $(b) \iff (d)$ and also $(a) \implies (b)$. To complete the proof it suffices to show that $(b) \implies (a)$. Let $r(v, F) < r_1 + r_2$ and $rad_V(F) \leq r_2$.

Claim : $B[v, r_1] \cap S_{r_2}(F) \cap V \neq \emptyset$.

If $r(v, F) \leq r_2$ then $v \in B[v, r_1] \cap S_{r_2}(F) \cap V$ and we are done. Let $r(v, F) > r_2$, then we have $rad_V(F) \leq r_2 < r(v, F) < r_1 + r_2$.

Let $\varepsilon = \frac{1}{3}(r_1 + r_2 - r(v, F))$ then $\varepsilon > 0$ and $V \cap S_{r_2+\frac{\varepsilon}{2}}(F) \neq \emptyset$ (since $rad_V(F) \leq r_2$) also we have $r(v, F) \leq r_2 + (r_1 - 3\varepsilon)$. By weak Property-$(R_1)$ there exists $x_0 \in V \cap B[v, r_1 - 2\varepsilon] \cap S_{r_2+\varepsilon}(F)$. By induction we will construct a sequence $(x_n) \subseteq V$ such that $\|x_n - x_{n+1}\| \leq \frac{\varepsilon}{2^n}$ and $r(x_n, F) \leq r_2 + \frac{\varepsilon}{2^n}$.

Suppose $(x_i)_{i=1}^n$ are chosen. Hence we have $r(x_n, F) \leq r_2 + \frac{\varepsilon}{2^n} = r_2 + \frac{\varepsilon}{2^n}(\frac{3}{4} + \frac{1}{4}) = \frac{3\varepsilon}{4.2^n} + (r_2 + \frac{\varepsilon}{4.2^n})$. Again by weak Property-(R_1) there exists $x_{n+1} \in V \cap$

$B[x_n, \frac{3\varepsilon}{4.2^n} + \frac{\varepsilon}{4.2^n}] \cap S_{r_2+\frac{\varepsilon}{4.2^n}+\frac{\varepsilon}{4.2^n}}$. Hence (x_n) is cauchy and hence $x_n \to x$ for some $x \in V$. We have $\|x - x_0\| \leq 2\varepsilon$ and $r(x, F) \leq r_2$. Finally, we have $\|x - v\| \leq \|x - x_0\| + \|v - x_0\| \leq 2\varepsilon + r_1 - 2\varepsilon = r_1$ which establishes the above CLAIM. □

PROPOSITION 2.3. *For $0 \leq \delta < \varepsilon$ and $F \in CB(X), V \in CC(X)$ we have,*

$$\delta - cent_V(F)_{\varepsilon-\delta} \subseteq \bigcap_{x\in F} B[x, rad_V(F) + \varepsilon] \cap V$$

In other words we have $\delta - cent_V(F)_{\varepsilon-\delta} \subseteq \varepsilon - cent_V(F)$.

PROOF. Let $y \in V$ such that $d(y, \delta - cent_V(F)) \leq \varepsilon - \delta$. Let $g_n \in \delta - cent_V(F)$ such that $\|y - g_n\| \to d(y, \delta - cent_V(F))$.

If $x \in F$ then

$$\begin{aligned} \|x - y\| &\leq \|x - g_n\| + \|y - g_n\| \\ &\leq r(g_n, F) + \|y - g_n\| \\ &\leq rad_V(F) + \delta + \|y - g_n\| \\ &\to rad_V(F) + \delta + d(y, \delta - cent_V(F)) \\ &\leq rad_V(F) + \varepsilon \end{aligned}$$

True for all $x \in F$, hence $r(y, F) \leq rad_V(F) + \varepsilon$. That is $y \in \varepsilon - cent_V(F)$. □

We now prove the main result of this Section.

THEOREM 2.4. *Let V be a subspace and $\mathcal{F} \subseteq CB(X)$ is closed under translation then for the triplet $(X, V, \mathcal{F})$ the following are equivalent*

(*a*) *$(X, V, \mathcal{F})$ has Property-(R_1).*

(*b*) *$d(v, cent_V(F)) = r(v, F) - rad_V(F)$, for all $v \in V, F \in \mathcal{F}$.*

(*c*) *$d(0, cent_V(F)) = r(0, F) - rad_V(F)$, for all $F \in \mathcal{F}$.*

(*d*) *$\delta - cent_V(F) = cent_V(F)_\delta \cap V$.*

(*e*) *For $F \in \mathcal{F}, r_1, r_2 \geq 0$ with $rad_V(F) \leq r_1 < r_2$ define $A_i = \{v \in V : r(v, F) = r_i\}, i = 1, 2$. Let $A_1 \neq \emptyset$ and $g_2 \in A_2$ then $d(g_2, A_1) = r_2 - r_1$.*

PROOF. $(b) \iff (d)$: Let $v \in \delta - cent_V(F)$. Then $r(v, F) = rad_V(F) + d(v, cent_V(F))$ but $r(v, F) \leq rad_V(F) + \delta$ also. Hence the result in (d) follows. Let $\delta = r(v, F) - rad_V(F), v \in V$. Then $v \in \delta - cent_V(F) = cent_V(F)_\delta \cap V$. That is $d(v, cent_V(F)) \leq \delta = r(v, F) - rad_V(F)$. Hence (b) follows.

$(b) \iff (c)$: Follows from obvious translation. Suppose (b) is true. Replace F by $F - v$. Clearly $r(0, F - v) = r(v, F)$.

$rad_V(F) = \inf\{r(v, F) : v \in V\} = \inf\{r(z, F - v) : z \in V\} = rad_V(F - v)$.

Now

$$\begin{aligned} d(0, cent_V(F - v)) &= \inf\{\|x\| : x \in cent_V(F - v)\} \\ &= \inf\{\|x\| : r(x + v, F) = rad_V(F)\} \\ &= \inf\{\|x\| : x + v \in cent_V(F)\} \\ &= \inf\{\|y - v\| : y \in cent_V(F)\} \\ &= d(v, cent_V(F)) \end{aligned}$$

$(a) \implies (e)$: Suppose $d(g_2, A_1) \neq r_2 - r_1$, hence $d(g_2, A_1) > r_2 - r_1$. Choose $\varepsilon > 0$ such that $d(g_2, A_1) > r_2 - r_1 + \varepsilon$. Let $\overline{r_1} = r_2 - r_1 + \varepsilon$ and $\overline{r_2} = r_1$. Then $r(g_2, F) = r_2 < \overline{r_1} + \overline{r_2}$ and $\emptyset \neq A_1 \subseteq S_{\overline{r_2}}(F) \cap V$. By (a) there exists $y \in V \cap B[g_2, \overline{r_1}] \cap S_{\overline{r_2}(F)}$.

Since $r(y,F) \leq \overline{r_2} = r_1$ and $r(g_2,F) = r_2 > r_1$, there exists $g_1 \in [y, g_2]$, the line joining y and g_2, such that $r(g_1,F) = r_1$ then $g_1 \in A_1$ and $\|g_2 - g_1\| = \|g_2 - y\| - \|g_1 - y\| \leq \|g_2 - y\| \leq \overline{r_1} < d(g_2, A_1)$. This contradiction ensures the result.

$(e) \Longrightarrow (d)$: Let $F \in \mathcal{F}, 0 \leq \varepsilon_1 < \varepsilon_2$ such that $\varepsilon_1 - cent_V(F) \neq \emptyset$. It remains to show that $\varepsilon_2 - cent_V(F) \subseteq \varepsilon_1 - cent_V(F)_{\varepsilon_2 - \varepsilon_1} \cap V$.

Let $v_0 \in \varepsilon_2 - cent_V(F)$, if $v_0 \in \varepsilon_1 - cent_V(F)$ then we are done. Hence assume that $v_0 \notin \varepsilon_1 - cent_V(F)$, then $r(v_0, F) > rad_V(F) + \varepsilon_1$. Let $r_1 = rad_V(F) + \varepsilon_1$, $r_2 = r(v_0, F)$. Define $A_i = \{v \in V : r(v,F) = r_i\}$, $i = 1, 2$. We have $rad_V(F) \leq r_1 < r_2$ and $v_0 \in A_2$. Since $\varepsilon_1 - cent_V(F) \neq \emptyset$ we have $A_1 \neq \emptyset$. By $A_1 \subseteq \varepsilon_1 - cent_V(F)$, the hypothesis in (e) together with $v_0 \in \varepsilon_1 - cent_V(F)$ we have that,

$$\begin{aligned} d(v_0, \varepsilon_1 - cent_V(F)) \leq d(v_0, A_1) &= r_2 - r_1 \\ &= r(v_0, F) - rad_V(F) - \varepsilon_1 \\ &\leq rad_V(F) + \varepsilon_2 - rad_V(F) - \varepsilon_1 \\ &= \varepsilon_2 - \varepsilon_1 \end{aligned}$$

$(d) \Longrightarrow (a)$: Let $r_1, r_2 \geq 0$ and $x \in V$ satisfying $B(x, r_2) \cap S_{r_1}(F) \neq \emptyset$ and also $V \cap S_{r_1}(F) \neq \emptyset$. First, it is clear that $rad_V(F) \leq r_1$.

If $r(x,F) \leq r_1$ then $x \in V \cap B[x, r_2] \cap S_{r_1}(F)$. Let $r_1 < r(x,F)$ and $\varepsilon_1 = r_1 - rad_V(F)$ and $\varepsilon_2 = r(x,F) - rad_V(F)$. Then $0 \leq \varepsilon_1 < \varepsilon_2$ and $x \in \varepsilon_2 - cent_V(F)$. Since $\varepsilon_1 - cent_V(F) = V \cap (\bigcap_{z \in F} B[z, r_1]) \neq \emptyset$, hence by (d) and $r(x,F) < r_1 + r_2$ we have $d(x, \varepsilon_1 - cent_V(F)) \leq \varepsilon_2 - \varepsilon_1 = r(x,F) - r_1 < r_2$.

Let $y_1 \in \varepsilon_1 - cent_V(F)$ such that $\|x - y_1\| < r_2$, since $y_1 \in Y \cap (\cap_{z \in F} B[z, r_1])$ we have $r(y,F) \leq r_1$. And finally we have $y_1 \in B[x, r_2] \cap S_{r_1}(F) \cap V$, which ensures the non emptyness of the last set. □

As an easy consequence of Theorem 2.4 we get the following.

THEOREM 2.5. *Let* $(X, V, \mathcal{F})$ *has Property-*(R_1) *then the map* $\delta - cent_V(.) : (\mathcal{F}, \tau_H) \to (CB(X), \tau_H)$ *is Lipschitz continuous, for all* $\delta \geq 0$.

PROOF. We show that for $G, H \in \mathcal{F}, d_H(\delta - cent_V(G), \delta - cent_V(H)) \leq 2d_H(G,H)$, for $\delta \geq 0$.

CASE 1: When $\delta = 0$.

It is clear that $|rad_V(G) - rad_V(H)| \leq d_H(G,H)$ and $|r(v,G) - r(v,H)| \leq d_H(G,H)$, for all $v \in V$. In fact for any $g \in G$ and $\varepsilon > 0$ get a $h \in H$ such that $\|g - h\| < d_H(G,H) + \varepsilon$. Now $\|v - g\| \leq \|v - h\| + \|g - h\| < r(v,H) + d_H(G,H) + \varepsilon$. True for all $g \in G$ which leads to that $r(v,G) \leq r(v,H) + d_H(G,H) + \varepsilon$. Replacing G by H and varying $\varepsilon > 0$ we get $|r(v,F) - r(v,G)| \leq d_H(G,H)$.

The above inequality also gives, $rad_V(H) \leq r(v,H) \leq r(v,G) + d_H(H,G)$. And hence $|rad_V(H) - rad_V(G)| \leq d_H(H,G)$.

CASE 2: When $\delta > 0$.

Let $v \in \delta - cent_V(G)$ then

$$\begin{aligned} r(v,G) &\leq rad_V(G) + \delta \\ \text{Hence } r(v,H) &\leq r(v,G) + d_H(G,H) \\ &\leq rad_V(G) + \delta + d_H(G,H) \\ &\leq rad_V(H) + \delta + 2d_H(G,H) \end{aligned}$$

That is $v \in (2d_H(G,H) + \delta) - cent_V(H)$.

CLAIM: For $r_1, r_2 > 0$, $(r_1 + r_2) - cent_V(F) \subseteq (r_1 - cent_V(F))_{r_2}$.

Suppose for $v \in V, r(v, F) \leq rad_V(F) + r_1 + r_2$ and also we have $r_1 - cent_V(F) \neq \emptyset$ hence by Property $(R_1), V \cap B[v, r_2] \cap (r_1 - cent_V(F)) \neq \emptyset$. Which implies $d(v, r_1 - cent_V(F)) \leq r_2$.

Hence we get $d(v, \delta - cent_V(H)) \leq 2d_H(G, H)$. Interchanging G and H we get the result. □

Let us recall the Definition 1.4 of Property-(P_1). Here note that Property (P_1) is weaker than Property (R_1); Theorem 2.4 (d) says that in Property (P_1) if corresponding $\delta(\varepsilon, F) = \varepsilon$ always, then it is Property (R_1).

3. Modulus of restricted chebyshev centres

DEFINITION 3.1. Let $V \in CL(X)$ and $\mathcal{F} \subseteq CB(X)$ be such that $(V, \mathcal{F})$ has r.c.p., the modulus of restricted chebyshev centres $\varepsilon : \mathcal{F} \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\varepsilon_V(F, t) = \inf\{r > 0 : t - cent_V(F) \subseteq cent_V(F) + rB_X\}$$

It is clear from the definition that for $t_1 \geq t_2 > 0, \varepsilon(F, t_1) \geq \varepsilon(F, t_2)$ for all $F \in \mathcal{F}$.

LEMMA 3.2. *Let $V \in CL(X)$ and $\mathcal{F} \subseteq CB(X)$ be such that $(V, \mathcal{F})$ has r.c.p.. Then for all $F \in \mathcal{F}, \varepsilon_V(F, .)$ is continuous and increasing function of t whenever $t > 0$. Moreover if $rad_V(F) = d$ and $t > s$ then $d_H(t - cent_V(F), s - cent_V(F)) \leq (t - s)\frac{2d+t}{t}$.*

PROOF. Suppose $rad_V(F)$ and $t > s$, it is enough to prove that

$$t - cent_V(F) \subseteq s - cent_V(F) + (t - s)\frac{2d + t}{t}B_X.$$

Let $\eta = t - s$ and $y \in t - cent_V(F)$ then $r(y, F) \leq rad_V(F) + t = d + s + \eta$.

Let $y_0 \in cent_V(F)$ and $\overline{y} = (1 - \lambda)y + \lambda y_0$, where $\lambda = \frac{\eta}{s+\eta}$.

Now

$$\begin{aligned} r(\overline{y}, F) &= \sup_{z \in F} \|\overline{y} - z\| \\ &\leq \sup_{z \in F} \|(1 - \lambda)y + \lambda y_0 - z\| \\ &\leq \sup_{z \in F}(1 - \lambda)\|y - z\| + \sup_{z \in F} \|y_0 - z\| \\ &\leq (1 - \lambda)(d + s + \eta) + \lambda r(y_0, F) \\ &= (1 - \lambda)(d + s + \eta) + \lambda d = d + s. \end{aligned}$$

That is $\overline{y} \in s - cent_V(F)$.

Also $\|y - \overline{y}\| = \lambda\|y - y_0\| \leq \frac{\eta}{s+\eta}(\|y - x\| + \|x - y_0\|) \leq \eta\frac{2d+t}{t}$.

This completes the proof. □

Lemma 3.2 now enable us to characterize the geometric aspects defined in Section 1 viz. Property (P_1), Property (R_1).

THEOREM 3.3. *Let $V \in CL(X)$ and $\mathcal{F} \subseteq CB(X)$ be such that $(V, \mathcal{F})$ has r.c.p.. Then*

(*a*) *$(V, \mathcal{F})$ has Property-(P_1) $\iff$ $\varepsilon(F, .)$ continuous at 0 for all $F \in \mathcal{F}$.*
(*b*) *$(V, \mathcal{F})$ has Property-(R_1) $\iff$ $\varepsilon(F, t) \leq t$ for all $F \in \mathcal{F}$.*

PROOF. (b). Apply Theorem 2.4(d). □

We now derive few results related to the Chebyshev radius of a finite subset of X and X^{**} relative to a subspace Y of X and also relative to $Y^{\perp\perp}$ in X^{**} respectively. These are in fact some applications of Principle of Local Reflexivity. These results along with Theorem 3.3 will enable us to derive some geometric results of a subspace Y having $1\frac{1}{2}$-ball property. Let us recall the following result viz. *Principle of local reflexivity* (PLR in short) which is relevant in the proof of Proposition 3.5.

THEOREM 3.4. *Let X be a Banach space. For every finite dimensional subspace E of X^{**} and finite dimensional subspace F of X^* and $\varepsilon > 0$ there exists an isomorphism $T : E \to X$ such that $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and for all $x^* \in F$. $T|_{E\cap X} = I$ and $\|T\|\|T^{-1}\| \leq 1+\varepsilon$.*

For a bounded subset F of X and $t > 0$, $\varepsilon_X(F,t), \varepsilon_{X^{**}}(F,t)$ represent the modulus of restricted chebyshev centre of F in the corresponding space.

PROPOSITION 3.5. *Let Y be a finite codimensional subspace of a Banach space X and $\mathcal{F}$ represents the set of all finite subsets of X. Then,*

(*a*) *$d(cent_X(F), Y) = d(cent_{X^{**}}(F), Y^{\perp\perp})$, for $F \in \mathcal{F}$, if $(X, Y, \mathcal{F})$ has Property-(R_1).*

(*b*) *Let $(X, Y, \mathcal{F})$ has Property-(P_1) then $d(y, cent_Y(F)) = d(y, cent_{Y^{\perp\perp}}(F))$, for $y \in Y$.*

(*c*) *Let $(X, Y, \mathcal{F})$ has Property-(P_1) then $\varepsilon_Y(F,t) \leq \varepsilon_{Y^{\perp\perp}}(F,t)$, for any $F \in \mathcal{F}$.*

PROOF. (*a*) We have $d(cent_X(F), Y) \geq d(cent_{X^{**}}(F), Y^{\perp\perp})$. Let us assume $d(cent_X(F), Y) > d(cent_{X^{**}}(F), Y^{\perp\perp})$ and hence there exist $\varepsilon, \delta > 0$ such that $d((cent_X(F))_\delta, Y) > d(cent_{X^{**}}(F), Y^{\perp\perp}) + \varepsilon$. Get $\Phi \in cent_{X^{**}}(F)$ and $y^{**} \in Y^{\perp\perp}$ such that $\|\Phi - y^{**}\| = d(cent_{X^{**}}(F), Y^{\perp\perp})$. Let $Z = span\{\{\Phi, y^{**}\} \cup F\}$ and $W = span\{y_i^* : 1 \leq i \leq n\}$. Then Z, W are finite dimensional subspace of X^{**} and X^* respectively. Choose $\eta > 0$ such that $\eta < \frac{\min\{\varepsilon, \delta/2\}}{\max\{\|\Phi - y^{**}\|, r(\Phi, F)\}}$ and by PLR get a $T : Z \to X$ an isomorphism into its range such that $\|T\|\|T^{-1}\| \leq 1+\eta$ and also satisfying $T(z) = z$, $z \in F$ and also $f(T(x^{**})) = x^{**}(f)$ whenever $x^{**} \in Z$, $f \in W$. It is clear that $Ty^{**} \in Y$.

CLAIM: $T\Phi \in (cent_X(F))_\delta$.

$$\begin{aligned}
\text{For } z \in F, \|T\Phi - z\| &= \|T(\Phi - z)\| \\
&\leq (1+\eta)\|\Phi - z\| \\
&\leq r(\Phi, F) + \delta/2 \\
&= rad_{X^{**}}(F) + \delta/2 = rad_X(F) + \delta/2.
\end{aligned}$$

Hence $r(T\Phi, F) < rad_X(F) + \delta/2$ and by Theorem 2.5 we have that $T\Phi \in cent_X(F) + \delta B_X = (cent_X(F))_\delta$.

Now $\|T\Phi - Ty^{**}\| \leq \|\Phi - y^{**}\|(1+\eta) < d(cent_{X^{**}}(F), Y^{\perp\perp}) + \varepsilon$, this contradict the fact that $d(cent_{X^{**}}(F), Y^{\perp\perp}) + \varepsilon < d((cent_X(F))_\delta, Y^{\perp\perp})$. Hence the result follows.

(*b*) For a given $\varepsilon > 0$ and $F \in \mathcal{F}$ get $\delta > 0$ from the Property-(P_1) of $(X, Y, \mathcal{F})$.

Choose $\eta > 0$ such that $\eta < \min\left\{\frac{(d(y, cent_Y(F)) - \varepsilon) - \|y - \Phi\|}{\|y - \Phi\|}, \frac{\delta}{rad_{Y^{\perp\perp}}(F)}\right\}$.

(c) If possible let for some $F \in \mathcal{F}$ and $t > 0$, $\varepsilon_{Y^{\perp\perp}}(F,t) < \varepsilon_Y(F,t)$. Get a $\varepsilon > 0$ and $y \in Y$ with $y \in t - cent_Y(F)$ such that $\varepsilon_{Y^{\perp\perp}}(F,t) < \varepsilon_Y(F,t) - \varepsilon$ and

$$d(y, cent_Y(F)) > \varepsilon_Y(F,t) - \varepsilon. \tag{1}$$

It is clear that $y \in t - cent_{Y^{\perp\perp}}(F)$ and

$$d(y, cent_{Y^{\perp\perp}}(F)) < \varepsilon_Y(F,t) - \varepsilon. \tag{2}$$

The last inequality follows from the fact that $\varepsilon_Y(F,t) - \varepsilon > \varepsilon_{Y^{\perp\perp}}(F,t)$, the modulus. The inequalities in equations 1, 2 contradicting each other. Hence the result follows. □

Corollary 3.6. *Let Y be a finite co-dimensional subspace of a L_1 predual space X which is strongly proximinal and F be a finite subset of X. Then $rad_Y(F) = rad_X(F) + d(cent_X(F), Y)$.*

Proof. From Proposition 3.5(a) it follows that $d(cent_X(F), Y) = d(cent_{X^{**}}(F), Y^{\perp\perp})$. Since X^{**} is a space of type $C(\Omega)$ for some compact Hausdorff space Ω and $Y^{\perp\perp}$ is strongly proximinal we have from [**5**, Theorem 2.1] the linear functionals which determine the subspace $Y^{\perp\perp}$ are finitely supported. Hence from [**11**, Theorem 2.2, Proposition 3.3] we have $rad_{Y^{\perp\perp}}(F) = rad_{X^{**}}(F) + d(cent_{X^{**}}(F), Y^{\perp\perp})$. Since $rad_{Y^{\perp\perp}}(F) = rad_Y(F)$ and $rad_{X^{**}}(F) = rad_X(F)$ we have the result. □

Theorem 3.7. *Let $V \in CL(X)$ and $\mathcal{F} \subseteq CB(X)$ be such that $(V, \mathcal{F})$ has r.c.p.. Then,*

- (a) *If $cent_V(.)$ is continuous at F then $\varepsilon_V(.,t)$ continuous at F for all $t > 0$.*
- (b) *If $(V, \mathcal{F})$ has Property-(P_1) and $\varepsilon_V(.,t)$ are continuous at F for all $t > 0$ then $cent_V(.)$ is continuous at F.*

Proof. Case 1: When $cent_V(F)$ is continuous.

Case 1.1: $cent_V(.)$ is lHsc at F implies $\varepsilon_V(.,t)$ usc at F for all $t > 0$.

Fix $t > 0$ and let $F_n \xrightarrow{H} F$. We show that $\limsup_n \varepsilon_V(F_n, t) \leq \varepsilon(F,t)$.

Let $d_n = rad_V(F_n)$. Let $\alpha > \varepsilon_V(F,t)$, choose $r > 0$ such that $\alpha > \alpha - r > \varepsilon_V(F,t)$. Hence $t - cent_V(F) \subseteq cent_V(F) + (\alpha - r)B_X$.

For any β with $t > \beta > 0$ and n large we have

$$(t - \beta) - cent_V(F_n) \subseteq t - cent_V(F) \subseteq cent_V(F) + (\alpha - r)B_X.$$

Now for large $n, cent_V(F) \subseteq cent_V(F_n) + \frac{r}{2}B_X$ and hence from above $(t - \beta) - cent_V(F_n) \subseteq cent_V(F_n) + (\alpha - \frac{r}{2})B_X$.

Hence $\alpha - \frac{r}{2} \geq \varepsilon_V(F_n, t - \beta) \geq \varepsilon_V(F_n, t) - \beta\frac{2d_n + t}{t}$.

Taking $n \to \infty$ as $d_n \to d = rad_V(F)$ we have $\limsup_n \varepsilon_V(F_n, t) \leq \alpha$.

This establishes Case 1.1.

Case 1.2: $cent_V(.)$ is uHsc at F implies $\varepsilon_V(.,t)$ is lsc at F for all $t > 0$.

Fix $t > 0$ and let $F_n \xrightarrow{H} F$. We show that $\liminf_n \varepsilon_V(F_n, t) \geq \varepsilon_V(F,t)$.

Let $\alpha < \varepsilon_V(F,t)$, choose $r > 0$ such that $\alpha < \alpha + r < \varepsilon_V(F,t)$.

If along some subsequence $\varepsilon_V(F_n, t) \leq \alpha$ then

$$t - cent_V(F_n) \subseteq cent_V(F_n) + (\alpha + \frac{r}{4})B_X.$$

Since $cent_V(.)$ is uHsc at F, for large n

$$cent_V(F_n) \subseteq cent_V(F) + \frac{r}{4}B_X.$$

Hence $t - cent_V(F_n) \subseteq cent_V(F) + (\alpha + \frac{r}{2})B_X$.

Now for any β with $t > \beta > 0$ and large n,

$$(t-\beta) - cent_V(F) \subseteq t - cent_V(F_n)$$

To prove the last statement let $p \notin t - cent_V(F_n)$ ie. $r(p, F_n) > rad_V(F_n) + t$ ie. $r(p, F) > r(p, F_n) - d_H(F_n, F) > rad_V(F_n) + t - d_H(F_n, F)$.

Since $rad_V(F_n) \to rad_V(F)$ we have for sufficiently large n

$$r(p, F) \geq rad_V(F) + (t - \beta) \text{ ie. } p \notin (t-\beta) - cent_V(F_n).$$

Hence $(t - \beta) - cent_V(F) \subseteq cent_V(F) + (\alpha + \frac{r}{2})B_X$. That is

$$(\alpha + \frac{r}{2}) \geq \varepsilon_V(F, t-\beta) \geq \varepsilon_V(F, t) - \beta\frac{2d+t}{t}$$

Since β is arbitrary we have $\varepsilon_V(F, t) \leq \alpha$, a contradiction. This establishes Claim 2.

Case 2: When $\varepsilon_V(., t)$ are all continuous at F for all $t > 0$.

By continuity of $\varepsilon_V(F, .)$ at 0 there exists $t_0 > 0$ such that $\varepsilon(F, t_0) < \frac{\varepsilon}{4}$. Now let $F_n \xrightarrow{H} F$, since $\varepsilon_V(., t_0)$ is continuous at F we have $\lim_n \varepsilon_V(F_n, t_0) = \varepsilon_V(F, t_0)$, hence for large n $\varepsilon_V(F_n, t_0) \leq \varepsilon_V(F, t_0) + \frac{\varepsilon}{4}$.

For large n,

$$cent_V(F_n) \subseteq t_0 - cent_V(F) \subseteq cent_V(F) + (\varepsilon_V(F, t_0) + \frac{\varepsilon}{4})B_X$$

and also

$$cent_V(F) \subseteq t_0 - cent_V(F_n) \subseteq cent_V(F_n) + \varepsilon_V(F_n, t_0)(1 + \frac{1}{n})B_X$$

Thus

$$d_H\left(cent_V(F), cent_V(F_n)\right) \leq \max\{\varepsilon_V(F, t_0) + \frac{\varepsilon}{4}, \varepsilon_V(F_n, t_0)(1+\frac{1}{n})\} \leq \varepsilon.$$

Hence the result follows. □

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