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2-local isometries on $C^{(n)}([0,1])$

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Abstract. Let $C^{(n)}([0,1])$ be the Banach space of all n-times continuously differentiable functions on $[0,1]$ with the norm

$$\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} |f^{(k)}(t)|/k!.$$

We prove that every 2-local isometry on $(C^{(n)}([0,1]), \|\cdot\|_C)$ is a surjective complex linear isometry. Two proofs, hopefully illuminating different aspects of the operator on the space, are presented.

1. Introduction

A mapping $T\colon N \to N$ on a normed linear space $(N, \|\cdot\|_N)$ over the complex number field $\mathbb{C}$ is called an *isometry* if $\|T(f) - T(g)\|_N = \|f - g\|_N$ for all $f, g \in N$ (neither the linearity nor the surjectivity of the mapping is assumed). Motivated by the notion of 2-local automorphisms and derivations due to Šemrl [**9**], Molnár introduced the notion of 2-local isometry in [**6**]. A mapping $S\colon N \to N$ is called a *2-local isometry* if for each $f, g \in N$ there exists a surjective complex linear isometry $T_{f,g}\colon N \to N$, depending on f and g, such that $S(f) = T_{f,g}(f)$ and $S(g) = T_{f,g}(g)$. Again neither the surjectivity nor the linearity of the mapping S is assumed. Characterizing 2-local isometries on various function spaces has been studied by several authors. For example, Győry [**1**] gave the characterization of 2-local isometries on $C_0(X)$, the Banach space of all continuous complex-valued functions vanishing at infinity defined on a first countable σ-compact locally compact Hausdorff space X.

This paper deals with the space $C^{(n)}([0,1])$ $(n \geq 1)$, the space of all n-times continuously differentiable functions on $[0,1]$. Several norms are known which makes the space a Banach space. Hosseini [**2**] investigated 2-local isometries on $C^{(n)}([0,1])$, assuming the surjectivity and the real-linearity of such mappings, with respect to the norm $\|f\|_n$ defined by $\|f\|_n = \max\{|f(0)|, |f'(0)|, \cdots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\}$, where $\|\cdot\|_\infty$ is the supremum norm on $[0,1]$. We study 2-local isometries, without the surjectivity/linearity assumption, on the space $C^{(n)}([0,1])$ with respect to the

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norms $\|\cdot\|_C$ and $\|\cdot\|_\Sigma$ defined by:

$$\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!} \quad \text{and} \quad \|f\|_\Sigma = \sum_{k=0}^{n} \|f^{(k)}\|_\infty$$

where $f^{(0)}(t) = f(t)$. We prove that every 2-local isometry on the space $(C^{(n)}([0,1]), \|\cdot\|_C)$ is a surjective complex linear isometry on the space. The same holds for 2-local isometries on $(C^{(n)}([0,1]), \|\cdot\|_\Sigma)$.

Two proofs of our theorem are presented: one is based on the idea due to Jiménez-Vargas and Villegas-Vallecillos in [**4**, Proof of Theorem 2.1] and the other is based on the Bartle-Graves theorem. We hope that these proofs with different viewpoints would give us a new perspective to the 2-local-reflexivity problem. It is most likely that the present method could be applied to study 2-local isometries on some other function spaces. Its full development is a subject of future study. It should be mentioned that the 2-local reflexivity of various function spaces including $C^{(1)}([0,1])$ has been studied in a general framework by Hatori and Oi in [**3**]. Studying the connection of our method with theirs is also a subject of further study.

Before proceeding, let us observe that every 2-local isometry $S : N \to N$ is an isometry; for each $f, g \in N$ there exists an isometry $T : N \to N$ such that $S(f) = T(f), S(g) = T(g)$ and we have

$$\|S(f) - S(g)\|_N = \|T(f) - T(g)\|_N = \|f - g\|_N.$$

In particular every 2-local isometry is a continuous injection.

2. Main result and its First Proof

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The following is our main theorem.

THEOREM 2.1. *Let $n \geq 1$ be an integer and let S be a 2-local isometry of $C^{(n)}([0,1])$ with respect to the norm $\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} |f^{(k)}(t)|/k!$. There exists a constant $c \in \mathbb{T}$ such that $S(f)(t) = cf(t)$ for all $f \in C^{(n)}([0,1])$ and $t \in [0,1]$, or $S(f)(t) = cf(1-t)$ for all $f \in C^{(n)}([0,1])$ and $t \in [0,1]$.*

Both of our proofs depend on the characterization of complex linear $\|\cdot\|_C$-isometries on $C^{(n)}([0,1])$ due to Pathak:

THEOREM 2.2. [**7**, Theorem 2.5] *Let $n \geq 1$ be an integer. For each surjective complex linear isometry T on $(C^{(n)}([0,1]), \|\cdot\|_C)$, there exists $c \in \mathbb{T}$ such that $T(f)(t) = cf(t)$ for all $f \in C^{(n)}([0,1])$ and $t \in [0,1]$, or $T(f)(t) = cf(1-t)$ for all $f \in C^{(n)}([0,1])$ and $t \in [0,1]$.*

This section gives the first proof of Theorem 2.1 on the basis of the idea due to Jiménez-Vargas and Villegas-Vallecillos in [**4**, Proof of Theorem 2.1]. For later use, let id be the identity function on $[0,1]$, $r : [0,1] \to [0,1]$ be the map given by $r(t) = 1 - t,\ t \in [0,1]$, and let $\mathcal{H} = \{\mathrm{id}, r\}$, the only isometries on the unit interval $[0,1]$. Note that

$$\varphi \circ \varphi = \mathrm{id}, \quad \varphi \in \mathcal{H}. \tag{2.1}$$

PROOF OF THEOREM 2.1. Let S be a 2-local isometry on $C^{(n)}([0,1]), \|\cdot\|_C$. We denote by $\mathbf{1}$ the constant function which takes the value 1. For each $g \in C^{(n)}([0,1])$ there exists a surjective complex linear isometry T_g on $C^{(n)}([0,1])$ such that $S(\mathbf{1}) = T_g(\mathbf{1})$ and $S(g) = T_g(g)$. By Theorem 2.2, there exist $c_g \in \mathbb{T}$ and $\phi_g \in \mathcal{H}$ such that $T_g(f)(t) = c_g f(\phi_g(t))$ for all $f \in C^{(n)}([0,1])$ and $t \in [0,1]$. Observe that $c_g = T_g(\mathbf{1}) = S(\mathbf{1})$ and in particular c_g does not depend on g. Define $S_0 \colon C^{(n)}([0,1]) \to C^{(n)}([0,1])$ by $S_0 = \overline{S(\mathbf{1})}S$. For each $g \in C^{(n)}([0,1])$, we have

$$S_0(g) = \overline{S(\mathbf{1})}S(g) = \overline{S(\mathbf{1})}T_g(g) = \overline{S(\mathbf{1})}c_g(g \circ \phi_g) = g \circ \phi_g.$$

Hence, we get

$$S_0(g) = g \circ \phi_g, \quad g \in C^{(n)}([0,1]). \tag{2.2}$$

For an arbitrary $t_0 \in [0,1]$, choose a function $h_0 \in C^{(n)}([0,1])$ such that

$$h_0(t_0) = 1 \qquad \text{and} \qquad 0 \le h_0(t) < 1 \quad \text{for all } t \in [0,1] \setminus \{t_0\}. \tag{2.3}$$

For a function $f \in C^{(n)}([0,1])$, we define the set $E_{t_0,f}$ by

$$E_{t_0,f} = \{t \in [0,1] : S_0(f)(t) = f(t_0)\} = \{t \in [0,1] : f(\phi_f(t)) = f(t_0)\}.$$

In what follows we prove that the set $\cap_{f \in C^{(n)}([0,1])} E_{t_0,f}$ is a singleton. More strongly we show the following equality for an arbitrary function h_0 satisfying (2.3):

$$\cap_{f \in C^{(n)}([0,1])} E_{t_0,f} = \{\phi_{h_0}(t_0)\}. \tag{2.4}$$

First observe from (2.3) that

$$\begin{aligned} E_{t_0,h_0} &= \{t \in [0,1] : h_0(\phi_{h_0}(t)) = h_0(t_0) = 1\} \\ &= \{t \in [0,1] : \phi_{h_0}(t) = t_0\} = \{\phi_{h_0}(t_0)\}. \end{aligned}$$

Thus for the proof of (2.4) it suffices to show that $\phi_{h_0}(t_0) \in E_{t_0,f}$ for each $f \in C^{(n)}([0,1])$. Take an arbitrary function $f \in C^{(n)}([0,1])$ and take a surjective complex linear isometry T_{f,h_0} on $C^{(n)}([0,1])$ such that $S(f) = T_{f,h_0}(f)$ and $S(h_0) = T_{f,h_0}(h_0)$. Theorem 2.2 implies $S(f) = c_{f,h_0}(f \circ \varphi_{f,h_0})$ and $S(h_0) = c_{f,h_0}(h_0 \circ \varphi_{f,h_0})$ for some $c_{f,h_0} \in \mathbb{T}$ and $\varphi_{f,h_0} \in \mathcal{H}$. Using (2.2), we obtain

$$\begin{aligned} f \circ \phi_f &= S_0(f) = \overline{S(\mathbf{1})}S(f) = \overline{S(\mathbf{1})}\, c_{f,h_0}(f \circ \varphi_{f,h_0}), \\ h_0 \circ \phi_{h_0} &= S_0(h_0) = \overline{S(\mathbf{1})}\, c_{f,h_0}(h_0 \circ \varphi_{f,h_0}). \end{aligned}$$

Recalling (2.1) and using the second equality we obtain

$$1 = h_0(t_0) = \overline{S(\mathbf{1})}\, c_{f,h_0} h_0(\varphi_{f,h_0}(\phi_{h_0}(t_0))).$$

Hence $S(\mathbf{1})\overline{c_{f,h_0}} = h_0(\varphi_{f,h_0}(\phi_{h_0}(t_0))) \geq 0$. This and $S(\mathbf{1})\overline{c_{f,h_0}} \in \mathbb{T}$ imply $S(\mathbf{1})\overline{c_{f,h_0}} = 1$. Thus

$$f \circ \phi_f = f \circ \varphi_{f,h_0} \qquad \text{and} \qquad h_0 \circ \phi_{h_0} = h_0 \circ \varphi_{f,h_0}. \tag{2.5}$$

Then (2.1) and the second equality of (2.5) show:

$$1 = h_0(t_0) = h_0(\phi_{h_0}(\phi_{h_0}(t_0))) = h_0(\varphi_{f,h_0}(\phi_{h_0}(t_0))),$$

and, by (2.3), we have $\varphi_{f,h_0}(\phi_{h_0}(t_0)) = t_0$. Combining (2.2) and (2.5), we obtain

$$S_0(f)(\phi_{h_0}(t_0)) = f(\phi_f(\phi_{h_0}(t_0))) = f(\varphi_{f,h_0}(\phi_{h_0}(t_0))) = f(t_0).$$

Consequently, $\phi_{h_0}(t_0) \in E_{t_0,f}$ for each $f \in C^{(n)}([0,1])$ which proves the equality (2.4). Since $\phi_{h_0}(t_0) = t_0$ or $1 - t_0$, what we have actually shown is:

$$\cap_{f \in C^{(n)}([0,1])} E_{t_0,f} = \{t_0\} \text{ or } \{1 - t_0\} \text{ for each } t_0 \in [0,1]. \tag{2.6}$$

The condition (2.4) allows us to define a map $\psi\colon [0,1] \to [0,1]$ with the property that

$$\cap_{f \in C^{(n)}([0,1])} E_{t,f} = \{\psi(t)\}.$$

By the definition of $E_{t,f}$, we have

$$S_0(f)(\psi(t)) = f(t), \quad f \in C^{(n)}([0,1]), t \in [0,1] \tag{2.7}$$

and by (2.6), $\psi(t) \in \{t, 1-t\}, t \in [0,1]$.

To finish the proof we need to show that $\psi \in \mathcal{H}$. For the proof let $E_1 = \{t \in [0,1] : \psi(t) = t\}$ and $E_{-1} = \{t \in [0,1] : \psi(t) = 1-t\}$. We see $[0,1] = E_1 \cup E_{-1}$ and $E_1 \cap E_{-1} = \{1/2\}$. We prove that

($*$) $E_1 \cap [0,1/2)$ is closed in $[0,1/2)$.

For the proof, let (t_n) be a sequence in $E_1 \cap [0,1/2)$ such that $t_n \to t_0 \in [0,1/2)$ and suppose that $t_0 \in E_{-1}$. Then

$$S_0(f)(t_0) = \lim_n S_0(f)(t_n) = \lim_n S_0(f)(\psi(t_n)) = \lim_n f(t_n) = f(t_0)$$

and also

$$S_0(f)(1-t_0) = S_0(f)(\psi(t_0)) = f(t_0),$$

hence $S_0(f)(t_0) = S_0(f)(1-t_0)$ for all $f \in C^{(n)}([0,1])$. Recalling (2.2) it follows that $f(\phi_f(t_0)) = f(\phi_f(1-t_0))$ with $\phi_f \in \mathcal{H}$. This implies $f(t_0) = f(1-t_0)$ for all $f \in C^{(n)}([0,1])$, a contradiction because $t_0 \neq 1-t_0$. Thus we have $t_0 \in E_1$ and ($*$) is proved.

Similarly $E_{-1} \cap [0,1/2)$ is closed in $[0,1/2)$ and thus

$$E_1 \supset [0,1/2] \quad \text{or} \quad E_{-1} \supset [0,1/2].$$

Likewise we have $E_1 \supset [1/2,1]$ or $E_{-1} \supset [1/2,1]$. Suppose that both inclusions $E_1 \supset [0,1/2]$ and $E_{-1} \supset [1/2,1]$ hold. Then we see $\psi(t) = t$ for $t \in [0,1/2]$ and $\psi(t) = 1-t$ for $t \in [1/2,1]$. Hence $\psi([0,1]) = [0,1/2]$. However by (2.2) and (2.7), we have

$$f(\phi_f(\psi(t))) = S_0(f)(\psi(t)) = f(t), \quad f \in C^{(n)}([0,1]), t \in [0,1],$$

hence we obtain $f(\phi_f([0,1/2])) = f([0,1])$ for each $f \in C^{(n)}([0,1])$, which is impossible. Similarly both of the inclusions "$E_{-1} \supset [0,1/2], E_1 \supset [1/2,1]$" do not hold and what remains is either of the equalities:

$$E_1 = [0,1] \quad \text{or} \quad E_{-1} = [0,1]$$

which means that $\psi \in \mathcal{H}$. Using (2.1) and (2.7) we have $S_0(f)(t) = S_0(f)(\psi(\psi(t))) = f(\psi(t))$ for $f \in C^{(n)}([0,1]), t \in [0,1]$. This completes the proof. □

By examining the above proof, we see readily that the role of the norm $\|\cdot\|_C$ in the theorem is just to ensure the characterization theorem, Theorem 2.2. Such a theorem has been known also for the norm $\|\cdot\|_\Sigma$ on the space $C^{(1)}([0,1])$ [**8**, Theorem 4.1]. Therefore we have

Corollary 2.3. *Every 2-local isometry on $C^{(1)}([0,1])$ with the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^{(1)}([0,1])$ is a surjective complex linear isometry on $(C^{(1)}([0,1]), \|\cdot\|_\Sigma)$, where $\|g\|_\infty = \sup_{t \in [0,1]} |g(t)|$.*

3. The second proof

This section gives an alternative proof of Theorem 2.1 applying the following form of the Bartle-Graves theorem [**5**].

THEOREM 3.1. [**5**, Example 1.3, Example 1.3*, Corollary on p.364, and Theorem 3.2”] *Let $u : E \to B$ be a bounded linear surjection between Banach spaces E and B. Then*

(1) *there exists a continuous map $s : B \to E$ such that $u \circ s = \mathrm{id}_B$ and $s(0) = 0$.*

(2) *The map $\Phi : \mathrm{Ker}\, u \times B \to E$ defined by*

$$\Phi(e, b) = e + s(b), \quad (e, b) \in \mathrm{Ker}\, u \times B$$

is a homeomorphism.

In particular the restrictions $\Phi|_{\mathrm{Ker}\, u\times\{0\}} : \mathrm{Ker}\, u\times\{0\} \to \mathrm{Ker}\, u$ and $\Phi|_{\mathrm{Ker}\, u\times(B\setminus\{0\})} : \mathrm{Ker}\, u \times (B \setminus \{0\}) \to E \setminus \mathrm{Ker}\, u$ are homeomorphisms. The inverse homeomorphism $\Phi^{-1} : E \to \mathrm{Ker}\, u \times B$ is given by

$$\Phi^{-1}(e) = (e - (s \circ u)(e), u(e)), \quad e \in E.$$

THE SECOND PROOF OF THEOREM 2.1. Let $S : C^{(n)}([0,1]) \to C^{(n)}([0,1])$ be a 2-local isometry. As is noticed in Section 1, S is a continuous operator. For each $g \in C^{(n)}([0,1])$, choose a complex linear $\|\cdot\|_C$-isometry $T_g : C^{(n)}([0,1]) \to C^{(n)}([0,1])$ such that $S(g) = T_g(g), S(\mathbf{1}) = T_g(\mathbf{1})$. Again by Theorem 2.2, there exist $c_g \in \mathbb{C}$ and $\phi_g : [0,1] \to [0,1]$ such that $T_g f(t) = c_g f(\phi_g(t)),\ f \in C^{(n)}([0,1])$, where $c_g \in \mathbb{T}$ and $\phi_g \in \mathcal{H}$. As in the first proof, we have $c_g = S(\mathbf{1})$ and thus $S(g)(t) = (S(\mathbf{1}))g(\phi_g(t))$ for $g \in C^{(n)}([0,1])$ and $t \in [0,1]$. Recall $\mathcal{H} = \{\mathrm{id}, r\}$ where $r(t) = 1 - t, t \in [0,1]$. As before, let $S_0(f) = \overline{S(\mathbf{1})}S(f) = f \circ \phi_f,\ f \in C^{(n)}([0,1])$ (see (2.2)). It is a continuous mapping.

For $g \in C^{(n)}([0,1])$ with $g(t_0) \neq g(1 - t_0)$ for some $t_0 \in [0,1]$, the map ϕ_g is uniquely determined, because $(S(\mathbf{1}))g(t) = (S(\mathbf{1}))g(1 - t)$ for each $t \in [0,1]$ forces $g(t) = g(1 - t)$ for each $t \in [0,1]$. Let

$$Z = \{g \in C^{(n)}([-1,1]) : g \circ r \neq g\}.$$

By the above remark, we have a well-defined map $Q\colon Z \to \{\pm 1\}$ given by

$$Q(g) = \begin{cases} \ \ 1 & \text{if} \quad \phi_g(t) = t, \\ -1 & \text{if} \quad \phi_g(t) = 1 - t \end{cases}.$$

We show that Q is continuous. It suffices to prove that $Q^{-1}(1)$ and $Q^{-1}(-1)$ are closed in Z. Suppose that a sequence (g_n) in $Q^{-1}(1)$ converges to $g \in Z$. By the continuity of S_0, we have $S_0(g_n) \to S_0(g)$ and thus $g_n \circ \phi_{g_n} \to g \circ \phi_g$. Since $Q(g_n) = 1$ for each n, we have $\phi_{g_n} = \mathrm{id}$ and thus $g = g \circ \phi_g$. Since $g \in Z$, we have $\phi_g = \mathrm{id}$ and $g \in Q^{-1}(1)$. The same argument shows that $Q^{-1}(-1)$ is closed in Z. Next we prove

(∗∗) Z is connected.

For the proof, let $R : C^{(n)}([0,1]) \to C^{(n)}([0,1])$ be the linear operator defined by

$$Rf = f \circ r - f, \quad f \in C^{(n)}([0,1]).$$

Then $Z = C^{(n)}([0,1]) \setminus \mathrm{Ker}\, R$. We have $\mathrm{Im}\, R = A := \{f \mid f \circ r = -f\}$: in fact, the inclusion $\mathrm{Im}\, R \subset A$ is readily verified. If $h \circ r = -h$, then $R(-\frac{1}{2}h) = \frac{1}{2}(-h \circ r + h) =$

h and $h \in \operatorname{Im} R$. Thus we see the above equality. The subspace A is a Banach space as a closed subspace of $C^{(n)}([0,1])$. Applying Theorem 3.1 and the remark after the theorem, we see

$$Z = C^{(n)}([0,1]) \setminus \operatorname{Ker} R \text{ is homeomorphic to } \operatorname{Ker} R \times (A \setminus \{0\}).$$

It is easy to see that $A \setminus \{0\}$ is connected because A is an infinite dimensional Banach space, and so is the product $\operatorname{Ker} R \times (A \setminus \{0\})$ and hence Z is connected which proves $(**)$.

The continuity of the map Q and $(**)$ imply that Q is a constant map. Therefore, there exists $\phi \in \mathcal{H}$ such that ,

$$S(f)(t) = (S(\mathbf{1}))f(\phi(t)), \ f \in Z, \ t \in [0,1]. \tag{3.1}$$

Furthermore the space Z is dense in $C^{(n)}([0,1])$. In fact, it is easy to find a non-zero function h_0 with $\|h_0\|_C$ being arbitrarily small such that $h_0 \neq h_0 \circ r$. Then for each $g \in C^{(n)}([0,1])$ with $g = g \circ r$, the function $g + h_0$ is an approximation of g with $g + h_0 \in Z$. By the continuity of S, we see that (3.1) holds for each $f \in C^{(n)}([0,1])$. This proves the theorem. □

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