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Into isometries of Banach spaces

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Abstract. In this article we consider several geometric properties of a Banach space X which are preserved under every isometric embedding of X into spaces of functions on discrete sets.

1. Introduction

Let X be a real Banach space and let $Y \subset X$ be a closed subspace. An important question in the isometric theory of Banach spaces is to study relative properties of Y in X, that are preserved by the range of every into isometry (called an embedding) of the space Y into X . Such an investigation of the finite dimensional structure of the range space of into isometries was done in [**19**].

Here we consider the following three relative properties of Y in X.

(1) For any $y_1, y_2, ..., y_n \in Y$ and $x \in X$ there is a $y_0 \in Y$ such that $\|y_i - y_0\| \leq \|y_i - x\|$ for $1 \leq i \leq n$.
(2) For any $x \in X$ there is a $y_0 \in Y$ such that $\|y - y_0\| \leq \|y - x\|$ for all $y \in Y$.
(3) There is a surjective linear projection $P : X \to Y$ such that $\|P\| = 1$.

Subspaces satisfying property 1) were called cental subspaces in [**3**] and spaces satisfying property 2) were called almost constrained subspaces in [**5**] and existence sets in [**11**]. See also [**9**].

By taking $y_0 = P(x)$ it is easy to see that 3) $\Rightarrow$ 2). Clearly 2) $\Rightarrow$ 1). Lindenstrauss ([**16**]) gave an example to show that 2) need not imply 3) by constructing a Banach space X and a closed subspace $Y \subset X$ such that for any $x \in X$ there is a projection of norm one from $span\{x, Y\}$ onto Y and there exists $x_1, x_2 \in X$ for which there is no projection of norm one from $span\{x_1, x_2, Y\}$ onto Y.

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Let c_0 and ℓ^∞ denote spaces of sequences of real numbers converging to 0 and the space of bounded sequences respectively, with both the spaces equipped with the supremum norm. Let $n > 2$, $y_1, y_2, ..., y_n \in c_0$, $x \in \ell^\infty$ and let $\epsilon = min_{1 \leq 1 \leq n}\{\|y_i - x\|\}$. There exists a $N > 0$ such that $|y_i(k)| \leq \epsilon$ for all $k > N$ and $1 \leq i \leq n$. Since for $n > 2$, any n-pair-wise intersecting intervals have a point in common, choose real numbers α_j such that $|y_i(j) - \alpha_j| \leq |y_i(j) - x(j)|$ for $1 \leq i \leq n$ and $1 \leq j \leq N$. Let $y_0 \in c_0$, be such that $y_0(j) = 0$ for $j > N$ and $y_0(j) = \alpha_j$ for $1 \leq j \leq N$. Now $\|y_0 - y_i\| \leq \|y_i - x\|$ for $1 \leq i \leq n$. Thus $c_0 \subset \ell^\infty$ is a central subspace.

On the other hand for the constant sequence 1 in ℓ^∞, if there is a $y_0 \in c_0$ such that $sup_{i \geq 1}|y(i) - y_0(i)| \leq sup_{i \geq 1}|y(i) - 1|$ for all $y \in c_0$, by taking $y_n = 2e_n$, where e_n is the sequence with 1 in the nth place and 0's elsewhere, we have $|2 - y_0(i)| \leq 1$ for all i, which is a contradiction since $y_0 \in c_0$. So c_0 is not an almost constrained subspace of ℓ^∞. See also Proposition 8.

The third property, $Y \subset X$ being the range of a contractive projection has been well studied in the literature. In this case one also says Y is one-complemented in X. We refer to the survey article [**20**] for several interesting results in this direction. We have given only few references that have appeared after [**20**]. We note that if $P : X \to X$ is a contractive projection, then $P(X)^*$ is isometric to $P^*(X^*)$. We note that all the three properties are transitive and for $Z \subset Y \subset X$, if the stronger of the properties is assumed between Y and X, the weaker property is transitive from Z to X.

We investigate the question when do these properties gets preserved under every embedding of Y in X? Let $\Delta \subset [0, 1]$ denote the Cantor set. It is known that (see [**10**] Theorem 4 in section 22) for the space of continuous functions, $C(\Delta)$ has an isometric embedding in $C([0, 1])$ which is the range of a projection of norm one and another embedding in $C([0, 1])$ where it is not even a complemented subspace (see [**1**], page 91 and Proposition 4.4.6). I am grateful to Professor Gilles Godefroy for bringing the results from [**1**] to my attention. To quote from [**20**] 'it seems that the one-complementability of a subspace $Y \subset X$ in fact depends on the way that Y is embedded in X'. For a discrete set Γ, let $\ell^\infty(\Gamma)$ and $c_0(\Gamma)$ denote the space of bounded functions and functions vanishing at ∞ on Γ respectively, equipped with the supremum norm and let $\ell^1(\Gamma)$ denote the space of countably supported and absolutely summable functions on Γ equipped with the ℓ^1-norm.

We show that for any infinite discrete set Γ and for a closed subspace $Y \subset c_0(\Gamma)$ all the three properties are equivalent. Moreover if Y has any of the properties, then it has them under every embedding of Y in $c_0(\Gamma)$. This uses a result of Ando and Douglas ([**2**], [**7**]), that the range of a projection of norm one in a L^1-space is again a L^1-space. These results extend the results from [**5**] and [**11**] from countable to uncountable discrete sets under a weaker assumption, by completely different methods. We also show that any almost constrained subspace of $\ell^\infty(\Gamma)$ is the range of a projection of norm one, improving a result from [**11**]. As an application of these results we give a new proof of a result of Lima [**12**] that distinguishes spaces in which any collection of 3-pair-wise intersecting closed balls having non-empty intersection from spaces where this holds for 4 closed balls.

We recall that a Banach space X is said to be a L^1-predual space if X^* is isometric to $L^1(\mu)$ for some positive measure μ. For a compact set K, the space $C(K)$ and for a discrete set Γ, since $\ell^\infty(\Gamma)$ is isometric to the space $C(\beta(\Gamma))$, where $\beta(\Gamma)$ is the Stone-$\hat{C}$ech compactification of Γ, and the spaces $c_0(\Gamma)$, are all L^1-predual spaces. See [**15**] and Chapter 7 of [**10**] for properties of these spaces and more examples. If Y is a L^1-predual space such that whenever $Z \subset Y$ is a L^1-predual then it is an almost constrained subspace, we show that Y is isometric to $c_0(\Gamma)$ for a discrete set Γ.

Let K be a compact set and let $\sigma : K \to K$ be a homeomorphism such that $\sigma(\sigma(k)) = k$ for all $k \in K$. Let $Y = \{f \in C(K) : f \circ \sigma = -f\}$. It is easy to see that $P : C(K) \to Y$ defined by $P(f) = \frac{f - f\circ\sigma}{2}$ is a contractive projection. Since $C(K)^*$ is a L^1-space and P^* is a contractive projection onto Y^*, by the Ando-Douglas' theorem Y is also a L^1-predual space. Such spaces are called C_σ-spaces. See Chapter 3, Section 10 of [**10**]. We show that if X a separable L^1-predual space with a non-separable dual such that the range of every self isometry is an almost constrained subspace, then X is isometric to a C_σ-space.

QUESTION 1. *For an uncountable compact metric space K, for the space $C(K)$, is there always a self embedding of $C(K)$ which is not an almost constrained subspace of $C(K)$?*

For an uncountable discrete set Γ we show that any finite dimensional central subspace of $\ell^1(\Gamma)$ is the range of a projection of norm one.

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2. $c_0(\Gamma)$ and $\ell^\infty(\Gamma)$ spaces

We need a well known Lemma on the structure of projections of norm one. See [**14**] . The proofs are included for the sake of completeness. In what follows we canonically embed a Banach space X in its bidual X^{**}. We recall that under this identification we have $Y \subset Y^{\perp\perp} \subset X^{**}$ and Y is a weak*-dense subspace of $Y^{\perp\perp}$. We also note that Y^{**} is canonically isometric to $Y^{\perp\perp}$. We denote by $Q : X^{***} \to X^* \subset X^{***}$ the canonical surjective projection, $\Lambda \to \Lambda|X$ and note that, $\|Q\| = 1$, $\ker(Q) = X^\perp$. Also Q is a weak*-weak* continuous map.

Lemma 2. *Let $Y \subset X$ be a closed subspace. Suppose $P : X^{**} \to Y^{\perp\perp}$ be a surjective projection with $\|P\| = 1$. There exists a contractive projection $R : X^* \to X^*$ such that $\ker(R) = Y^\perp$ and thus $R(X^*)$ is isometric to Y^*.*

Proof. Let $Q : X^{***} \to X^{***}$ be the canonical projection. Let $R = Q \circ P^*|X^*$. It is easy to see that R is a contractive projection. Let x^* be such that $R(x^*) = 0$. For any $y \in Y$, by the definition of Q, $0 = R(x^*)(y) = P^*(x^*)(y) = P(y)(x^*) = x^*(y)$, since $y \in Y^{\perp\perp}$. Thus $x^* \in Y^\perp$. On the other hand for $x^* \in Y^\perp$, $\Lambda \in X^{**}$, $P^*(x^*)(\Lambda) = x^*(P(\Lambda)) = 0$. Thus $R(x^*) = Q(P^*(x^*)) = Q(0) = 0$. Now the quotient space $X^*/Ker(R) = X^*/Y^\perp$ is isometric to Y^*, therefore $R(X^*)$ is isometric to Y^*. □

In what follows we consider the duality, $c_0(\Gamma)^* = \ell^1(\Gamma)$, $c_0(\Gamma)^{**} = \ell^\infty(\Gamma)$. Our next lemma deals with the structure of projections of norm one in $\ell^1(\Gamma)$.

Lemma 3. *Let $P : \ell^1(\Gamma) \to \ell^1(\Gamma)$ be a projection of norm one. Then range of P is a weak*-closed subspace. In particular if $Y \subset c_0(\Gamma)$ is such that $Y^{\perp\perp}$ is the range of a projection of norm one in $\ell^\infty(\Gamma)$, then Y is the range of a projection of norm one in $c_0(\Gamma)$.*

Proof. Let $P : \ell^1(\Gamma) \to \ell^1(\Gamma)$ be a projection of norm one. It is well known that $P(\ell^1(\Gamma)) = \ell^1(\Gamma')$ for some discrete set Γ'. We also recall that under the canonical embedding,

$$\ell^\infty(\Gamma)^* = \ell^1(\Gamma) \oplus_1 (c_0(\Gamma))^\perp$$

(an ℓ^1-direct sum). Similarly

$$\ell^1(\Gamma')^{\perp\perp} = \ell^\infty(\Gamma')^* = \ell^1(\Gamma') \oplus_1 (c_0(\Gamma'))^\perp.$$

It now follows from the proof of Proposition IV.1.10 in [**6**] that $\ell^1(\Gamma')$ is a weak*-closed subspace of $\ell^1(\Gamma)$. If $Y \subset c_0(\Gamma)$ is such that $Y^{\perp\perp}$ is the range of a projection of norm one in $\ell^\infty(\Gamma)$, by Lemma 1, there is a projection $R : \ell^1(\Gamma) \to \ell^1(\Gamma)$ of norm one such that $ker(R) = Y^\perp$. Now the range and null space of R are weak*-closed subspaces. Let $M \subset c_0$ be a closed subspace such that $M^\perp = range(R)$. Thus by an application of the separation theorem and the open mapping theorem we have $c_0 = M \bigoplus Y$. If P is the projection associated with this decomposition with range as Y, we get that $P^* = R$ is a weak*-continuous map. Thus there exists a projection $P : c_0(\Gamma) \to c_0(\Gamma)$ of norm one such that $range(P) = Y$. □

Let X be a Banach space suppose $Y \subset X$ is isometric to $\ell^\infty(\Gamma)$ for some discrete set Γ. Then ignoring the embedding let $\ell^\infty(\Gamma) \subset X$. For $\gamma \in \Gamma$, let $e_\gamma : \ell^\infty(\Gamma) \to \mathbb{R}$ be defined by $e_\gamma(x) = x(\gamma)$ for $x \in \ell^\infty(\Gamma)$. Let $e'_\gamma \in X^*$ be a norm preserving extension of e_γ, for $\gamma \in \Gamma$. Let $P : X \to \ell^\infty(\Gamma)$ be defined by $P(x)(\gamma) = e'_\gamma(x)$. It is easy to see that P is a projection of norm one with range $\ell^\infty(\Gamma)$.

We recall that a closed subspace $M \subset X$ is said to be a M-ideal if there exists a linear projection $P : X^* \to X^*$ such that $ker(P) = M^\perp$

and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. Such a projection P is called an L-projection. See Chapter I of [**6**] for several examples and properties of these spaces. In particular if X is a L^1-predual space, and $J \subset X$ is a M-ideal then both J and the quotient space X/J are L^1-predual spaces. If P is a projection such that $\|x\| = max\{\|P(x)\|, \|x - P(x)\|\}$ for all $x \in X$, we say that P is a M-projection and the range of P is called a M-summand of X. It is also known that P is an L-projection if and only if P^* is a M-projection. See Chapter 1 of [**6**]. If $Y \subset X$ is a M-ideal then $P^{**} : X^{**} \to X^{**}$ is a M-projection whose range is $Y^{\perp\perp} = Y^{**}$.

THEOREM 4. *Let $Y \subset c_0(\Gamma)$ be an infinite dimensional closed subspace. The following statements are equivalent.*

(1) *Y is isometric to $c_0(\Gamma')$ for some discrete set Γ'.*
(2) *Let $\Psi : Y \to c_0(\Gamma)$ be any into isometry. $\Psi(Y)$ is the range of a projection of norm one in $c_0(\Gamma)$*
(3) *Let $\Psi : Y \to c_0(\Gamma)$ be any into isometry. Then $\Psi(Y)$ is a central subspace of $c_0(\Gamma)$.*

PROOF. 1) $\Rightarrow$ 2): Suppose Y is isometric to $c_0(\Gamma')$ for some discrete set Γ'. It is enough to show that Y is the range of a projection of norm one in $c_0(\Gamma)$. Now $Y^{\perp\perp} = Y^{**} = \ell^\infty(\Gamma')$. So by our observation before the theorem, it follows that $Y^{\perp\perp}$ is the range of a projection of norm one in $\ell^\infty(\Gamma)$. The conclusion now follows from Lemma 3. 2) $\Rightarrow$ 3): Easy to see. 3) $\Rightarrow$ 1): We may assume without loss of generality that $Y \subset c_0(\Gamma)$ is a central subspace. We will first show that for any four pairwise intersecting balls $\{B(a_i, r_i)\}_{1 \le i \le 4}$ in Y, $\cap_1^4 B(a_i, r_i) \neq \emptyset$. Since $a_i \in c_0(\Gamma)$ and $|a_i(\alpha) - a_j(\alpha)| \le r_i + r_j$ for all $\alpha \in \Gamma$ and for all i, j. By arguments similar to the ones given in the Introduction (in the case of c_0 and ℓ^∞), it is easy to see that there exists $a \in c_0(\Gamma)$ such that $\|a - a_i\| \le r_i$ for $1 \le i \le 4$. Now by hypothesis there exists $a_0 \in Y$ such that $\|a_i - a_0\| \le \|a_i - a\| \le r_i$ for $1 \le i \le 4$. Hence the claim. Therefore by a well known characterization of L^1-predual spaces in terms of 4-ball intersection property ([**10**] Theorem 6 in Section 21 and [**15**]), Y^* is isometric to $L^1(\mu)$ for some positive measure μ. Since Γ is a discrete set, it is easy to see that μ is a purely atomic measure. Thus Y^* isometric to $\ell^1(\Gamma')$ for some discrete set Γ' (see [**10**] Theorem 6 in Section 22). So by arguments similar to the ones given during the proof of Lemma 3, we get that Y^* is a weak*-closed subspace of $\ell^1(\Gamma)$.

We next show that any M-ideal J in Y is a M-summand. It would then follow from the results in [**18**] that Y is isometric to $c_0(\Gamma')$ for some discrete set Γ'. Let $J \subset Y$ be a M-ideal. It follows from the remarks made above that $J^* \subset Y^* \subset \ell^1(\Gamma)$, J^* is isometric to $\ell^1(A)$ for some discrete set A. Thus J^* is a weak* closed subspace of $\ell^1(\Gamma)$ and hence of Y^*. Therefore the L-projection corresponding to $J^\perp$ is weak*-continuous in Y^*, so that J is the range of a M-projection. □

Proposition 5. *Let Γ be an infinite discrete set and let X be a L^1-predual space. Let $\Phi : X \to c_0(\Gamma)$ be an into isometry. $Y = \Phi(X)$ is the range of a projection of norm one in $c_0(\Gamma)$.*

Proof. We note that $Y^{\perp\perp} = Y^{**} = \Phi((X))^{**} = \Phi^{**}(X^{**}) \subset \ell^\infty(\Gamma) = c_0(\Gamma)^{**}$. Since X^{**} is a $\mathcal{P}_1$-space (see [**15**]), there is a projection of norm one $R : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$ with $range(R) = Y^{\perp\perp}$. By Lemma 3, we get that there is a projection of norm one $R : c_0(\Gamma) \to Y$ such that $range(R) = Y$. □

Our next proposition shows that a weaker form of this property determines isometrically spaces of the form $c_0(\Gamma)$ among L^1-predual spaces.

Proposition 6. *Let Y be a L^1-predual space such that for any L^1-predual space X and for any isometry $\Phi : X \to Y$, $\Phi(X)$ is an almost constrained subspace of Y, then Y is isometric to $c_0(\Gamma)$ for a discrete set Γ.*

Proof. We will show that any M-ideal in Y is a M-summand. It would then follow that Y is isometric to $c_0(\Gamma)$ for some discrete set Γ, Let $Z \subset Y$ be any M-ideal. Since Y is a L^1-predual space, so is Z. Applying the hypothesis for the inclusion map, we get that Z is an almost constrained subspace of Y. Therefore by Proposition 3.17 of [**4**] Z is a M-summand in Y. □

It is easy to see that if $Y \subset X$ is the range of a projection of norm one, then so is $Y^{\perp\perp}$ in X^{**}. Our analysis shows that the converse is true when $X = c_0(\Gamma)$.

Corollary 7. *Let Y be a L^1-predual space. Suppose for every closed subspace $X \subset Y$ such that $X^{\perp\perp}$ is one-complemented subspace of Y^{**}, X is one-complemented in Y. Then Y is isometric to $c_0(\Gamma)$ for some discrete set Γ.*

Proof. In view of Proposition 6, ignoring the isometric embedding, let $X \subset Y$ be a L^1-predual space. Since X^{**} is a $\mathcal{P}_1$-space (see [**15**]), $X^{\perp\perp}$ is the range of a projection of norm one on Y^{**}. Therefore by hypothesis, X is a one complemented subspace of Y and hence is an almost constrained subspace. Now the conclusion follows from Proposition 6. □

We next note that any almost constrained subspace of $\ell^\infty(\Gamma)$ is the range of a projection of norm one. The authors of [**11**] (see page 126) use strong geometric techniques to deduce the same when Γ is a finite set.

Proposition 8. *Let $Y \subset \ell^\infty(\Gamma)$ be an almost constrained subspace. Then Y is the range of a projection of norm one on $\ell^\infty(\Gamma)$.*

Proof. Let $I : Y \to Y$ be identity map. We first claim that any collection $\{B(y_i, r_i)\}_{i \in I}$ of pair-wise intersecting closed balls in Y have non-empty intersection. To see this consider the family of larger balls in $\ell^\infty(\Gamma)$ with the same centers and radii. It is easy to see that any finitely many of them intersect in $\ell^\infty(\Gamma)$. Since ℓ^∞ is a dual space, by weak*-compactness of closed balls there exists a $\alpha \in \ell^\infty$ such that $\|y_i - \alpha\| \leq r_i$ for all $i \in I$. Since Y is an almost constrained subspace, there exists a $y_0 \in Y$ such that $\|y_i - y_0\| \leq r_i$ for all $i \in I$.

Now consider the identity map $I : Y \to Y$. For any $\tau \in \ell^\infty(\Gamma)$, since the closed balls $\{B(y, \|y - \tau\|)\}_{y \in Y}$ pair-wise intersect, let $y_0 \in Y$ be such that $\|y_0 - y\| \leq \|y - \tau\|$ for all $y \in Y$, it is easy to see that $I' : span\{\tau, Y\} \to Y$ defined by $I'(y + a\tau) = y + ay_0$ for any real number a, is a norm preserving extension of I and is a projection onto Y.

An application of Zorn's Lemma based exhaustion argument can be used now to get a contractive projection $P : \ell^\infty(\Gamma) \to Y$. □

As an application we give a new proof of a weaker form of Theorem 4.3 from [**12**]. Our analysis is intricately related to the structure of finite dimensional subspaces.

We note that if X is a finite dimensional space such that X^* contains an isometric copy of $\ell^\infty(k)$ for some $k > 1$, then since there is a contractive projection $P : X^* \to \ell^\infty(k)$, we get that X contains an isometric copy of $\ell^1(k)$ as the range of a projection of norm one. By our observations on ranges of projections of norm one, if a Banach space has an isometric copy $\ell^1(k)$ as the range of a projection of norm one, then X^* has a copy of $\ell^\infty(k)$ as the range of a projection of norm one. We also note that $(x, y) \to (\frac{x+y}{2}, \frac{x-y}{2})$ is an isometry of $\ell^\infty(2)$ and $\ell^1(2)$. For $k > 2$, since the unit ball of $\ell^\infty(k)$ has 2^k extreme points with coordinates coming from $\{\pm 1\}$ where as the unit ball of $\ell^1(k)$ has $2k$ extreme points with only one coordinate as ± 1 and the rest 0 , we see that these spaces are not isometric.

Theorem 9. *Let X be a real Banach space. If X^* contains an isometric copy of $\ell^\infty(3)$ then there exists 4 pair-wise intersecting closed balls in X having empty intersection. Suppose X is finite dimensional and for any 3 pair-wise intersecting balls in X intersect and there exists 4 pair-wise intersecting closed balls in X having empty intersection. Then X^* has an isometric copy of $\ell^\infty(3)$.*

Proof. Suppose X^* contains an isometric copy of $\ell^\infty(3)$ and every collection of 4 pair-wise intersecting closed balls in X has non-empty intersection. Then by Theorem 6.1 in [**16**], X^* is isometric to $L^1(\mu)$. Therefore from our remarks made earlier any isometric copy of $\ell^\infty(3)$ in X^* will be a range of a projection of norm one in $L^1(\mu)$. Hence by Ando-Douglas' theorem we get $\ell^\infty(3)$ is isometric to $\ell^1(3)$. A contradiction. So X has a set of 4 pair-wise intersecting balls whose intersection is empty.

Conversely suppose that X is a finite dimensional space and in X any set of 3 pair-wise intersecting balls intersect and has a set of 4 pair-wise intersecting balls whose intersection is empty. It follows from Corollary

7.4 of [**8**] that X is a combination of ℓ^1 and ℓ^∞ sums of one dimensional subspaces of X. If X has no copy of $\ell^1(3)$ then in Hansen and Lima's decomposition ([**8**]) of X only finite dimensional $\ell^\infty(k)$ spaces appear. It is easy to see that in this case any set of 4 pair-wise intersecting balls have non-empty intersection. So X has an isometric copy of $\ell^1(3)$. Since X is a combination of ℓ^1 and ℓ^∞ sums of one dimensional subspaces of X, there is an isometric copy of $\ell^1(3)$ which is the range of a projection of norm one. Thus X^* has an isometric copy of $\ell^\infty(3)$. □

3. Large separable L^1-predual spaces

Conjecture: If X is a separable L^1-predual space such that X^* is not separable, then there is a self isometry of X whose range is not almost constrained in X?

We only have the following 'positive' result. We recall that for a set $C \subset X$, $c_0 \in C$ is said to be a centre of symmetry if for all $c \in C$, $2c_0 - c \in C$. In what follows we will use a result of Lima and Uttersrud, [**13**], that a Banach space X is isometric to a C_σ space if and only if any three intersecting balls in X have a centre of symmetry.

THEOREM 10. *Let X be a separable L^1-predual space with a non-separable dual. Suppose the range of every isometric embedding of X is an almost constrained subspace of X. Then X is a C_σ space.*

PROOF. Let Δ denote the Cantor set. Since X is a separable L^1-predual space with X^* non-separable, there is an isometry $\Psi : C(\Delta) \to X$ (see [**10**] Theorem 4 in section 22). Since X is a separable Banach space, let $\Phi : X \to C(\Delta)$ be the canonical embedding. Now $\Psi \circ \Phi : X \to X$ is an isometric embedding, so that by hypothesis, $(\Psi \circ \Phi)(X)$ is an almost constrained subspace of X and hence is an almost constrained subspace of $\Psi(C(\Delta))$. Since $\Psi(C(\Delta))$ is a $C(K)$ space for some compact set K, we next show that an almost constrained subspace Z of a $C(K)$ space is a C_σ space. This completes the proof.

Let $\{B(z_i, r_i)\}_{1 \leq i \leq 3}$ be closed balls in Z with $\|z_0 - z_i\| \leq r_i$ for $1 \leq i \leq 3$. Since $C(K)$ is a C_σ space, let f_0 be a centre of symmetry for the intersection of these 3 closed balls in $C(K)$ with centres at z_i. Since Z is an almost constrained subspace, let $P : span\{Z, f_o\} \to Z$ be a projection of norm one. We claim that $P(f_0)$ is a centre of symmetry for the intersecting balls in Z.

Let $z \in Z$ and $\|z - z_i\| \leq r_i$ for $1 \leq i \leq 3$. $\|2P(f_0) - z - z_i\| = \|P(2f_0 - z - z_i)\| \leq r_i$, for all i, as f_0 is a centre of symmetry. Thus $2P(f_0) - z$ is in the intersection of the 3 balls. Hence $P(f_0)$ is a centre of symmetry. Therefore Z is isometric to a C_σ space. □

4. Subspaces of $\ell^1(\Gamma)$

In this section we consider the equivalence of the 3 properties considered here for $\ell^1(\Gamma)$ for an infinite discrete set Γ. We continue to use the duality $c_0(\Gamma)^* = \ell^1(\Gamma)$. Since Γ is possibly uncountable, our arguments differ from traditional arguments based on basic sequences, see [**17**], Proposition 2.a.1.

THEOREM 11. *Let Γ be an infinite discrete set. Let $Y \subset \ell^1(\Gamma)$ be a finite dimensional central subspace. Then Y is the range of a projection of norm one. In particular Y is isometric to $\ell^1(k)$ where $k = dim(Y)$.*

PROOF. We will first indicate a separable reduction procedure. Let $y_1, ..., y_k$ be a basis for Y. Since every element of $\ell^1(\Gamma)$ has countable support, there exists a countable set $A \subset \Gamma$ such that $Y \subset \ell^1(A)$ and $\ell^1(\Gamma) = \ell^1(A) \bigoplus_1 \ell^1(\Gamma - A)$. Let $P : \ell^1(\Gamma) \to \ell^1(A)$ be the contractive projection associated with this decomposition. We next note that Y is an almost constrained subspace of $\ell^1(A)$. It would then follow from Theorem 2.4 in [**11**] that there exists a surjective contractive projection $R : \ell^1(A) \to Y$. Therefore $R \circ P$ is the required contractive projection onto Y. Thus by Ando-Douglas' theorem again, Y is isometric to $\ell^1(k)$ where $k = dim(Y)$.

Since P is a contractive projection, clearly $Y \subset \ell^1(A)$ is a central subspace. Let $x \in \ell^1(A)$. Consider the family of closed balls $\{B(y, \|y-x\|)\}_{y\in Y}$ in Y. For any $y_1, .., y_n \in Y$, since Y is a central subspace of $\ell^1(A)$, there exists a $y_0 \in Y$ such that $\|y_i - y_0\| \leq \|y_i - x\|$ for $1 \leq i \leq n$. Thus any finite collection of sets from $\{B(y, \|y - x\|)\}_{y\in Y}$ intersect. Therefore by compactness we get a $y_0 \in Y$ such that $\|y - y_0\| \leq \|y - x\|$ for all $y \in Y$. Hence Y is an almost constrained subspace of $\ell^1(A)$. ☐

REMARK 12. *Arguments similar to the ones given above can be used to show that if $Y \subset \ell^1(\Gamma)$ is a separable almost constrained subspace then Y is the range of a projection of norm one on $\ell^1(\Gamma)$.*

QUESTION 13. *If $Y \subset \ell^1(\Gamma)$ is an almost constrained subspace, is it a weak*-closed subspace, w. r. t the duality $c_0(\Gamma)^* = \ell^1(\Gamma)$?*

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