Probabilistic Methods in Geometry, Topology and Spectral Theory

CRM Workshops
Probabilistic Methods in Spectral Geometry PDE
August 22–26, 2016
Probabilistic Methods in Topology
November 14–18, 2016
Centre de Recherches Mathématiques, Université de Montréal, Québec, Canada

Yaiza Canzani
Linan Chen
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Preface

Thematic Semester on Probabilistic Methods in Geometry, Topology and Spectral Theory was held at Centre de Recherches Mathématiques in Montreal from August until December 2016. Probabilistic methods have played an increasingly important role in many areas of mathematics, from the study of random groups and random simplicial complexes in topology, to the theory of random Schrödinger operators in mathematical physics. The Thematic Semester at CRM (organized by the CRM Mathematical Analysis Laboratory and the CRM Probability Laboratory) included five intensive one-week workshops: Frontiers in Mathematical Physics workshop in Honour of Barry Simon’s 70th Birthday; Probabilistic Methods in Spectral Geometry and PDE held on August 22–26, 2016; Random Growth Problems and Random Matrices; Probabilistic Methods in Dynamical Systems and Applications; and Probabilistic Methods in Topology, held on November 14-18. Workshops on Probabilistic Methods in Spectral Geometry and PDE, and on Probabilistic Methods in Topology are described below in more detail. The thematic semester featured an emphasis on interconnections and cross-fertilization of ideas between these topics, leading to new investigations and fruitful collaborations between participants. There were three Aisenstadt chairs during the semester: Yuval Peres, Scott Sheffield and Nalini Anantharaman. Nalini Anantharaman gave her lectures during the workshop on Probabilistic Methods in Spectral Geometry and PDE, whose participants contributed many papers to this volume. Other papers in this volume are related to the workshop on Probabilistic Methods in Topology, which included a series of introductory lectures by Matthew Kahle, and a lecture by Misha Gromov.

The workshop on Probabilistic Methods in Spectral Geometry and PDE was organized by Yaiza Canzani (Harvard), Linan Chen (McGill), Dmitry Jakobson (McGill), Armen Shirikyan (Cergy-Pontoise), Lior Silberman (UBC) and John A. Toth (McGill). The workshop brought together some of the leading researchers in quantum chaos, semi-classical theory, ergodic theory and dynamical systems, partial differential equations, probability, random matrix theory, mathematical physics, conformal field theory, and random graph theory. Its emphasis was on the use of ideas and methods from probability in different areas, such as quantum chaos (study of spectra and eigenstates of chaotic systems at high energy); geometry of random metrics and related problems in quantum gravity; solutions of partial differential equations with random initial conditions. The workshop was part of a series of related workshops held at the CRM and elsewhere, including a workshop on infinite-dimensional geometry (MSRI, December 2013), a workshop on the geometry of eigenvalues and eigenfunctions, and a workshop on manifolds of metrics and probabilistic methods in geometry and analysis (both held at the CRM in 2012).
Many of the talks concerned delocalization (equidistribution) of eigenfunctions at high energy, or the so-called quantum ergodicity. In particular this was the main topic of Nalini Anantharaman’s first two Aisenstadt lectures. Quantum ergodicity is also the subject of three papers in the current volume.

In the paper *Quantum Limits of Eisenstein Series in $\mathbb{H}^3$*, Niko Laaksonen studies the quantum limits of Eisenstein series off the critical line for $\text{PSL}_2(\mathbb{O}_K)$ where $K$ is an imaginary quadratic field of class number one. He generalizes previous results of Petridis, Raulf and Risager on $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$.

In the paper *Observability and Quantum Limits for the Schrödinger Equation on $S^d$*, Fabricio Macià and Gabriel Riviére describe their recent results on semiclassical measures for the Schrödinger evolution on Zoll manifolds. The authors focus on the particular case of eigenmodes of the Schrödinger operator on the sphere endowed with its canonical metric. They also explore the relation of this problem with the observability question from control theory.

In the paper *Entropy Bounds and Quantum Unique Ergodicity for Hecke Eigenfunctions on Division Algebras*, Lior Silberman and Akshay Venkatesh prove the Arithmetic Quantum Unique Ergodicity conjecture for non-degenerate sequences of Hecke eigenfunctions on quotients $\Gamma \backslash G/K$, where $G \simeq \text{PGL}_d(\mathbb{R})$, $K$ is a maximal compact subgroup of $G$ and $\Gamma < G$ is a lattice associated to a division algebra over $\mathbb{Q}$ of prime degree $d$. More generally, they introduce a new method of proving positive entropy of quantum limits, which applies to higher-rank groups.

Several talks at the workshop were devoted to the study of “random” eigenfunctions; that subject was considered in three papers in this volume.

Linan Chen and Na Shu contributed the paper *A Geometric Treatment of Log-correlated Gaussian Fields*. One way to regularize a log-correlated Gaussian free field (GFF) is to consider (functionals of) its spherical averages. In even dimensions, this regularization approach has been adopted in the construction of the Liouville Quantum Gravity (LQG) measure and the proof of the Knizhnik-Polyakov-Zamolodchikov (KPZ) formula. In this article, the authors combine the Fourier-Bessel expansion with the spherical averages of the GFF to extend such a regularization approach to treat log-correlated GFFs in odd dimensions. In particular, the authors established the existence of the LQG measure and the KPZ formula under this setting.

In the paper *Tangent Nodal Sets for Random Spherical Harmonics*, Suresh Eswarathasan considers a fixed vector field $V$ on $S^2$ and studies the distribution of points which lie on the nodal set (of a random spherical harmonic) where $V$ is also tangent. He shows that the expected value of the corresponding counting function is asymptotic to the eigenvalue with a leading coefficient that is independent of the vector field $V$. This demonstrates, in some form, a universality for vector fields up to lower order terms.

In the survey paper *Random Nodal Lengths and Wiener Chaos*, Maurizia Rossi collects some of the recent results on the “nodal geometry” of random eigenfunctions on Riemannian surfaces. She focuses on the asymptotic behaviour, for high energy levels, of the nodal length of Gaussian Laplace eigenfunctions on the torus (arithmetic random waves) and on the sphere (random spherical harmonics). She gives some insight on both Berry’s cancellation phenomenon and the nature of nodal length second order fluctuations (non-Gaussian on the torus and Gaussian on the sphere) in terms of chaotic components. Finally she considers the general
case of monochromatic random waves, i.e. Gaussian random linear combination of
eigenfunctions of the Laplacian on a compact Riemannian surface with frequencies
from a short interval, whose scaling limit is Berry’s Random Wave Model.

Another subject discussed at the workshop was the study of random metrics
and random maps. This is related to two papers in the current volume: the paper
of Linan Chen and Na Shu described earlier; and the paper contributed by V. Kon-
akov, S. Menozzi and S. Molchanov, titled *The Brownian Motion on $\text{Aff}(\mathbb{R})$ and
Quasi-Local Theorems*. This paper is concerned with Random walk approximations
of the Brownian motion on the Affine group $\text{Aff}(\mathbb{R})$. The authors are in particular
interested in the case where the innovations are discrete. In this framework, the
return probabilities of the walk have fractional exponential decay in large time, as
opposed to the polynomial one of the continuous object. The authors prove that
integrating those return probabilities on a suitable neighbourhood of the origin, the
expected polynomial decay is restored.

The workshop on Probabilistic Methods in Topology was the last workshop
held during the thematic semester on Probabilistic Methods in Geometry, Topol-
yogy and Spectral Theory at CRM. It was organized by D. Wise (McGill), M. Pichot
(McGill), L. Silberman (UBC), P. Przytycki (McGill), I. Rivin (Temple), A. Nabu-
tovsky (Toronto), M. Kahle (Ohio State). It brought together researchers working
on random simplicial complexes and geometry of spaces of triangulations (with
connections to manifold learning); topological statistics, and geometric probability;
theory of random groups and their properties; random knots; and other problems.

Many talks at the workshop concerned the study of random graphs, random
simplicial complexes and their properties, including the talks by Kahle, Bubenik,
Addario-Berry, Abu Fraiha, Bobrowski, Behrstock, Lishak, Farber, Peled and
Luczak. That was also the subject of the last Aisenstadt lecture by Nalini Anan-
tharaman, who discussed asymptotic equidistribution of Laplace eigenvectors on
large graphs. Two of the papers in the current volume are devoted to the study of
random graphs.

In the paper *Formal Zeta Function Expansion and the Frequency of Ramanujan
Graphs*, Joel Friedman shows that logarithmic derivative of the Zeta function of any
regular graph is given by a power series about infinity whose coefficients are given
in terms of the traces of powers of the graph’s Hashimoto matrix. He then makes
a formal calculation that suggests that for fixed $d$ there is an $f(d) > 1/2$ such that
a $d$-regular graph on $n$ vertices is Ramanujan with probability at least $f(d)$ for $n$
sufficiently large. A related argument for random covering graphs of degree $n$
over a fixed, regular “base graph” suggests that for $n$ large, a strict majority of random
covering graphs are relatively Ramanujan.

The talks by Rivin and Even-Zohar concerned the study of random knots. An
important tool in the study of knots are knot polynomials. The study of those
polynomials is related to the study of Tutte (dichromatic) polynomials of graphs;
and to the study of Bollobás-Riordan polynomials of *ribbon graphs* (graphs that
are cellurally embedded into surfaces). In their paper *Rank and Bollobás-Riordan
Polynomials: Coefficient measures and zeros*, D. Jakobson, T. Langsetmo, I. Rivin
and L. Turner discuss some (numerical and theoretical) results about the coefficients
and zeros of Tutte (dichromatic) polynomial of graphs of bounded degree
whose size increases. In particular, certain natural coefficient measures converge
to a delta-function, provided that the corresponding graphs converge in the sense
of Benjamini-Schramm. The authors also establish related results for Bollobás-Riordan polynomials for large ribbon graphs.

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This volume contains the proceedings of the CRM Workshops on Probabilistic Methods in Spectral Geometry and PDE, held from August 22–26, 2016 and Probabilistic Methods in Topology, held from November 14–18, 2016 at the Centre de Recherches Mathématiques, Université de Montréal, Montréal, Quebec, Canada.

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The workshop Probabilistic Methods in Topology brought together researchers working on random simplicial complexes and geometry of spaces of triangulations (with connections to manifold learning); topological statistics, and geometric probability; theory of random groups and their properties; random knots; and other problems.

This volume covers recent developments in several active research areas at the interface of Probability, Semiclassical Analysis, Mathematical Physics, Theory of Automorphic Forms and Graph Theory.