CONTEMPORARY MATHEMATICS

755

Centenary of the Borel Conjecture

Marion Scheepers
Ondřej Zindulka
Editors

AMERICAN MATHEMATICAL SOCIETY

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10 9 8 7 6 5 4 3 2 1 25 24 23 22 21 20

Contents

Preface

It is a common phenomenon in mathematics that an innocuous observation made in the process of a wider research project stimulates long term and broad impact on the development of Mathematics. What has been called the Borel Conjecture is a prime example of this phenomenon. In a 1919 paper the mathematician Émile Borel, exploring the then recently established notion of measurability in terms of the Lebesgue measure, noted that countable sets of real numbers have a certain covering property, and he speculated that only countable sets of real numbers have that property.

In 1919 set theory was still a young field, with axioms recently formulated, and driven by several very basic questions, including the question whether the Continuum Hypothesis holds. Less than ten years after Borel's paper appeared, information about Borel's Conjecture emerged from the exploration of consequences of the Continuum Hypothesis: Sierpiński proved that the Continuum Hypothesis implies the negation of Borel's Conjecture. When Gödel established a few years later that the Continuum Hypothesis is consistent, one of the consequences was of course that Borel's Conjecture cannot be proven from the adopted axiom system. Almost half a century later, in 1976, using the breakthrough technique of forcing, Laver proved that also the negation of Borel's Conjecture cannot be proven from the adopted axiom system: Borel's Conjecture is independent of the axioms adopted as foundation for Mathematics. This independence proof also introduced the new technique of countable support iterated forcing onto the mathematical arena.

The independence of Borel's Conjecture from the foundational axioms opens a large field of exploration of alternative versions of the mathematical universe, depending on whether Borel's Conjecture is adopted as an axiom, or whether its negation is adopted as an axiom. One might ask whether Borel's Conjecture or its negation would have any significant or interesting impact on the corresponding mathematical universe. And to a large extent the unabated research on the mathematical objects Borel's Conjecture is about sheds light on this question. As the reader will see from the papers in this volume, the impact of Borel's Conjecture on mathematics is broad and interesting: There are equivalents of Borel's Conjecture in terms of infinite two-person games, there are equivalents of Borel's Conjecture in terms of properties of certain topological groups, in terms of Ramsey theory, and numerous other mathematical notions that at first glance appear unrelated. In addition, the basic idea underlying Borel's Conjecture inspired, in light of the many mathematical statements found to be equivalent to Borel's Conjecture, several innovative mathematical concepts, and related questions of consistency and independence, as the reader will see from the papers in this volume.

It was not possible to represent all the developments related to the Borel Conjecture in the time-frame envisioned for putting this volume together. For example, the reader will notice that there is no paper that reports on BMax (for those familiar with this specialized terminology), or on the several generalizations of the covering property at the center of Borel's Conjecture beyond the world of metrizable topological spaces and groups. On the time-scale of mathematics as a field, exploration of the Borel Conjecture and its relatives is still in its infancy. With this volume we hope to initiate a tradition of updating, at appropriate times, the developments in mathematics inspired by the Borel Conjecture.

The papers in this volume are listed, as is the standard in mathematics, alphabetically by last name of the authors.

We thank the American Mathematical Society for publishing this volume in the Contemporary Mathematics series. In particular, we thank Christine Thivierge of the American Mathematical Society for her excellent guidance throughout the whole process of preparing this volume and we thank Mike Saitas for his very efficient and accommodating management of the production of this volume. Last, but not least, we thank the referees of the papers appearing in this volume for their selfless work in helping to maintain standards towards a continuing credible record of mathematical research.

Marion Scheepers
Ondřej Zindulka

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Borel's Conjecture entered the mathematics arena in 1919 as an innocuous remark about sets of real numbers in the context of a new covering property introduced by Émile Borel. In the 100 years since, this conjecture has led to a remarkably rich adventure of discovery in mathematics, producing independent results and the discovery of countable support iterated forcing, developments in infinitary game theory, deep connections with infinitary Ramsey Theory, and significant impact on the study of topological groups and topological covering properties.

The papers in this volume present a broad introduction to the frontiers of research that has been spurred on by Borel's 1919 conjecture and identify fundamental unanswered research problems in the field. Philosophers of science and historians of mathematics can glean from this collection some of the typical trends in the discovery, innovation, and development of mathematical theories.

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